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A NEW TYPE OF BOUNDARY INTEGRAL EQUATION FOR PLANE PROBLEMS OF ELASTICITY INCLUDING ROTATIONAL FORCES

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Abstract—Based on the idea proposed by Hu [*Scientia Sinica Series A* **XXX**, 385–390 (1987)], a new type of boundary integral equation for plane problems of elasticity including rotational forces is derived and its boundary element formulation is presented. Numerical results for a rotating hollow disk are given to demonstrate the accuracy of the new type of boundary integral equation.

1. INTRODUCTION

The boundary element method has emerged as a powerful computational tool for stress analysis [1]. The boundary element is based on a boundary integral equation of the problem. Recently, Hu [2] proposed a new type of boundary integral equation for the theory of elasticity to solve the plane problems of elasticity without rotational loads. This new type of boundary integral equation is different from the other type which was initiated by Rizzo [3] and Cruse [4] who used the method of weighted residuals. It is generated using a two-state conservation integral of elasticity. It expresses the stresses of the elastic body with the displacements and tractions on the boundary. The boundary integral equation by the above method is of the second-kind integral equation on the boundary where boundary displacements are given. This result is contrary to that of the Rizzo-type. So one can choose a Rizzo-type of boundary integral equation or a Hu-type depending on one's needs, because this new boundary integral equation complements the Rizzo-type equation [5]. In this paper, we applied the new type of boundary integral equation proposed in [2] into a plane problem of elasticity with rotational body forces. The boundary element formulation is presented in detail. A numerical example of a rotating hollow disk is given to demonstrate the accuracy of the new boundary integral equation.

2. TWO-STATE CONSERVATION INTEGRAL INCLUDING DISTRIBUTED LOADS

To consider the plane problems of elastic body, we chose Cartesian coordinates. The components of displacements, strains, stresses and distributed loads in the region and tractions on the boundary are u_i, ϵ_{ij}, σ_{ij}, f_i and p_i, respectively. Basic equations of elasticity are

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1a}$$

$$\sigma_{ij} = A_{ijlm}\epsilon_{lm} \tag{1b}$$

$$\sigma_{ij,j} + f_i = 0. \tag{1c}$$

For an isotropic elastic material, A_{ijlm} is a constant.

The equilibrium condition, when applied to the boundary B of the domain Ω, yields

$$p_i = \sigma_{ij}n_j \tag{2}$$

in which n_j represents the direction cosines of the outside normal of B.

The strain energy density $U(\epsilon)$ is

$$U(\epsilon) = \tfrac{1}{2}A_{ijlm}\epsilon_{ij}\epsilon_{lm}. \tag{3}$$

If u_i, ϵ_{ij}, σ_{ij}, f_i and p_i satisfy (1), they constitute an exact elasticity state. Differentiating such a state with respect to x_1 and x_2, respectively, we get two other exact states

$$u_i^k = u_{i,k} \tag{4a}$$

$$\epsilon_{ij}^k = \epsilon_{ij,k} \tag{4b}$$

$$\sigma_{ij}^k = \sigma_{ij,k} \tag{4c}$$

$$f_i^k = f_{i,k} \tag{4d}$$

$$p_i^k = n_j\sigma_{ij}^k = n_j\sigma_{ij,k}, \quad (k = 1, 2). \tag{4e}$$

Note that p_i^k is found by using the equilibrium condition, not by differentiating p_i with respect to x_1 and x_2.

Substituting an exact state and one of its generated exact states in (4) into virtual work principle, we have

$$\int_\Omega A_{ijlm}\epsilon_{ij}\epsilon^k_{lm}\,d\Omega = \int_\Omega f_i u^k_i\,d\Omega + \int_B p_i u^k_i\,dB. \quad (5)$$

From (3) and (4), the left-hand side of (5) is

$$\int_\Omega A_{ijlm}\epsilon_{ij}\epsilon^k_{lm}\,d\Omega = \int_\Omega [U(\epsilon)]_{,k}\,d\Omega = \int_B n_k U(\epsilon)\,dB. \quad (6)$$

Substituting (6) into (5), we get

$$\int_\Omega f_i u_{i,k}\,d\Omega = \int_\Omega [n_k U(\epsilon) - p_i u_{i,k}]\,dB. \quad (7)$$

Equation (7) is a single-state conservation integral.

