TRACES OF NONSMOOTH FUNCTIONS ON POLYHEDRAL DOMAINS

DING HUA (丁 桦)
(*Institute of Mechanics, Academia Sinica, Beijing* 100080, *PRC*)

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ABSTRACT

Introduced in this paper are the definitions of the traces for a class of nonsmooth functions on polyhedral domains. By analyzing their properties we get the structures of these traces.

Keywords: traces, polyhedral domain, nonsmooth function.

I. INTRODUCTION

The concept of traces plays a very important role in treating boundary value problems of partial differential equations. The existence of traces of the functions in a given space is concerned directly with the well-posedness of the problems in this space.

The analysis of traces is a classical problem, which entered into people's consideration when they was starting to treat boundary value problems in functional spaces. A lot of work has been done by numerous authors.

According to Sobolev's imbedding theorem the functions belonging to $W^s_p(\Omega)$ are continuous when $s > n > p$[1]. They are even continuous up to the boundary. This fact enables us to investigate the values on the boundary of such functions. However, in many cases, we must consider boundary values of functions in a weaker sense. Lions and Magenes[2] gave a systematic result on functions in $W^s_p(\Omega)$ with a relaxed condition on s when Ω was smooth enough. Thus, the existence of traces for function in Sobolev's spaces on smooth domains is proved completely. But for nonsmooth domains, it is difficult to give a general result. For the Lipschitz domains, in a special case, i.e. for $W^1_p(\Omega)$, Gagliardo[3] proved the existence of the traces. Because of the limitation of differentiability of the Lipschitz boundaries it is very difficult to define higher or der Sobolev's spaces. Grisvard[4] studied the problem on a polygonal domain for the Sobolve space $W^s_p(\Omega)$, $s > 1/p$. He gained an explicit idea of the properties of the traces and the compatibility conditions at the vertexes.

For polygonal and polyhedral domains, the normal directions of the boundary at the vertexes or edges cannot be defined uniquely. This undefiniteness makes it hard to define the traces of nonsmooth functions. For smooth functions, these vertexes and edges do not cause serious troubles because the measures of these vertexes and edges are all zero (relative to the measure for the domain), so we only need to consider the compatibility conditions.

In this paper we consider the traces of functions belonging to the maximum domain $D_\Delta(L^p(\Omega))$ of Laplace operator Δ in $L^p(\Omega)$, $p > 1$ on a polyhedral domain Ω. Similar results can be obtained easily for linear elastic system[1)].

II. Preliminary Results

Let Ω be a bounded open set of $\mathbb{R}^n$, with a Lipschitz boundary $\partial\Omega$. Assume that u is a continuous (up to the boundary) function on Ω, i.e. $u \in C(\bar{\Omega})$. If we put

$$\gamma u = u|_{\partial\Omega},$$

then

$$\gamma u \in C(\partial\Omega).$$

But when $u \in W_p^1(\Omega)$, we cannot guarantee that u is continuous (up to the boundary) on Ω. Instead of $\gamma u \in C(\partial\Omega)$, we have[3]

Theorem 1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. Then the mapping $u \to \gamma u$ which is defined for $u \in C^{0,1}(\bar{\Omega})$ has a unique continuous extension as an operator from $W_p^1(\Omega)$ onto $W_p^{1-1/p}(\Gamma)$. This operator has a right continuous inverse independent of p.*

In the following, we shall always denote by γ the extended operator defined on the whole of $W_p^1(\Omega)$ (as it is unique), and we shall call it the trace operator. When we consider the traces in a weaker sense, we need the following Green's formula[5]:

Theorem 2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. Then for every $u \in W_p^2(\Omega)$ and $v \in W_q^2(\Omega)$ with $1/p + 1/q = 1$, we have*

$$\int_\Omega \Delta uv dx - \int_\Omega u\Delta v dx = \int_{\partial\Omega} \gamma \frac{\partial u}{\partial n} \gamma v d\sigma - \int_{\partial\Omega} \gamma u \gamma \frac{\partial v}{\partial n} d\sigma,$$

where n is the unit outward normal of $\partial\Omega$, which is defined a.e. on $\partial\Omega$.

