

MAGNETOSTATIC CONFIGURATION OF A CONFINED MAGNETIC FLUX TUBE

X. ZHANG and W. R. HU

Institute of Mechanics, Chinese Academy of Sciences, Beijing, China

(Received 5 August, 1991)

Abstract. In this paper an isolated magnetic flux tube confined in stratified atmosphere is studied for slender and axisymmetric model. The functions of the pressure, density, and temperature are expanded as a Taylor series of magnetic surface function ψ . Several models of an isolated magnetic flux tube confined in a stratified atmosphere are constructed, and the external pressure of the stratified atmosphere decreases reasonably with increasing height. The distribution of thermal dynamic quantities and the magnetic pressure in the flux tube are also obtained.

1. Introduction

In astrophysics, most of the theoretical models of magnetic field deal with the continuous fields. However, observations, for example, in the solar atmosphere, show that the magnetic field often appears generally in concentrated configuration known as discrete and isolated flux tubes with magnetic field of a few of kG. These flux tubes are embedded in a surrounding medium without or with very weak field (Parker, 1979; Spruit, 1981). The case is perhaps similar for other celestial bodies. An isolated magnetic flux tube is generally the basic configuration of cosmic magnetic fields.

The problem of magnetostatic equations under the boundary condition of total pressure conservation is a nonlinear problem with free boundary. Therefore, it is difficult to be tackled analytically even in the simple case of a potential field. Various limitations have been made in order to get approximate solutions. The simplest model is one-dimensional, which assumes all physical quantities be uniform across the flux tube and depend only on height in flux tube (cf. Roberts and Webb, 1978; Unno and Ribes, 1979). Parker (1979) considered a cylindrical flux tube which properties are uniform in the direction of the symmetric axis. However, the two-dimensional model is more acceptable. Browning and Priest (1983), Pneuman *et al.* (1986) describe a method in which all quantities are expanded as a power series of the radial distance from the axis. Wilson (1977) gave a two-dimensional magnetostatic model of a tapering flux tube. The problem is simplified in these models by prescribing the shape of the flux tube and, then, giving the external pressure $p_e(z)$ from boundary condition, though, the pressure is physically expected to be prescribed. Hu employed the perturbation method to get the two-dimensional features of flux tubes and obtained an asymptotic expression based on the small angle between the field and the symmetric axis of the slender tube (Hu, 1987, 1989).

In the present paper we discuss a slender flux tube whose longitudinal length scale is large compared with the radius. Since the radial variation will be weak and ψ is an

intrinsic quantity which describes the boundary shape, expanding all quantities in a series of ψ can be applied. In Section 2, the general magnetostatic equations are presented. Mathematical treatment are included in Section 3. Section 4 gives a sort of model. The last section is the discussion and conclusions.

2. Flux Tube Model

The magnetohydrostatic equations may be written (cf. Hu, 1986) as

$$\frac{1}{4\pi} (\nabla \times \mathbf{B}^*) \times \mathbf{B}^* - \rho^* \mathbf{g} - \nabla p^* = 0, \quad (2.1)$$

$$\nabla \cdot \mathbf{B}^* = 0, \quad (2.2)$$

$$p^* = \rho^* T^* k / m, \quad (2.3)$$

where \mathbf{B}^* is the magnetic field, p^* , ρ^* , and T^* are, respectively, the thermodynamical pressure, density, and temperature of plasma, \mathbf{g} the gravity acceleration, m the mass of particle, and k the Boltzmann constant. In the cylindrical coordinate we have

$$\mathbf{g} = \frac{GM^*}{(r^2 + z^2)^{3/2}} (\mathbf{e}_r + \mathbf{e}_z) = -GM^* \nabla \frac{1}{(r^2 + z^2)^{1/2}}, \quad (2.4)$$

where G and M^* are, respectively, the gravitational constant and the mass of the star. We consider the polytropic process for simplification, and it requires

$$p^* = c \rho^{*n}, \quad (2.5)$$

where c is a constant and n the polytropic index.

