

## CRACK TIP STRESS SINGULARITIES IN BIMATERIAL WITH AN INCLINED INTERFACE

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In so-called advanced materials such as structural ceramics, ceramic and metal-matrix composites, polycrystalline intermetallic alloys, interfacial and intergranular fracture are common and may, in large part, determine the material's overall mechanical response. The crack tip stress singularities are an important factor for interfacial and intergranular fracture. The stresses near the tips of the crack are proportional to  $r^{\lambda-1}$  where  $\lambda$  is the eigenvalue. Therefore, determination of the eigenvalue is an interesting research topic.

Williams [1] was probably the first to carry out tip singularity analysis in dissimilar media by using the eigenfunction method. He has shown that the eigenvalues for interface cracks in linear elasticity are complex and the real part of the lowest eigenvalue is 1/2. Later, Zak and Williams [2] used the same method to analyse the stress singularities of a crack tip in a material which terminates perpendicularly at the interface between two materials. The result shows that the eigenvalues are real, the lowest one, however, is not 1/2 and depends on elastic constants of the bimaterial. It gives strictly a monotonic decay of stress. This behaviour is characteristically different from the associated case [1] when the crack lies along the interface.

This paper gives numerical results of the eigenvalues of a crack tip which terminates at the inclined interface (Fig. 1). The inclined angles  $\phi = 0^\circ$  and  $\phi = 90^\circ$  correspond to the cases studied in [1,2].

Consider a semi-infinite crack in material 1. The crack terminates at the inclined interface between material 1 and material 2 as shown in Fig. 1. Assume a polar coordinate system at the crack tip with the interface along  $\theta = \phi$ . Poisson's ratio and the shear modulus for each material are  $\nu_i$  and  $\mu_i$ ,  $i = 1, 2$ .

Using William's method [1,2] and introducing product solution of the biharmonic equation

$$U_i(r, \theta) = r^{\lambda+1} F_i(\theta), \quad i = 1, 2, 3 \quad (1)$$

where

$$F_i(\theta) = a_i \sin(\lambda + 1) + b_i \cos(\lambda + 1) + c_i \sin(\lambda - 1) + d_i \cos(\lambda - 1) \quad (2)$$

Three stress functions are required in order to pose this three region problem. The stresses and displacements are:

$$\begin{aligned} \sigma_{ri} &= \frac{\partial^2 U_i}{r^2 \partial \theta^2} + \frac{1}{r} \frac{\partial U_i}{\partial r} = r^{\lambda-1} [F_i''(\theta) + (\lambda + 1) F_i(\theta)] \\ \sigma_{\theta i} &= \frac{\partial U_i}{\partial r^2} = \lambda(\lambda + 1) r^{\lambda-1} F_i(\theta) \\ \tau_{r\theta i} &= -\frac{1}{r} \frac{\partial^2 U_i}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^2 U_i}{\partial \theta^2} = -\lambda r^{\lambda-1} F_i'(\theta) \\ u_{ri} &= \frac{1}{2\mu_i} r^\lambda \{ -(\lambda + 1) F_i(\theta) + (1 + \kappa_i) [c_i \sin(\lambda - 1)\theta + d_i \cos(\lambda - 1)\theta] \} \\ u_{\theta i} &= \frac{1}{2\mu_i} r^\lambda \{ -F_i'(\theta) - (1 + \kappa_i) [c_i \cos(\lambda - 1)\theta - d_i \sin(\lambda - 1)\theta] \} \end{aligned} \quad (3)$$

The boundary conditions-traction free crack surface and continuity of traction and displacements across the interface provide twelve equations in twelve unknowns ( $a_i, b_i, c_i, d_i, i = 1, 2, 3$ ).

