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The mechanism of pattern selection in coupled map lattices ☆

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Abstract

The pattern selection of one-dimensional coupled map lattices is studied in this paper. It is shown by spatiotemporal variable separation that there exists a threshold wavelength in pattern selection which possesses wave-like structures in space and periodic chaotic motion in time.

1. Introduction

As an abstraction of real nonlinear phenomena, coupled map lattices (CMLs) are proposed to investigate space–time complexity in the current paper, for their possible applications to turbulence, pattern formation in natural systems, etc. Here we would like to examine the following simplest model, the one-dimensional CML with periodic boundary conditions,

$$u_{n+1}^{k}=(1-\epsilon)f(u_n^k)+\tfrac{1}{2}\epsilon[f(u_n^{k+1})+f(u_n^{k-1})]\,,$$
$$u_n^0=u_n^N,\quad u_n^1=u_n^{N+1},\quad k=1,2,...,N\,, \tag{1}$$

where ϵ is the coupling parameter, $0\leqslant\epsilon\leqslant\frac{1}{2}$, n denotes the discrete time step, k the kth lattice point and f is the nonlinear map, here taken as the logistic map

$$f(x)=1-ax^2\,, \tag{2}$$

in which a is the nonlinearity parameter. Kaneko has obtained the following evolution series of space–time patterns by numerical simulation when ϵ is not small and a increases [1]: pattern freezing (space–time order) → pattern selection → pattern competition intermittence → fully developed turbulence (space–time chaos).

Although the above space–time patterns are characterized by some quantifier including spectrum, entropy and Lyapunov exponents, their dynamical mechanism has not been clearly understood yet. We have discussed the mechanism of pattern freezing by the coarse-grain method [2]. In this paper, we are concerned with the mechanism of pattern selection in CMLs whose precise description can be given as follows:

(1) The temporal motion in each domain is almost period-2^k, that is, 2^k-periodic chaos, in which $k=1,2,...$.

(2) The spatial patterns have wave-like structures and there does exist a threshold wavelength or structural wavelength L independent of the initial values such that the length of each domain is less than $\frac{1}{2}L$.

Here the method of space–time variable separation is used in CMLs such that the space and time states are formulated independently. The generalized dispersion relation is obtained, which shows that temporal motions are dependent on spatial struc-

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tures. With these results in mind we can investigate the mechanism of pattern selection and space–time periodic structures.

2. Space–time variable separation in CMLs

For wave-like structures of pattern selection in CML (1) with periodic boundary conditions, we may introduce the space–time state variable separation with a drift,

$$u_n^k = A + B_n \cos(k\omega) , \qquad (3)$$

where A is a constant, $\omega = 2m\pi/N$ is the wave number determined by the periodic boundary condition.

Substituting (3) into (1), we have

$$A + B_{n+1}\cos(k\omega) = 1 - aA^2 - aB_n(\alpha - \beta B_n) , \qquad (4)$$

in which

$$\alpha = 2A[1 - 2\epsilon \sin^2(\tfrac{1}{2}\omega)]\cos(k\omega) ,$$

$$\beta = \tfrac{1}{2}[1 + (1 - 2\epsilon \sin^2\omega)\cos(2k\omega)] .$$

Let $A = [(1+4a)^{1/2} - 1]/2a$. We have

$$B_{n+1}\cos(k\omega) = -aB_n(\alpha + \beta B_n) . \qquad (5)$$

If $\cos(k\omega) = 0$, $u_n^k = A$, the kth lattice is just a node of the wave-like pattern. If $\cos(k\omega) \neq 0$, we introduce the scalar transformation in (5) as follows,

$$B_n = -(\alpha/\beta)x_n .$$

Therefore

$$x_{n+1} = \lambda x_n(x_n - 1) = -\lambda x_n(1 - x_n) , \qquad (6)$$

where

$$\lambda = [(1+4a)^{1/2} - 1][1 - 2\epsilon \sin^2(\tfrac{1}{2}\omega)] . \qquad (7)$$

(6) is the so-called generalized logistic map with $-\lambda \in [-2, 0]$, which represents the temporal motions of wave-like structures in pattern selection. The bifurcation parameter μ in the logistic map $x \to \mu x(1-x)$ is usually taken to be in the range between 0 and 4, and we have the well-known bifurcation diagram showing how the steady states depend on μ. If the range of the parameter μ is extended to the interval $[-2, 0]$, a similar bifurcation diagram is obtained by numerical simulation (see Fig. 1). As a matter of fact, let $\lambda = \mu - 2$, $\mu \in [2, 4]$, then the bifurcation structures of (6) are readily obtained from the results of the logistic map. Especially, for the periodic chaotic interval $\mu_1 = 3.569954 < \mu < \mu_2 = 3.678573$ in the logistic map [3], we can find the corresponding interval $\lambda_1 = 1.569954 < \lambda < \lambda_2 = 1.678573$ in (6).