Now we introduce the dimensionless quantities

$$\begin{aligned} \mathbf{B} &= \frac{\mathbf{B}^*}{B_0}, & p &= \frac{p^*}{p_0}, & \rho &= \frac{\rho^*}{\rho_0}, & T &= \frac{T^*}{T_0}, \\ r &= \frac{r^*}{r_0}, & \beta &= \frac{p_0}{B_0^2/4\pi}, & \sigma &= \frac{CM^*\rho_0/r_0}{B_0^2/4\pi}, & \delta &= \frac{\sigma}{\beta}, \end{aligned} \quad (2.6)$$

where the subscript 0 denotes the typical value.

The dimensionless magnetostatic equations inside the flux tube can be written as

$$-\beta \rho \nabla \left[Q - \frac{\delta}{(r^2 + z^2)^{1/2}} \right] + (\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \quad (2.7)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.8)$$

$$p = \rho T, \quad (2.9)$$

$$p = \rho^n, \quad (2.10)$$

where

$$Q = \int \frac{1}{\rho} dp. \quad (2.11)$$

In the region outside the flux tube, the magnetic field is relatively weak and the dimensionless static equilibrium condition given by the stratified atmosphere mode requires:

$$\frac{dp_e}{dz} + \rho_e \mathbf{g} = 0, \quad (2.12)$$

$$p_e = f(\rho_e), \quad (2.13)$$

where subscript e denotes the value outside the flux tube.

The boundary of magnetic flux tube Γ should be a 'contact' discontinuous surface. It requires that

$$\left[p + \frac{B^2}{2\beta} \right]_{\Gamma} = p_e, \quad (2.14)$$

here we assume that the magnetic field inside the flux tube has no influence on the distribution of stratified atmosphere outside. In this case, the problem of solving Equations (2.7)–(2.13) under boundary condition (2.14) may be decoupled, and we solve only equations of flux tubes (2.7)–(2.13) under boundary condition (2.14), where p_e is given from (2.12) and (2.13). Even in this decoupled problem, the exact solution is not easy to be obtained due to the nonlinear property in equations and in boundary condition. In the cylindrical-coordinate system, the axisymmetric magnetic field may be expressed as

$$\mathbf{B} = \left(-\frac{1}{r} \frac{\partial \psi}{\partial z}, B_{\theta}, \frac{1}{r} \frac{\partial \psi}{\partial r} \right) = B_{\theta} \mathbf{e}_{\theta} + \nabla \times \left(\frac{\psi}{r} \mathbf{e}_{\theta} \right), \quad (2.15)$$

where function ψ is the magnetic surface function. According to Equations (2.7)–(2.10), we have

$$\mathbf{e}_{\theta} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \quad (2.16)$$

$$\mathbf{B} \cdot \nabla \left[Q - \frac{\delta}{(r^2 + z^2)^{1/2}} \right] = 0. \quad (2.17)$$

Thus we obtain

$$B_{\theta} = \frac{1}{r} G(\psi), \quad (2.18)$$

$$\int \frac{1}{\rho} dp - \frac{\delta}{(r^2 + z^2)^{1/2}} = f(\psi). \quad (2.19)$$

By use of (2.18) and (2.19), Equation (2.7) becomes

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = - \left(r^2 \beta \rho \frac{\partial f}{\partial \psi} + G \frac{\partial G}{\partial \psi} \right). \quad (2.20)$$

It shows that the magnetic surface function ψ is an intrinsic quantity of the problem. This is the reason why we use it as an expansion quantity.

3. Mathematical Treatment

The magnetic surface function ψ will be small for a slender tube, if we take its zero value at the symmetric axis. We can express

$$r^l = \sum_{n=0}^{\infty} s_{n+1}(z) \psi^{n+1}, \quad (3.1)$$

where $\psi = 0$ associate with $r = 0$.

The functions $f(\psi)$, $G(\psi)$, and $\rho(\psi, z)$ may be expanded as

$$f(\psi) = \sum_{n=0}^{\infty} f_{n/l} \psi^{n/l}, \quad (3.2)$$

$$G(\psi) = \sum_{n=0}^{\infty} G_{n/l} \psi^{n/l}, \quad (3.3)$$

$$\rho = \sum_{n=0}^{\infty} \rho_{n/l}(z) \psi^{n/l}, \quad (3.4)$$

where f_i , G_i are suitable constants and l is a constant no less than 2 due to the requirement of regularity. By substituting (3.1)–(3.4) into (2.20), we find that only $l = 2$ is appropriate. A neglect of terms larger than second-order of magnitude in the expansions, requires that