Suppose $u_i^{(1)}$, $\epsilon_{ij}^{(1)}$, $\sigma_{ij}^{(1)}$, $f_i^{(1)}$, $p_i^{(1)}$ and $u_i^{(2)}$, $\epsilon_{ij}^{(2)}$, $\sigma_{ij}^{(2)}$, $f_i^{(2)}$, $p_i^{(2)}$ are two exact elasticity states, their linear combination

$$u_i^{(0)} = u_i^{(1)} + a u_i^{(2)} \quad (8a)$$

$$\epsilon_{ij}^{(0)} = \epsilon_{ij}^{(1)} + a\epsilon_{ij}^{(2)} \quad (8b)$$

$$\sigma_{ij}^{(0)} = \sigma_{ij}^{(1)} + a\sigma_{ij}^{(2)} \quad (8c)$$

$$f_i^{(0)} = f_i^{(1)} + a f_i^{(2)} \quad (8d)$$

$$p_i^{(0)} = p_i^{(1)} + a p_i^{(2)} \quad (8e)$$

is also an exact state, where a is an arbitrary constant. Substituting (8) into (7), we have

$$\int_\Omega f_i^{(0)} u_{i,k}^{(0)}\,d\Omega = \int_B [n_k U(\epsilon^{(0)}) - p_i^{(0)} u_{i,k}^{(0)}]\,dB. \quad (9)$$

Because (9) is an identity with respect to a, the coefficients of the terms linear in a on the two sides of this equation must be equal, we obtain

$$\int_\Omega [f_i^{(1)} u_{i,k}^{(2)} + f_i^{(2)} u_{i,k}^{(1)}]\,d\Omega = \int_\Omega [n_k A_{ijlm}\epsilon_{ij}^{(1)}\epsilon_{lm}^{(2)} - p_i^{(1)} u^2_{i,k} - p_i^{(2)} u_{i,k}^{(1)}]\,dB. \quad (10)$$

This equation is an identity satisfied by two exact elasticity states. It is called the two-state conservation integral.

3. BOUNDARY INTEGRAL OF DOMAIN DISPLACEMENTS AND STRESSES BASED ON THE TWO-STATE CONSERVATION INTEGRAL

For plane strain applications where the fundamental solution corresponds to unit point loads applied within an infinite plane (Kelvin), the components of the fundamental displacements and tractions are given by

$$u_i^h = \frac{1}{8\pi G(1-\nu)}\left[(3-4\nu)\ln\frac{1}{r}\delta_{ih} + r_{,i}r_{,h}\right] \quad (11)$$

$$p_i^h = -\frac{1}{4\pi(1-\nu)r}\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ih} + 2r_{,i}r_{,h}] - (1-2\nu)[r_{,i}n_h - r_{,h}n_i]\right\}, \quad (12)$$

where $r = r(\xi, x)$ represents the distance between the load point ξ and the field point x and its derivatives are taken with reference to the coordinates. Applying Betti's reciprocal theorem of work to an unknown elasticity state and the fundamental solution, we have

$$\int_\Omega f_i^h u_i\,d\Omega + \int_B p_i^h u_i\,dB = \int_\Omega f_i u_i^h\,d\Omega + \int_B p_i u_i^h\,dB. \quad (13)$$

So

$$u_h(\xi) = \int_B (p_i u_i^h - p_i^h u_i)\,dB + \int_\Omega f_i u_i^h\,d\Omega \quad (14)$$

in which ξ is a point in the domain Ω, and the domain integral of body forces can be suitably transformed into a surface integral. Equation (14) is the expression of boundary integral of displacements. In order to obtain the expressions of domain stresses, we first have to get the derivative of domain displacements. We choose two states of the conservation integral as the outstanding state and fundamental solutions, respectively

$$\int_\Omega (f_i u_{i,k}^h + f_i^h u_{i,k})\,d\Omega = \int_B (n_k A_{ijlm}\epsilon_{ij}\epsilon_{lm}^h - p_i u_{i,k}^h - p_i^h u_{i,k})\,dB \quad (15)$$

$$u_{h,k}(\xi) = \int_B (n_k A_{ijlm}\epsilon_{ij}\epsilon_{lm}^h - p_i u_{i,k}^h - p_i^h u_{i,k})\,dB - \int_\Omega f_i u_{i,k}^h\,d\Omega. \quad (16)$$

Here we obtain the expressions of domain displacement derivatives represented by the boundary integral. Equation (16) is the main formula of the new type of boundary integral equation.