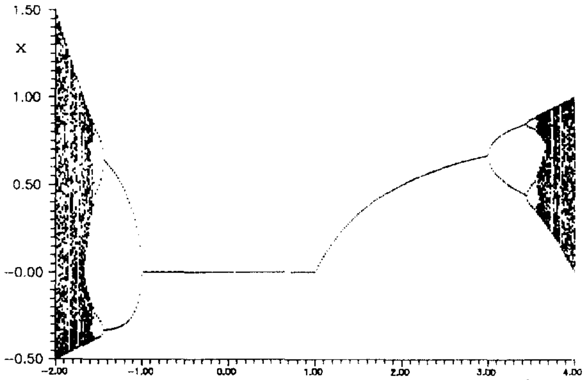

Fig. 1

(7) shows that the temporal motions in CMLs depend on the nonlinearity parameter a, the coupling coefficient ϵ and the spatial wave number ω. In fact, Eq. (7) is a generalized dispersion equation which reflects the temporal motion dependence of the spatial wave numbers. Consequently, (7) can be used to study the relation of spatial waves and temporal chaos.

Based on the previous argument, the space–time periodic structures of CMLs are easily investigated by means of (6) and (7). When $N=4$, $a=0.96$ and $\epsilon=0.2$, for example, numerical calculation shows $u_n^k = 0.9981849$ and $u_{n+1}^k = 0.0438170$, $k = 1, 2, 3, 4$, which is the period-2 pattern with wavelength $L=1$ or $\omega = 2\pi$. On the other hand, the corresponding parameter $\lambda = 1.2$ from (7) just falls in the range of the period-2 motions in Fig. 1, which explains the relation between space–time motions of CMLs.

3. The mechanism of pattern selection in CMLs

The mechanism of pattern selection in CMLs is studied by means of (6) and (7) in this section. For pattern selection in a CML, x_n in Eq. (6) remains in the periodic chaotic state. Therefore, λ satisfies the following inequalities according to the discussion in section 2: $\lambda_1 \leqslant \lambda \leqslant \lambda_2$. Then

$$\frac{1-\lambda_1/[(1+4a)^{1/2}-1]}{2\epsilon} \geqslant \sin^2(\omega) \geqslant \frac{1-\lambda_2/[(1+4a)^{1/2}-1]}{2\epsilon},$$

which leads to the bounds of the circular frequency by the above inequality,

$$\omega_{\max} \geqslant \omega \geqslant \omega_{\min}, \qquad (8)$$

where

$$\omega_{\max} = 2 \arcsin\left(\frac{1-\lambda_1/[(1+4a)^{1/2}-1]}{2\epsilon}\right)^{1/2},$$

$$\omega_{\min} = 2 \arcsin\left(\frac{1-\lambda_2/[(1+4a)^{1/2}-1]}{2\epsilon}\right)^{1/2}.$$

In order that inequality (8) holds, we have to set

$$1 \geqslant \frac{1-\lambda_1/[(1+4a)^{1/2}-1]}{2\epsilon} \geqslant 0,$$

$$1 \geqslant \frac{1-\lambda_2/[(1+4a)^{1/2}-1]}{2\epsilon} \geqslant 0,$$

that is

$$(1-2\epsilon)[1+(1+4a)^{1/2}] \leqslant \lambda_1,$$

$$a \geqslant \tfrac{1}{4}[(\lambda_2+1)^2-1] = 1.543689. \qquad (9)$$

Inequalities (9) determine the parameter region of pattern selection in CMLs (see Fig. 2) which is produced by numerical simulation in Ref. [1].

Then the range of wavelength is given by (8),

$$2\pi/\omega_{\max} \leqslant L \leqslant 2\pi/\omega_{\min},$$

from which the threshold wavelength can be derived,

$$L = 2\pi/\omega_{\min}.$$

The curves for $L = 2\pi/\omega_{\min}(a, \epsilon) = 5, 10, 15, 20, 25, 30$ on the parameter plane (a, ϵ) (see Fig. 3) illustrate how the parameters a and ϵ determine the largest structure wavelength. Satisfactorily enough, it is in good agreement with the results of Kaneko [1] that there exists some special L in the parameter region of pattern selection.