$$G_0 = 0, \quad G_{1/2} = 0, \quad G_{3/2} = 0, \quad (3.5)$$

$$f_{1/2} = 0, \quad f_{3/2} = 0, \quad \rho_{1/2} = 0, \quad \rho_{3/2} = 0, \quad (3.6)$$

$$s_1^{-2}(8s_2 + s_1 s_{1zz} - 2s_{1z}^2) - \beta s_1 \rho_0 f_1 = G_1^2, \quad (3.7)$$

$$s_1^{-3}(-16s_2^2 - 2s_1 s_2 s_{1zz} + 10s_2 s_{1z}^2 + s_1^2 s_{2zz} - 6s_1 s_{1z} s_{2z} + 12s_1 s_3) - \beta(s_2 \rho_0 f_1 + s_1 \rho_1 f_1 + 2s_1 \rho_0 f_2) = 3G_1 G_2. \quad (3.8)$$

Thus, the magnetic field components are expanded as

$$B_\theta^2 = G_1^2 s_1^{-1} \psi + (2G_1 G_2 s_1^{-1} - G_1^2 s_2 s_1^{-2}) \psi^2 + o(\psi^3), \quad (3.9)$$

$$B_z = 2s_1^{-1} - 4s_2 s_1^{-2} \psi - 4s_2^2 s_1^{-3} \psi^2 + o(\psi^3), \quad (3.10)$$

$$B_r^2 = s_1^{-3} s_{1z}^2 \psi + (2s_1^{-3} s_{1z} s_{2z} - 5s_1^{-5} s_{1z}^2 s_2) \psi^2 + o(\psi^3). \quad (3.11)$$

The magnetic pressure is

$$B^2 = 4s_1^{-2} - (16s_2 s_1^{-3} - G_1^2 s_1^{-1} - s_{1z}^2 s_1^{-3}) \psi + (2G_1 G_2 s_1^{-1} - G_1^2 s_2 s_1^{-2} + 2s_1^{-3} s_{1z} s_{2z} - 5s_1^{-5} s_{1z}^2 s_2) \psi^2 + o(\psi^3). \quad (3.12)$$

According to the observations, the boundary condition of flux tube at, for example, $z = z_0$ may be given by observations (Hu, 1989), then the undertermined function $s_i(z)$, $\rho_i(z)$ and constants f_i , G_i may be obtained order-by-order of magnitude.

Therefore, Equations (2.9), (2.10), and (2.19) can be solved and they give two types of solutions:

$$T = 1, \quad (3.13)$$

$$\rho_0 = \exp\left(\frac{\delta}{z} + f_0\right), \quad (3.14)$$

$$p = \rho = \rho_0 + \rho_0 f_1 \psi + \rho_0 f_2 \psi^2 + o(\psi^3), \quad (3.15)$$

for uniform temperature model $n = 1$, where

$$f_0 = \ln \rho_0(z_0) - \frac{\delta}{z_0}, \quad f_1 = \frac{\rho_1(z_0)}{\rho_0(z_0)}, \quad f_2 = \frac{\rho_2(z_0)}{\rho_0(z_0)}$$

and

$$\rho_0 = \left[\frac{n-1}{n} \left(\frac{\delta}{z} + f_0 \right) \right]^{1/(n-1)}, \quad (3.16)$$

$$T = \rho_0^{n-1} + \frac{n-1}{n} f_1 \psi + \frac{n-1}{n} f_2 \psi^2 + o(\psi^3), \quad (3.17)$$

$$\rho = \rho_0 + \frac{1}{n} \rho_0^{2-n} f_1 \psi + \frac{1}{n} \rho_0^{2-n} f_2 \psi^2 + o(\psi^3), \quad (3.18)$$

$$p = \rho_0^n + \rho_0 f_1 \psi + \rho_0 f_2 \psi^2 + o(\psi^3), \quad (3.19)$$

for a polytropic model of non-uniform temperature distribution $n \neq 1$, where

$$f_0 = \frac{n}{n-1} \rho_0^{n-1}(z_0) - \frac{\delta}{z_0}, \quad f_1 = n \rho_1(z_0) \rho_0^{n-2}(z_0),$$

$$f_2 = n \rho_2(z_0) \rho_0^{n-2}(z_0).$$

By use of the solutions (3.13)–(3.15) or (3.16)–(3.19), the pressure, and the other quantities of the stratified atmosphere, may be given by the total pressure at the flux tube boundary (2.14) and Equations (2.12) and (2.13).