$$\sigma_{r1} = 0 \quad \theta = +\pi \quad (4.1)$$

$$\tau_{r\theta 1} = 0 \quad \theta = +\pi \quad (4.2)$$

$$\sigma_{r1} = \sigma_{r2} \quad \theta = \phi \quad (4.3)$$

$$\tau_{r\theta 1} = \tau_{r\theta 2} \quad \theta = \phi \quad (4.4)$$

$$u_{r1} = u_{r2} \quad \theta = \phi \quad (4.5)$$

$$u_{\theta 1} = u_{\theta 2} \quad \theta = \phi \quad (4.6)$$

$$\sigma_{r2} = \sigma_{r3} \quad \theta = \phi - \pi \quad (4.7)$$

$$\tau_{r\theta 2} = \tau_{r\theta 3} \quad \theta = \phi - \pi \quad (4.8)$$

$$u_{r2} = u_{r3} \quad \theta = \phi - \pi \quad (4.9)$$

$$u_{\theta 2} = u_{\theta 3} \quad \theta = \phi - \pi \quad (4.10)$$

$$\sigma_{r3} = 0 \quad \theta = -\pi \quad (4.11)$$

$$\tau_{r\theta 3} = 0 \quad \theta = -\pi \quad (4.12)$$

Because the equations are homogeneous, a solution exists only for those where the determinant of the coefficient matrix for the system of linear equations vanishes. The values which make the determinant zero are the eigenvalues of the problem. The determinant can be written as

$$f(\lambda) = \det |A| = 0 \quad (5)$$

It is a tiresome task to derive the characteristic equation from (5). The derived analytic equation will be a complicated transcendental equation which can only be solved by the iterative method. Therefore, we use directly the Gaussian elimination method with partial pivoting to evaluate numerically determinant (5) and Muller's iterative method [3] to find the eigenvalues instead of deriving the characteristic equation.

The Muller's method is based on quadratic interpolation of the last three estimates. We give the solution procedure as follows:

$\lambda_0, \lambda_1, \lambda_2$  are the initial estimates.

$$f_0 = f(\lambda_0), \quad f_1 = f(\lambda_1), \quad f_2 = f(\lambda_2) \quad (6)$$

$$z_2 = (\lambda_2 - \lambda_1)/(\lambda_1 - \lambda_0) \quad (7)$$

Iterative algorithm: ( $i \geq 2$ )

$$\delta_i = 1 + z_i$$

$$g_i = f_{i-2}z_i^2 - f_{i-1}\delta_i^2 + f_i(z_i + \delta_i)$$

$$z_{i+1} = -\frac{2f_i\delta_i}{g_i \pm \sqrt{g_i^2 - 4f_i\delta_i z_i(f_{i-2}z_i - f_{i-1}\delta_i + f_i)}} \quad (8)$$

$$h_{i+1} = z_{i+1} - h_i$$

$$\lambda_{i+1} = \lambda_i + h_{i+1}$$

we are concerned only with those values of  $\lambda$  which may lead to singularities in the stress at the crack point. It can easily be shown that this condition corresponds to a requirement that the real part of  $\lambda$  lies between 0 and 1.

Numerical results are shown in Tables 1,2,3. Tables 1-3 give the eigenvalues for three cases: (1)  $v_1 = v_2 = 0.3$ ; (2)  $v_1 = 0.3, v_2 = 0.35$  and (3)  $v_1 = 0.35, v_2 = 0.3$ . Based on the results obtained in the study, the following conclusions can be drawn:

1) It is interesting to note that as material M1 becomes harder with respect to M2, that is M1 has a larger elastic modulus than M2, the strength of the singularity, which is equal to  $(\lambda - 1)$ , increases. In fact in the limit as  $\mu_1/\mu_2 \rightarrow \infty$ , it can be shown that the strongest singularity,  $\lambda \rightarrow 0$ , is produced.

2) There is a critical angle  $\phi_c$  for various material combinations  $\mu_1/\mu_2, v_1/v_2$ . The eigenvalues are real if  $\phi > \phi_c$  and otherwise it will be complex. For example,

$$\phi_c = 10^\circ \text{ when } v_1 = v_2 = 0.3 \quad \mu_1/\mu_2 \geq 5$$

$$\phi_c = 0^\circ \text{ when } v_1 = v_2 = 0.3 \quad \mu_1/\mu_2 = 1$$

$$\phi_c = 90^\circ \text{ when } v_1 = 0.3, v_2 = 0.35, \quad \mu_1/\mu_2 = 1$$

3) In most cases we have either two real eigenvalues or a pair of conjugate complex. For  $\phi = 90^\circ$  the eigenvalue is a real number which is consistent with results obtained in [2,4].