For the plane problems of elasticity, $f_i = 0$ is assumed, we have a further rearrangement to (16)

$$u_{h,1}(\xi) = \int_B (\sigma_{12}^h u_{1,s} + \sigma_{22}^h u_{2,s} - p_i u_{i,1}^h)\,dB \quad (17)$$

$$u_{h,2}(\xi) = \int_B (-\sigma_{11}^h u_{1,s} - \sigma_{12}^h u_{2,s} - p_i u_{i,2}^h)\,dB. \quad (18)$$

Substituting (17) and (18) into the constitutive equations of elasticity, we have the expressions of boundary integral of domain stress. For the plane strain problems of isotropic elastic body, the domain stresses are

$$\sigma_{11}(\xi) = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \int_B \left[u_{1,s}\left(\sigma^1_{12} - \frac{\nu}{1-\nu}\sigma^2_{11}\right) + u_{2,s}\left(\sigma^1_{22} - \frac{\nu}{1-\nu}\sigma^2_{12}\right) - p_i\left(u^1_{i,1} + \frac{\nu}{1-\nu}u^2_{i,2}\right)\right] \mathrm{d}B \qquad (19)$$

$$\sigma_{22}(\xi) = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \int_B \left[u_{1,s}\left(\sigma^2_{11} - \frac{\nu}{1-\nu}\sigma^1_{12}\right) + u_{2,s}\left(\sigma^2_{12} - \frac{\nu}{1-\nu}\sigma^1_{22}\right) - p_i\left(u^2_{i,2} + \frac{\nu}{1-\nu}u^1_{i,1}\right)\right] \mathrm{d}B \qquad (20)$$

$$\sigma_{12}(\xi) = \frac{E}{4(1+\nu)} \int_B [u_{1,s}(\sigma^2_{12} - \sigma^1_{11}) + u_{2,s}(\sigma^2_{22} - \sigma^1_{12}) - p_i(u^2_{i,1} + u^1_{i,2})] \,\mathrm{d}B \qquad (21)$$

in which E is the modulus of elasticity, ν is Poisson's ratio.

An important remark is now due: all the expressions presented here are assumed to be valid for plane strain problems, the plane stress case can be dealt with by the same equations, providing E is replaced by

$$\bar{E} = \frac{(1+2\nu)}{(1+\nu)^2}E$$

and ν by

$$\bar{\nu} = \frac{\nu}{(1+\nu)}.$$

4. BOUNDARY INTEGRAL EQUATION OF ISOTROPIC ELASTIC BODY

When the point $\xi \in \Omega$ is (17) and (18) approaches boundary B, the boundary integral in these formulae becomes singular at ξ. So we need to extend the formulae to the boundary. Suppose (ξ_1, ξ_2) is a point on the boundary B, where the boundary curve has continuous tangent. Let B_ϵ be a half circle with radius ϵ, centred at (ξ_1, ξ_2). Let Ω_ϵ be the half curve region surrounded by B_ϵ, as shown in Fig. 1. Apply eqn (10) on Ω_ϵ with one outstanding state and another being the fundamental solution of elasticity body. Since in Ω_ϵ, $f^{(1)}_i = f^{(2)}_i = 0$, so

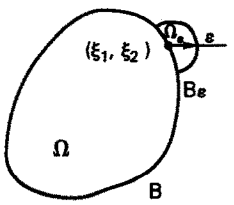

Fig. 1.