In this approach, the magnetostatic problem of the isolated flux tube confined in the stratified atmosphere, may be, in principle, solved.

4. Types of Models

Now we take

$$r^2 = s_1 \psi + s_2 \psi^2 + o(\psi^3), \quad (4.1)$$

where the functions $s_i(z)$ are expressed as

$$s_1 = \frac{a_1}{a_2 + (a_3 z + a_4) e^{-a_5 z}}, \quad (4.2)$$

$$\begin{aligned} s_2 &= \frac{1}{8} s_1^2 (G_1^2 + \beta f_1 \rho_0 s_1) + 2 s_{1z}^2 - s_1 s_{1zz} = \\ &= \frac{1}{8} s_1^3 [s_1^{-1} + \frac{1}{2} \rho_0 + a_1^{-1} (a_3 a_5^2 z + a_4 a_5^2 - 2 a_3 a_5) e^{-a_5 z}]; \end{aligned} \quad (4.3)$$

where we will take

$$\delta = 1, \quad \beta = \frac{1}{2}, \quad n = \frac{5}{3}, \quad z_0 = 0.5, \quad \rho_0(z_0) = 1, \quad \rho_1(z_0) = 0.6, \quad G_1 = 1.$$

By appropriate choice of the parameters a_i , which keep the order of terms s_i not greater than $o(1)$, a sort of models of slender flux tubes in a stratified atmosphere may be investigated. From (4.1)–(4.3), it was shown that the twist has no effect on the shape of the tube to the first-order of magnitude. But the field becomes more twisted, accompanying with increasing G_1 to the second-order, and the configuration of the flux tube will be expanded.

The results are displayed in detail in Figures 1, 2, 3, and 4 for $a_1 = 2$ and $a_2 = a_3 = a_4 = a_5 = 1$. The field geometry of the model is plotted in Figure 1. A bundle of forced lines show that the flux tube is divergent. A definite r corresponds to a definite value of ψ , and the larger the radial distance, the greater the value of ψ at a fixed level z . The tube will be less slender for smaller a_2 , a_3 , and a_4 or larger a_1 and a_5 , and *vice versa*.

The curves in Figure 2 give the profiles of the ratio of azimuthal and longitudinal magnetic field components B_θ/B_z . According to (3.9) and (3.10), we have

$$\frac{B_\theta}{B_z} = \frac{1}{2} G_1 s_1^{1/2} \psi^{1/2} + o(\psi^{3/2}).$$

Since ψ is small, s_1 is of order $o(1)$ and G_1 be a constant, the azimuthal field B_θ is not strong compared with the longitudinal field B_z . But the ratio will be larger if ψ increases, it means that the twisting becomes stronger, and the azimuthal field B_θ will play an important role in the fat tube. This can also be seen from the figure that the twisting affect the configuration of the flux tube.

The profiles of the magnetic pressure and the external plasma pressure at different levels are presented in Figures 3 and 4. It seems that both of them decrease with the