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## REFERENCES

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Table 1  
Singularity Values  $\lambda$  for Range of Elastic Constants and Angleless  $\phi$ . ( $\nu_1=0.3$ ,  $\nu_2=0.3$ )

$\mu_1/\mu_2$	10°	20°	30°	40°	60°	80°	90°
0.00100	.53034+.09634i	.56694+.092592i	.61195+.074368i	.62672	.71030	.62033	.95131
0.00125	.53033+.09628i	.56692+.092530i	.61191+.074291i	.62657	.71029	.62025	.95033
0.00167	.53031+.09619i	.56689+.092426i	.61184+.074164i	.62632	.71028	.62011	.94873
0.00250	.53029+.09692i	.56682+.092219i	.61171+.073911i	.62583	.71026	.61984	.94564
0.00500	.53021+.09547i	.56662+.091601i	.61133+.073154i	.62439	.71016	.61905	.93717
0.01000	.53004+.09440i	.56623+.090375i	.61055+.071655i	.62161	.70986	.61747	.92273
0.01250	.52996+.09387i	.56603+.089769i	.61016+.070914i	.62027	.70967	.61670	.91638
0.01667	.52983+.09299i	.56570+.088767i	.60952+.069689i	.61810	.70929	.61542	.90671
0.02500	.52956+.09125i	.56504+.086794i	.60823+.067282i	.61400	.70834	.61291	.88987
0.05000	.52874+.08626i	.56306+.081123i	.60439+.060378i	.60320	.70427	.60583	.85073
0.10000	.52712+.07708i	.55913+.070789i	.59680+.047845i	.58620	.69293	.59330	.79597
0.12500	.52631+.07285i	.55718+.066077i	.59308+.042125i	.57928	.68641	.58772	.77463
0.16667	.52497+.06627i	.55395+.058823i	.58699+.033239i	.56939	.67500	.57926	.74435
0.25000	.52230+.05467i	.54760+.046275i	.57532+.016678i	.55395	.65172	.56490	.69652
0.50000	.51449+.02880i	.52976+.020248i	.52797 .56142	.52584	.58884	.53462	.60272
1.00000	.5	.5	.5	.5	.5	.5	.5
1.25000	.49337+.00798i	.48337	.49215	.47336	.49355	.46815	.49272
1.66667	.48319+.01568i	.45504	.48582	.43611	.48667	.42639	.48406
2.50000	.46571+.01711i	.40739	.48074	.38114	.47909	.36835	.47275
5.00000	.38546	.47283	.31932	.47713	.29072	.46855	.27788
10.0000	.29955	.47993	.23886	.47448	.21429	.45544	.20385
12.5000	.27356	.48155	.21615	.47310	.19334	.44956	.18376
16.6667	.24206	.48319	.18941	.47060	.16892	.44016	.16041
25.0000	.20206	.48464	.15651	.46525	.13916	.42249	.13205
50.0000	.14617	.48476	.11202	.44860	.099302	.37734	.09416
100.000	.10457	.48140	.079695	.41721	.070542	.31439	.06687
125.000	.093748	.47935	.071369	.40295	.063153	.29204	.05986
166.667	.081379	.47577	.061883	.38136	.054743	.26307	.05188
250.000	.066603	.46842	.050589	.34538	.044739	.22374	.04240
500.000	.047207	.44680	.035816	.27541	.031665	.16495	.03001
1000.00	.033420	.40793	.025341	.20787	.022401	.11912	.02123

Table 2  
Singularity Values  $\lambda$  for Range of Elastic Constants and Angles  $\phi$  ( $\nu_1=0.3$ ,  $\nu_2=0.35$ )