$$\int_\Omega [f_i u^h_{i,k} + f^h_i u_{i,k}]\,\mathrm{d}\Omega + \int_{\Omega_\epsilon} 0\,\mathrm{d}\Omega = \int_{B-B_\epsilon} [n_k A_{ijlm}\epsilon_{ij}\epsilon^h_{lm} - p_i u^h_{i,k} - p^h_i u_{i,k}]\,\mathrm{d}B + \int_{B_\epsilon} [n_k A_{ijlm}\epsilon_{ij}\epsilon^h_{lm} - p_i u^h_{i,k} - p^h_i u_{i,k}]\,\mathrm{d}B. \qquad (22)$$

On the boundary, substituting (11) into the integral in (22) and letting $\epsilon \to 0$, we have

$$\tfrac{1}{2} u_{h,k}(\xi) = \int_{\dot{B}} (n_k A_{ijlm}\epsilon_{ij}\epsilon^h_{lm} - p_i u^h_{i,k} - p^h_i u_{i,k})\,\mathrm{d}B, \qquad (23)$$

where $\dot{B}$ represents the integral in the sense of Cauchy principal value. Equation (23) can be further written as

$$\tfrac{1}{2} u_{h,1}(\xi) = \int_{\dot{B}} (\sigma^h_{12} u_{1,s} + \sigma^h_{22} u_{2,s} - p_i u^h_{i,1})\,\mathrm{d}B \qquad (24)$$

and

$$\tfrac{1}{2} u_{h,2}(\xi) = \int_{\dot{B}} (-\sigma^h_{11} u_{1,s} - \sigma^h_{12} u_{2,s} - p_i u^h_{i,2})\,\mathrm{d}B. \qquad (25)$$

Substituting (24) and (25) into the constitutive equations, we obtain

$$\tfrac{1}{2}\sigma_{11}(\xi) = \int_{\dot{B}} \left\{ \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \left[u_{1,s}\left(\sigma^1_{12} - \frac{\nu}{1-\nu}\sigma^2_{11}\right) + u_{2,s}\left(\sigma^1_{22} - \frac{\nu}{1-\nu}\sigma^2_{12}\right)\right] - p_i\sigma^i_{11}\right\} \mathrm{d}B \qquad (26)$$

$$\tfrac{1}{2}\sigma_{22}(\xi) = \int_{\dot{B}} \left\{ \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \left[u_{1,s}\left(\sigma^2_{11} - \frac{\nu}{1-\nu}\sigma^1_{12}\right) + u_{2,s}\left(\sigma^2_{12} - \frac{\nu}{1-\nu}\sigma^1_{22}\right)\right] - p_i\sigma^i_{22}\right\} \mathrm{d}B \qquad (27)$$

$$\tfrac{1}{2}\sigma_{12}(\xi) = \int_{\dot{B}} \left\{ \frac{E}{4(1+\nu)} [u_{1,s}(\sigma^2_{12} - \sigma^1_{11}) + u_{2,s}(\sigma^2_{22} - \sigma^1_{12})] - p_i\sigma^i_{12}\right\} \mathrm{d}B. \qquad (28)$$

Let n_1, n_2 be the direction cosines of the outward normal of the boundary curve on (ξ_1, ξ_2). Applying boundary conditions, we have

$$\tfrac{1}{2}p_1(\xi)=n_1\cdot\tfrac{1}{2}\sigma_{11}(\xi)+n_2\cdot\tfrac{1}{2}\sigma_{12}(\xi)$$

$$=\int_B\Big\{u_{1,s}\Big[n_1\frac{(1-v)E}{(1+v)(1-2v)}\Big(\sigma_{12}^1-\frac{v}{1-v}\sigma_{11}^2\Big)$$

$$+n_2\frac{E}{(4(1+v)}(\sigma_{12}^2-\sigma_{11}^1)\Big]$$

$$+u_{2,s}\Big[n_1\frac{(1-v)E}{(1+v)(1-2v)}\Big(\sigma_{22}^1-\frac{v}{1-v}\sigma_{12}^2\Big)$$

$$+n_2\frac{E}{4(1+v)}(\sigma_{22}^2-\sigma_{12}^1)\Big)\Big]-p_ip_1^i\Big\}\,\mathrm{d}B \tag{29}$$

$$\tfrac{1}{2}p_2(\xi)=n_1\cdot\tfrac{1}{2}\sigma_{12}(\xi)+n_2\cdot\tfrac{1}{2}\sigma_{22}(\xi)$$