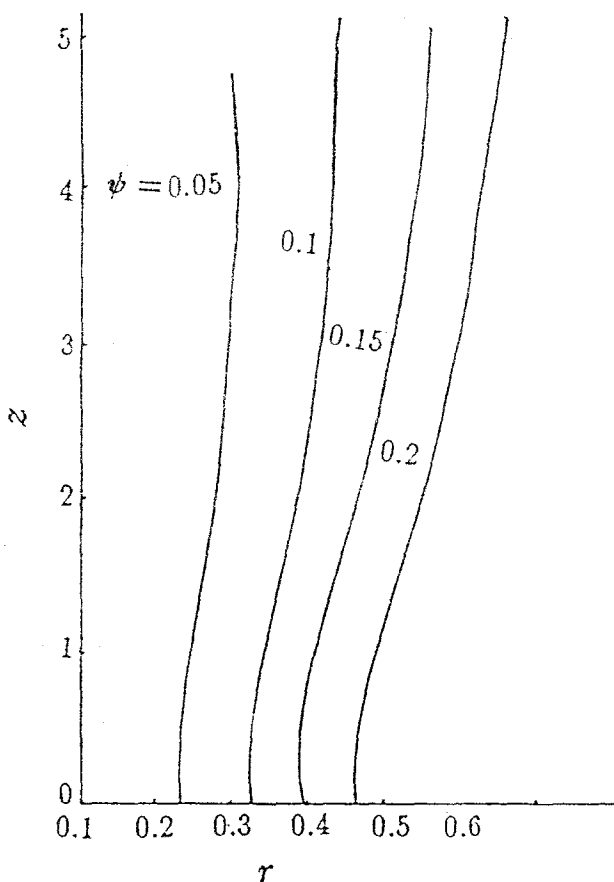


Fig. 1. The configuration of the flux tubes for various values of ψ .

increasing height. As ψ increases, the magnetic pressure at the boundary will increase in case of the presence of B_θ . However, the present model gives that the twisting will be associated with the decreasing of the longitudinal field B_z due to (3.10) and (4.3). The magnetic pressure will decrease and the thermodynamic pressure increase from the axis to the boundary to balance the external pressure.

A type of isothermal models of the flux tube is similarly obtained by substituting Equations (3.13)–(3.15) into (4.1)–(4.3) with $n = 1$. Furthermore, the model of force-free twisted tubes results from neglect of the internal thermodynamic quantity could be analyzed also. For this simplified case of force-free field, our result is consistent with Browning and Priest (1983) who studied the same physical problem by expanding all variables in r . They discussed how twisting affects the tube's shape and concluded that a fairly slender tube would be expected to have its shape not significantly affected by twisting.

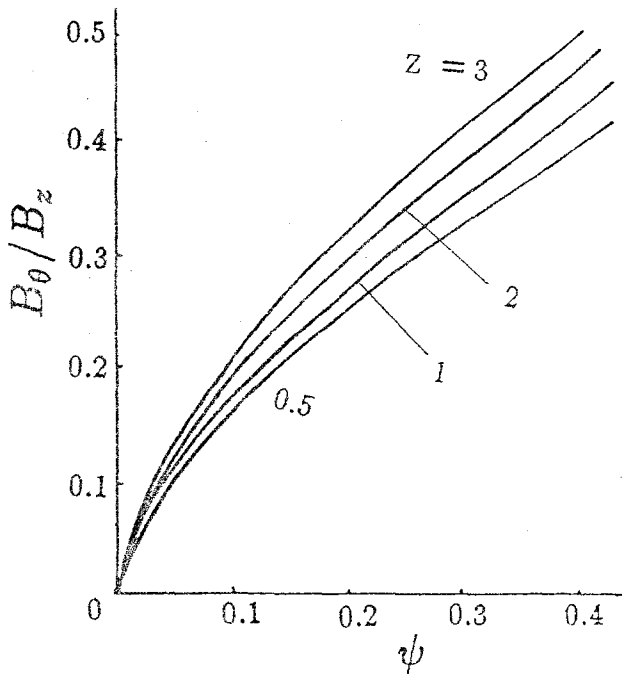


Fig. 2. The ratio of B_0/B_z for various values of z .

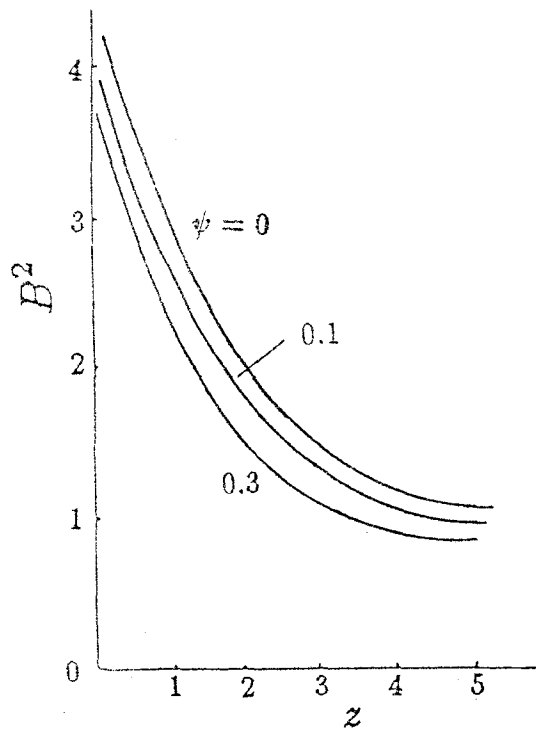


Fig. 3. Magnetic pressure B^2 for various values of ψ .