$\mu_1/\mu_2$	10°	20°	40°	60°	75°	90°
0.00100	.53034+.096371i	.56695+.092634i	.62685	.71020	.62038	.95164
0.00125	.53033+.096326i	.56693+.0925882i	.62674	.71017	.62032	.65342
0.00167	.53032+.096249i	.56690+.092496i	.62655	.71012	.62021	.65334
0.00250	.53029+.096097i	.56684+.092324i	.62617	.71001	.61998	.65321
0.00500	.53022+.095643i	.56665+.091810i	.62504	.70969	.61932	.65295
0.01000	.53007+.094742i	.56628+.090794i	.62285	.70898	.61801	.92488
0.01250	.52999+.094295i	.56609+.090288i	.62179	.70861	.61737	.91887
0.01667	.52986+.093556i	.56579+.089454i	.62006	.70795	.61630	.90969
0.02500	.52961+.092098i	.56517+.087812i	.61676	.70651	.61421	.89363
0.05000	.52885+.087877i	.56332+.083083i	.60784	.70150	.60825	.85606
0.10000	.52735+.080076i	.55965+.074438i	.59332	.68950	.59754	.80309
0.12500	.52659+.076464i	.55783+.070485i	.58727	.68298	.59270	.78233
0.16667	.52534+.070826i	.55481+.064393i	.57849	.67180	.58530	.75278
0.25000	.52286+.060800i	.54889+.053840i	.56450	.64936	.57251	.70584
0.50000	.51557+.038140i	.53219+.032015i	.58859	.58859	.54476	.61272
1.00000	.50199+.011679i	.50396+.011235i	.50762+	.009330i	.51047+	.006200i
1.25000	.49206 .49940	.48444 .49994	.47685	.49959	.47826	.49729
1.66667	.47609 .49603	.45416 .49657	.43346	.49258	.43408	.48282
2.50000	.45089 .48764	.40841 .49041	.37486	.48129	.37508	.45956
5.00000	.38237 .48449	.32276 .48421	.28336	.46046	.28427	.41164
10.0000	.30155 .48633	.24270 .47987	.20811	.43182	.20956	.35038
12.5000	.27619 .48699	.21987 .47813	.18764	.41960	.18915	.32829
16.6667	.24510 .48769	.19289 .47531	.16384	.40109	.16536	.29901
25.0000	.20522 .48820	.15958 .46970	.13490	.36953	.13633	.25785
50.0000	.14892 .48743	.11435 .45308	.096216	.30387	.097371	.19340
100.000	.10670 .48376	.081404 .42217	.068333	.23474	.069207	.14106
125.000	.095694 .48169	.072908 .40812	.061173	.21389	.061965	.12697
166.667	.083095 .47815	.063226 .38675	.053024	.18876	.053719	.11066
250.000	.068029 .47099	.051693 .35094	.043333	.15708	.043908	.090929
500.000	.048234 .44998	.036602 .28063	.030668	.11321	.031080	.064710
1000.00	.034152 .41209	.025899 .21214	.021695	.080827	.021989	.045904

Table 3  
Singularity Values  $\lambda$  for Range of Elastic Constants and Angles  $\phi$ . ( $\nu_1=0.35$ ,  $\nu_2=0.3$ )