$$=\int_B\Big\{u_{1,s}\Big[n_1\frac{E}{4(1+v)}(\sigma_{12}^2-\sigma_{11}^1)$$

$$+n_2\frac{(1-v)E}{(1+v)(1-2v)}\Big(\sigma_{11}^2-\frac{v}{1-v}\sigma_{12}^1\Big)\Big]$$

$$+u_{2,s}\Big[n_1\frac{E}{4(1+v)}(\sigma_{22}^2-\sigma_{12}^1)+n_2\frac{(1-v)E}{(1+v)(1-2v)}$$

$$\times\Big(\sigma_{12}^2-\frac{v}{1-v}\sigma_{22}^1\Big)\Big]-p_ip_2^i\Big\}\,\mathrm{d}B, \tag{30}$$

where

$$p_1^i=n_1\sigma_{11}^i+n_2\sigma_{12}^i \tag{31a}$$

$$p_2^i=n_1\sigma_{12}^i+n_2\sigma_{22}^i. \tag{31b}$$

Equations (29) and (30) are the final forms of the new type of boundary integral equations in plane elasticity.

5. BOUNDARY DISPLACEMENT DERIVATIVE

In this section, we discuss how to express $u_{1,s}$ and $u_{2,s}$ in (29) and (30) by using u_i and p_i on the boundary. Boundary tractions are

$$p_1=n_1\sigma_{11}+n_2\sigma_{12} \tag{32a}$$

$$p_2=n_1\sigma_{12}+n_2\sigma_{22}. \tag{32b}$$

For the plane strain problems of elasticity, the stresses are

$$\sigma_{11}=\frac{(1-v)E}{(1+v)(1-2v)}\Big(u_{1,1}+\frac{v}{1-v}u_{2,2}\Big) \tag{33a}$$

$$\sigma_{22}=\frac{(1-v)E}{(1+v)(1-2v)}\Big(u_{2,2}+\frac{v}{1-v}u_{1,1}\Big) \tag{33b}$$

$$\sigma_{12}=\frac{E}{2(1+v)}(u_{1,2}+u_{2,1}). \tag{33c}$$

Since the boundary is divided into several elements. In each element we have

$$\frac{\partial u_i}{\partial\eta_k}=\frac{\partial u_i}{\partial x_m}\cdot\frac{\partial x_m}{\partial\eta_k}, \tag{34}$$

where η_k is the local coordinate on one element.

To linear element, displacement u_i and coordinate x_m of arbitrary point on boundary can be represented as

$$u_i=\tfrac{1}{2}(u_i^{j+1}+u_i^j)+\tfrac{1}{2}(u_i^{j+1}-u_i^j)\eta_2 \tag{35a}$$

$$x_m=\tfrac{1}{2}(x_m^{j+1}+x_m^j)+\tfrac{1}{2}(x_m^{j+1}-x_m^j)\eta_2, \tag{35b}$$

respectively, where j and $j+1$ are the nodal numbers of the element.

Since (35) has nothing to do with local coordinate η_1, there are

$$\frac{\partial u_i}{\partial\eta_2}=\tfrac{1}{2}(u_i^{j+1}-u_i^j) \tag{36a}$$

$$\frac{\partial x_m}{\partial\eta_2}=\tfrac{1}{2}(x_m^{j+1}-x_m^j). \tag{36b}$$

The displacement derivative can be represented as

$$u_{1,1}(x_1^{j+1}-x_1^j)+u_{1,2}(x_2^{j+1}-x_2^j)=u_1^{j+1}-u_1^j \tag{37a}$$

$$u_{2,1}(x_1^{j+1}-x_1^j)+u_{2,2}(x_2^{j+1}-x_2^j)=u_2^{j+1}-u_2^j. \tag{37b}$$

Solving simultaneous equations (32) and (37), substituting into prescribed conditions, we will obtain the displacement derivative $u_{i,m}$. Since

$$u_{1,s}=n_1u_{1,1}+n_2u_{1,2} \tag{38a}$$

$$u_{2,s}=n_1u_{2,1}+n_2u_{2,2}. \tag{38b}$$

So we can get $u_{1,s}$ and $u_{2,s}$ represented by using boundary displacement u_i and traction p_i.