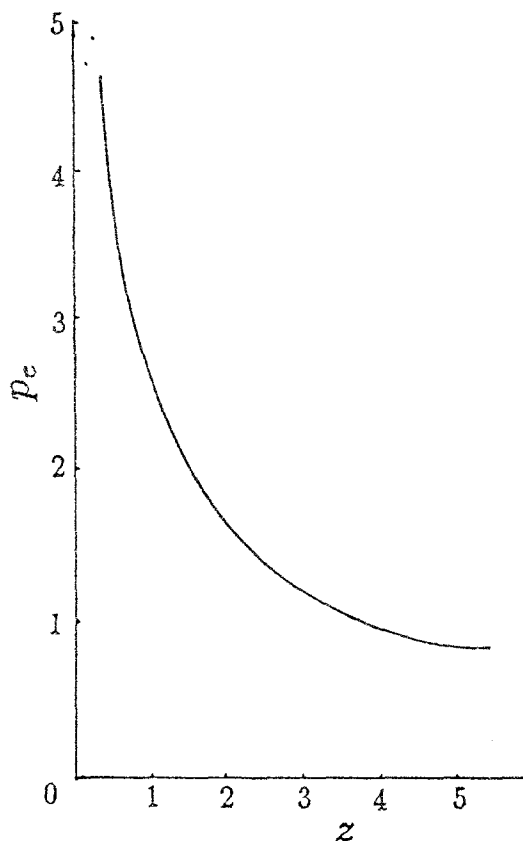


Fig. 4. The profile of external plasma pressure p_e .

5. Discussion and Conclusions

In the preceding sections we have obtained an approximate solution by calculating the pressure, density, and temperature from a prescribed cylindrical-symmetric field using an expansion procedure. This mathematical method allows us discussing a wide-ranging area with different similar parameter. Here we extend Browning and Priest's work of the force-free field to the non-force-free field and obtained the general property of the slender flux tube. It seems that the configuration of tube in the force-free field is relatively more slender than the one of the non-force-free field. The results show that this expansion is effective.

From the viewpoint of physics, it is readily seen from the figures that the external pressure is reasonably decreased with the increasing height. We could construct the theoretical model of the flux tube confined by atmosphere, which will agree with the observations. This provide the flux tube with divergent configuration. If the tube is less slender, twisting does affect the shape of flux tube. This result is practically comparing with the realistic solar atmosphere (Allen, 1973).

A shortcoming of the present method is that we confine ourselves to discuss the configuration of the slender flux tube in the physical model, and the problem is not solved due to a coupled distribution of atmospheric pressure p_e in the mathematical model. In reality, the configuration of free surface is a main point of the problem to be answered when the thermal dynamic quantity can be acquired by observation. These problems will be treated in further studies by a numerical method.

References

- Allen, C. W.: 1973, *Astrophysical Quantities* (3rd ed.), Athlone Press, New York.
Browning, P. K. and Priest, E. R.: 1983, *Astrophys. J.* **266**, 848.
Hu, W. R.: 1986, *Cosmical Magnetohydrodynamics*, Ch. 4, Science Press, China (in Chinese).
Hu, W. R.: 1987, *Plasma Phys. J.* **37**, 323.
Hu, W. R.: 1989, *Scientia Sinica* **32**, 1353.
Parker, E. R.: 1979, *Cosmical Magnetic Fields*, Ch. 9, Oxford University Press, Oxford.
Pneuman, G. W., Solanki, S. K., and Stenflo, J. O.: 1986, *Astron. Astrophys.* **154**, 231.
Roberts, B. and Webb, A. R.: 1978, *Solar Phys.* **56**, 5.
Spruit, H. C.: 1981, in S. Jordan (ed.), *The Sun as a Star*, NASA SP-450, p. 385.
Unno, W. and Ribes, E.: 1979, *Astron. Astrophys.* **73**, 314.
Wilson, P. R.: 1977, *Astrophys. J.* **214**, 611.