The stability of the pattern in the form (3) can be examined according to the maximum eigenvalue ρ of

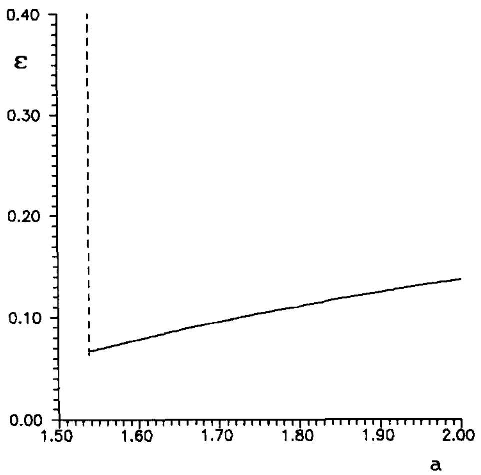

Fig. 2.

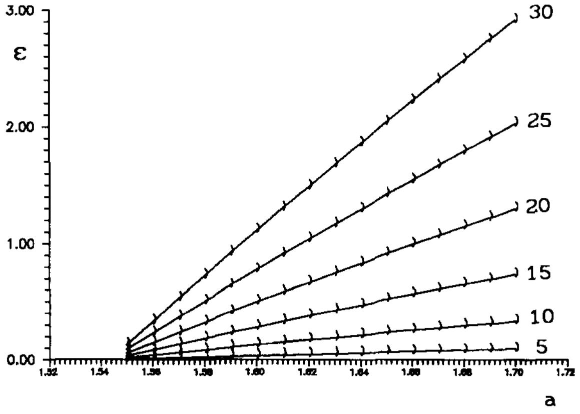

Fig. 3.

the coefficient matrix in the following linearized equation of (1),

$$\delta u_{n+1}^k = (1-\epsilon) f'(u_n^k) \delta u_n^k + \tfrac{1}{2}\epsilon [f'(u_n^{k+1}) \delta u_n^{k+1} + f'(u_n^{k-1}) \delta u_n^{k-1}],$$

$$u_n^0 = u_n^N, \quad u_n^1 = u_n^{N+1}, \quad k = 1, 2, \ldots, N,$$

where $u_n^k = A + B_n \cos(k\omega)$. The pattern in the form (3) is stable if and only if the modulus of ρ is less than 1. Although it is very difficult to determine ρ analytically, we can estimate its upper limit. Based on the well-known Gerschgorin theorem [4], we obtain

$$|p| \leqslant \max\{|(1-\epsilon)f'(u_n^k)| + \tfrac{1}{2}\epsilon[|f'(u_n^{k+1})| + |f'(u_n^{k-1})|]\}$$
$$= 2a \max\{(1-\epsilon)|u_n^k| + \tfrac{1}{2}\epsilon(|u_n^{k+1}| + |u_n^{k-1}|)\}$$
$$= 2a \max\{(1-\epsilon)|A+B_n\cos(k\omega)| + \tfrac{1}{2}\epsilon[|A+B_n\cos((k+1)\omega)| + |A+B_n\cos((k-1)\omega)|]\},$$

which leads to the conclusion that if the following inequality,

$$2a \max\{(1-\epsilon)|A+B_n\cos(k\omega)| + \tfrac{1}{2}\epsilon[|A+B_n\cos((k+1)\omega)| + |A+B_n\cos((k-1)\omega)|]\} < 1,$$

holds, the pattern in the form (3) is stable. It is easily found from the above inequality that the stability of the pattern in the form of (3) depends on a, ϵ and the selected frequency ω or the wavelength L.

4. Conclusion and remarks

The method of space–time variable separation seems to be an effective approach to study spatiotemporal complexity. The key point is to derive a generalized dispersion relation which links the parameters a, ϵ, ω. As an example, the one-dimensional CML of the logistic map with periodic boundary conditions has been examined in detail. Our theory has been employed to find the parameter range and threshold wavelength, which turn out to be in good agreement with Kaneko's numerical results so that the mechanism of evolution from pattern selection to fully developed turbulence is partially elucidated. Further investigation is absolutely needed to reveal why turbulence occurs in the present CML model.

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