$\mu_1/\mu_2$	10°	20°	30°	40°	60°	75°	90°
0.00100	.53048+ .075320i	.56762+ .064664i	.57301	.77278	.59041	.90973	.62411
0.00125	.53047+ .075269i	.56760+ .064602i	.57295	.77266	.59035	.90917	.62404
0.00167	.53045+ .075182i	.56756+ .064499i	.57283	.77245	.59024	.90825	.62391
0.00250	.53042+ .075010i	.56749+ .064292i	.57261	.77203	.59003	.90644	.62365
0.00500	.53033+ .074496i	.56727+ .063676i	.57194	.77079	.58940	.90117	.62288
0.01000	.53016+ .073476i	.56683+ .062454i	.57063	.76831	.58816	.89133	.62135
0.01250	.53007+ .072971i	.56661+ .061848i	.56998	.76707	.58755	.88670	.62059
0.01667	.52992+ .072137i	.56624+ .060847i	.56893	.76501	.58654	.87936	.61935
0.02500	.52963+ .070492i	.56551+ .058874i	.56687	.76091	.58457	.86586	.61690
0.05000	.52874+ .065752i	.56332+ .053177i	.56115	.74878	.57896	.83195	.60992
0.10000	.52698+ .057060i	.55898+ .042662i	.55139	.72551	.56897	.78104	.59734
0.12500	.52610+ .053064i	.55683+ .037770i	.54718	.71444	.56450	.76058	.59164
0.16667	.52464+ .046861i	.55329+ .030016i	.54097	.69686	.55769	.73121	.58286
0.25000	.52175+ .035922i	.54637+ .014769i	.53086	.66482	.54603	.68425	.56756
0.50000	.51334+ .011075i	.50959+ .54476	.51159	.58892	.52104	.59116	.53334
1.00000	.49789+ .011548i	.49590+ .010930i	.49247+	.008746i	.49004+	.0056165i	.48898+
1.25000	.49090+ .018838i	.48327+ .013936i	.46414	.48194	.45772	.47863	.46046
1.66667	.48023+ .025068i	.45944+ .47169	.41903	.47722	.41474	.46510	.42270
2.50000	.46212+ .024284i	.40212+ .47583	.35990	.46806	.35742	.44305	.37014
5.00000	.37875+ .47111	.31134+ .47554	.26986	.44908	.26981	.39610	.28550
10.0000	.29147+ .47983	.23153+ .47360	.19726	.42023	.19835	.33535	.21285
12.5000	.26568+ .48157	.20917+ .47221	.17768	.40762	.17891	.31358	.19259
16.6667	.23463+ .48325	.18317+ .46959	.15498	.38852	.15629	.28491	.16880
25.0000	.19548+ .48466	.15118+ .46387	.12748	.35618	.12875	.24496	.13954
50.0000	.14113+ .48454	.10807+ .44603	.090817	.29037	.091883	.18308	.099941
100.0000	.10087+ .48077	.076842+ .41271	.064462	.22290	.065279	.13327	.071135
125.0000	.090410+ .47852	.068805+ .39773	.057701	.20283	.058443	.11991	.063710
166.667	.078466+ .47164	.059653+ .37524	.050009	.17874	.050661	.10446	.055248
250.0000	.064206+ .46673	.048761+ .33824	.040864	.14852	.041404	.085801	.045170
500.0000	.045499+ .44359	.034517+ .26784	.028917	.10689	.029305	.061035	.031983
1000.00	.032208+ .40254	.020126+ .24421	.020456	.076259	.020732	.043288	.022631

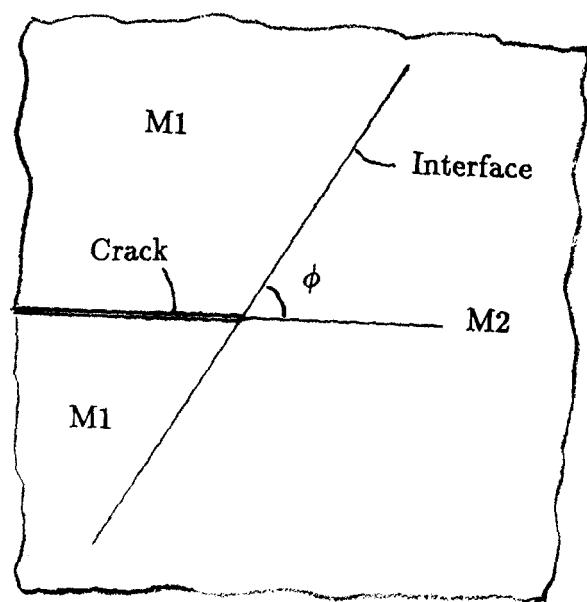


Figure 1.