6. BODY FORCE

When $f_i\neq 0$, the boundary integral equation has one more item than the equation which f_i is equal to zero. This item is the body integral of body forces

$$B_{hk}=\int_\Omega f_iu_{i,k}^h\,\mathrm{d}\Omega. \tag{39}$$

Because B_{hk} is domain integral, the calculation of B_{hk} requires the domain to be divided into internal cells; it would lose the advantage of boundary element method.

For rotational problems, the body force is a centrifugal load. If the axis of rotation passes through the origin of the coordinate system, the problem is equivalent to a prescribed body force of the form

$$f_i = \rho\omega^2 x_i \tag{40}$$

in which ω is the angular velocity, ρ is the material density.

Let us call G_{ij} the Galerkin tensor which is related to the fundamental solution u_i^h by the following expression

$$u_i^h = G_{ih,jj} - \frac{G_{ij,jh}}{2(1-\nu)}.$$

For the two-dimensional plane strain state

$$G_{ih} = \frac{1}{8\pi G} r^2 \ln\frac{1}{r}\,\delta_{ih}.$$

So

$$u_i^h = \frac{1}{8\pi G(1-\nu)} \times \left[(3-4\nu)\ln\frac{1}{r}\,\delta_{ih} - \frac{7-8\nu}{2}\,\delta_{ih} + r_{,i}r_{,h}\right] \tag{41}$$

the difference between (41) and (11) simply corresponds to a rigid-body translation. The above expressions, (40) and (41), when substituted into (39) lead to the following boundary integral

$$B_{hk} = \int_\Omega \rho\omega^2 x_i u_{i,k}^h \,\mathrm{d}\Omega = \rho\omega^2 \int_\Omega \{[x_i u_i^h]_{,k} - u_k^h\}\,\mathrm{d}\Omega. \tag{42}$$

There are two ways to deal with $\rho\omega^2\int_\Omega \{[x_i u_i^h]_{,k} - u_k^h\}\,\mathrm{d}\Omega$, the first is

$$b_1 = \rho\omega^2 \int_B \left[G_{kh,j} - \frac{G_{kj,h}}{2(1-\nu)}\right] n_j \,\mathrm{d}B, \tag{43a}$$

the second is

$$b_2 = \rho\omega^2 \int_B \left[G_{kh,j}n_j - \frac{G_{kj,j}}{2(1-\nu)}\,n_h\right]\mathrm{d}B. \tag{43b}$$

Also we get two expressions of B_{hk}

$$B_{hk}^{(1)} = \frac{\rho\omega^2}{8\pi G(1-\nu)} \times \int_B \left\{x_i n_k\left[(3-4\nu)\ln\frac{1}{r}\,\delta_{ih} - \frac{7-8\nu}{2}\,\delta_{ih} + r_{,i}r_{,h}\right] - (1-\nu)\left(2r\ln\frac{1}{r} - r\right) \times \left[\delta_{kh} r_{,j} n_j - \frac{1}{2(1-\nu)}\,r_{,h} n_k\right]\right\}\mathrm{d}B \tag{44a}$$

$$B_{hk}^{(2)} = \frac{\rho\omega^2}{8\pi G(1-\nu)} \times \int_B \left\{x_i n_k\left[(3-4\nu)\ln\frac{1}{r}\,\delta_{ih} - \frac{7-8\nu}{2}\,\delta_{ih} + r_{,i}r_{,h}\right] - (1-\nu)\left(2r\ln\frac{1}{r} - r\right) \times \left[\delta_{kh} r_{,j} n_j - \frac{1}{2(1-\nu)}\,r_{,k} n_h\right]\right\}\mathrm{d}B. \tag{44b}$$

The small difference of the two expressions will be discussed later. Therefore, the displacement derivatives are

$$u_{h,1}(\xi) = \int_B (\sigma_{12}^h u_{1,s} + \sigma_{22}^h u_{2,s} - p_i u_{i,1}^h)\,\mathrm{d}B - B_{h1} \tag{45a}$$

$$u_{h,2}(\xi) = \int_B (-\sigma_{11}^h u_{1,s} - \sigma_{12}^h u_{2,s} - p_i u_{i,2}^h)\,\mathrm{d}B - B_{h2}. \tag{45b}$$

When the point ξ approaches boundary B

$$u_{h,k}(\xi) \to \tfrac{1}{2} u_{h,k}(\xi).$$

We can prove that the integral of B_{hk} on B_s is also zero. But the integral limit is changed to $\hat{B}$. Now the boundary integral including body forces is

$$\tfrac{1}{2} u_{h,1}(\xi) = \int_{\hat{B}} (\sigma_{12} u_{1,s}^h + \sigma_{22} u_{2,s}^h - p_i^h u_{i,1})\,\mathrm{d}B - B_{h1} \tag{46a}$$

$$\tfrac{1}{2} u_{h,2}(\xi) = \int_{\hat{B}} (-\sigma_{11} u_{1,s}^h - \sigma_{12} u_{2,s}^h - p_i^h u_{i,2})\,\mathrm{d}B - B_{h2}. \tag{46b}$$

7. THE BOUNDARY INTEGRAL EQUATIONS INCLUDING BODY FORCES

Substituting (46) into the constitution equations, we have the boundary integral of stresses

$$\tfrac{1}{2}\sigma_{11}(\xi) = \int_{\hat{B}} \left\{\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left[u_{1,s}\left(\sigma_{12}^1 - \frac{\nu}{1-\nu}\sigma_{11}^2\right) + u_{2,s}\left(\sigma_{22}^1 - \frac{\nu}{1-\nu}\sigma_{12}^2\right)\right] - p_i\sigma_{11}^i - \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(B_{11} + \frac{\nu}{1-\nu}B_{22}\right)\right\}\mathrm{d}B \tag{47}$$

$$\tfrac{1}{2}\,\sigma_{22}(\xi)=\int_{B}\left\{\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left[u_{1,s}\left(\sigma^{2}_{11}-\frac{\nu}{1-\nu}\sigma^{1}_{12}\right)+u_{2,s}\left(\sigma^{2}_{12}-\frac{\nu}{1-\nu}\sigma^{1}_{22}\right)\right]-p_i\sigma^{i}_{22}-\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(B_{22}+\frac{\nu}{1-\nu}B_{11}\right)\right\}\mathrm{d}B \tag{48}$$

$$\tfrac{1}{2}\sigma_{22}(\xi)=\int_{B}\left\{\frac{E}{4(1+\nu)}[u_{1,s}(\sigma^{2}_{12}-\sigma^{1}_{11})+u_{2,s}(\sigma^{2}_{22}-\sigma^{1}_{12})]-p_i\sigma^{i}_{12}-\frac{E}{4(1+\nu)}(B_{12}+B_{21})\right\}\mathrm{d}B. \tag{49}$$

So

$$\tfrac{1}{2}p_1(\xi)=n_1\cdot\tfrac{1}{2}\sigma_{11}(\xi)+n_2\cdot\tfrac{1}{2}\sigma_{12}(\xi)$$

$$=\int_{B}\left\{u_{1,s}\left[n_1\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(\sigma^{1}_{12}-\frac{\nu}{1-\nu}\sigma^{2}_{11}\right)+n_2\frac{E}{4(1+\nu)}(\sigma^{2}_{12}-\sigma^{1}_{11})\right]+u_{2,s}\left[n_1\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(\sigma^{1}_{22}-\frac{\nu}{1-\nu}\sigma^{2}_{12}\right)+n_2\frac{E}{4(1+\nu)}(\sigma^{2}_{22}-\sigma^{1}_{12})\right]-p_ip^{i}_{1}-\left[n_1\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(B_{11}+\frac{\nu}{1-\nu}B_{22}\right)+n_2\frac{E}{4(1+\nu)}(B_{12}+B_{21})\right]\right\}\mathrm{d}B \tag{50}$$

$$\tfrac{1}{2}p_2(\xi)=n_1\cdot\tfrac{1}{2}\sigma_{12}(\xi)+n_2\cdot\tfrac{1}{2}\sigma_{22}(\xi)$$

$$=\int_{B}\left\{u_{1,s}\left[n_1\frac{E}{4(1+\nu)}(\sigma^{2}_{12}-\sigma^{1}_{11})+n_2\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(\sigma^{2}_{11}-\frac{\nu}{1-\nu}\sigma^{1}_{12}\right)\right]+u_{2,s}\left[n_1\frac{E}{4(1+\nu)}(\sigma^{2}_{12}-\sigma^{1}_{11})+n_2\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(\sigma^{2}_{12}-\frac{\nu}{1-\nu}\sigma^{1}_{22}\right)\right]-p_1p^{i}_{2}-\left[n_1\frac{E}{4(1+\nu)}(B_{12}+B_{21})+n_2\frac{(1-\nu)E}{(1+\nu)(-2\nu)}\times\left(B_{22}+\frac{\nu}{1-\nu}B_{11}\right)\right]\right\}\mathrm{d}B. \tag{51}$$

The above two equations, (50) and (51), are the boundary integral equations including rotational loads, in which $u_{1,s}$ and $u_{2,s}$ consist of boundary components u_i and p_i, σ^{h}_{ij} can be deduced from the fundamental solutions of elasticity. Dividing the whole boundary into several elements and applying (50) and (51) to each element, we can deduce matrix equations, including boundary displacements and tractions. Substituting the prescribed boundary conditions separating the known and unknown variables on each side of the equations and solving the equations, we can find the unknown boundary displacements and tractions. Once the nodal values of boundary displacements and tractions are calculated, the values of displacements and stresses can be computed at any internal points of domain by simply using (14) and (19)–(21). The whole problem can be solved.

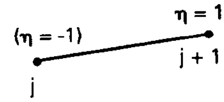

Fig. 2.

Now we discuss the small difference of $B^{(1)}_{hk}$ and $B^{(2)}_{hk}$, we use (50) as an example. The influence of rotational forces on the equation is

$$R_1=\int_{B}\left[n_1\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(B_{11}+\frac{\nu}{1-\nu}B_{22}\right)+n_2\frac{E}{4(1+\nu)}(B_{12}+B_{21})\right]\mathrm{d}B,$$

since $B^{(1)}_{11}=B^{(2)}_{11}$, $B^{(1)}_{12}=B^{(2)}_{21}$, $B^{(1)}_{21}=B^{(2)}_{12}$, $B^{(1)}_{22}=B^{(2)}_{22}$, it is identical in choosing $B^{(1)}_{hk}$ or $B^{(2)}_{hk}$ and has no influence on the final result.

8. NUMERICAL RESULTS

Calculating a rotating hollow as shown in Fig. 3 we only deal with a 30° sector of the disk, the shape parameters are: $\theta=30°$, $r_a=10$ mm, $r_b=100$ mm; the physical parameters are given in Tables 1 and 2. The exact solutions of displacements and stresses can be found in every textbook of elasticity mechanics. Tables 1 and 2 show the relative errors of displacement u_r and stress σ_θ, respectively.

9. CONCLUSION

From the numerical results given in Tables 1 and 2, we see that the errors decrease when the nodal

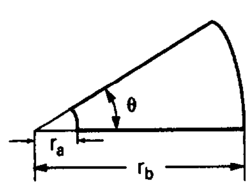

Fig. 3.

Table 1. Relative errors of displacement u_r

R, θ	55, 0	55, 30	100, 15	10, 15	55, 15
Element A	3.15	3.16	3.23	3.41	2.84
Element B	2.92	2.92	3.05	3.32	2.75

$E = 16{,}000$ MPa, $\nu = 0.3$, $\rho = 8.01$ g/cm^3, $\omega = 58{,}000$ rev/min. Nodal number of element A = 40, nodal number of element B = 52, unit: R, mm; θ, degree.

Table 2. Relative errors of stress σ_θ

R, θ	55, 0	55, 30	100, 15	10, 15	55, 15
Element A	3.14	2.46	4.47	3.31	5.74
Element B	2.82	2.25	4.02	3.08	4.83

numbers of elements increase. The new type of boundary integral equations provide better numerical results for the case when boundary displacements are given than the ones for the case that the boundary tractions are given. The new type of boundary integral equations have a great potential. It is worth extending the applicable regions of the new method.

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