From this theorem we can obtain

Corollary 1. *Let Ω be a bounded polyhedral domain of $\mathbb{R}^n$ with its boundary $\partial\Omega = \sum_{j=1}^{J} \bar{\Gamma}_j$, where Γ_j is the jth face of Ω. Then for every $u \in W_p^2(\Omega)$ and $v \in W_q^2(\Omega)$ with $1/p + 1/q = 1$, we have*

1) This work was done when the author was preparing his doctoral degree in France, under the direction of Prof. P. Grisvard.

$$\int_{\Omega}\Delta uv dx-\int_{\Omega}u\Delta v dx=\sum_{j=1}^{J}\left\{\int_{\Gamma j}\gamma_j\frac{\partial u}{\partial n_j}\gamma_j v d\sigma-\int_{\Gamma j}\gamma_j u\gamma_j\frac{\partial v}{\partial n_j}d\sigma\right\},$$

where $\gamma_j=\gamma|_{\Gamma j}$.

In the following sections we also need the following result[4].

Lemma 1. *Let G be a bounded open subset of* $\mathbb{R}^{n-1}$ *with a Lipschitz boundary* ∂G. *Assume that* $(\varphi_0,\varphi_1)\in\mathscr{D}(G)$, *then there exists a function* $v^{\varphi}\in W_q^2(\mathbb{R}^n)$ *such that* $\gamma_n v^{\varphi}|_G=\varphi_0$ *and* $\gamma_n\left.\frac{\partial v^{\varphi}}{\partial x_n}\right|_G=\varphi_1$, *and*

$$\|v^{\varphi}\|_{W_q^2}\leqslant C\{\|\tilde{\varphi}_0\|_{W_q^{2-1/q}}+\|\tilde{\varphi}_1\|_{W_q^{1-1/q}}\},$$

where $\mathscr{D}(G)$ *is the set of all the infinitely differentiable functions which have compact supports in G and* $\gamma_n u=u|_{x_n=0}$. *And*

$$\tilde{\varphi}(x')=\begin{cases}\varphi(x'), & x'\in G,\\ 0, & x'\in\mathbb{R}^{n-1}\backslash G.\end{cases}$$

Hereinafter we will denote by $D_{\Delta}(L^p(\Omega))$ the maximal domain of Laplace operator in $L^p(\Omega)$, i.e.

$$D_{\Delta}(L^p(\Omega))=\{u|u\in L^p(\Omega);\ \Delta u\in L^p(\Omega)\}.$$

III. Functional Representation of Traces

We shall still denote by v the restriction on Ω of a function v which is defined on $\mathbb{R}^n$. And we define

Definition 1. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. For a function $u\in D_{\Delta}(L^p(\Omega))$ we define a generalized function $\chi u\in W_p^{-2}(\mathbb{R}^n)$ by

$$\langle\chi u,\ v\rangle_{W_p^{-2}\times W_q^2(\mathbb{R}^n)}=(u,\Delta v)_{L^p\times L^q(\Omega)}-(\Delta u,v)_{L^p\times L^q(\Omega)}$$

for every $v\in W_q^2(\mathbb{R}^n)$, where $1/p+1/q=1$.

Remark 1. If we denote by $\tilde{w}$ the zero extension on $\mathbb{R}^n$ of a function w defined on Ω (i.e. $\tilde{w}|_{\Omega}=w,\tilde{w}|_{\mathbb{R}^n\backslash\Omega}=0$), then in the sense of distribution, we have

$$\chi u=\Delta\tilde{u}-(\Delta u)^{\sim}.$$

Proposition 1. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with a Lipschitz boundary and let* $u\in D_{\Delta}(L^p(\Omega))$, *then*

$$\operatorname{supp}\chi u\subset\partial\Omega,$$

where suppχu *denotes the support set of* χu.

Proof. For $v\in\mathscr{D}(\mathbb{R}^n\backslash\partial\Omega)$, using Theorem 2, we can easily obtain

$$\langle\chi u,\ v\rangle=(u,\ \Delta v)-(\Delta u,\ v)=0.$$

The proposition follows.

Using the integrabilities of $u\Delta v$ and $v\Delta u$ we can give an approximate approach of χu:

Proposition 2. *Under the conditions of Proposition* 1, *for a sequence of subdomains of* Ω $\{\Omega_\varepsilon\}$ *such that* $\Omega_\varepsilon \to \Omega(\varepsilon \to 0)$ (i.e. $\mathrm{mes}(\Omega - \Omega_\varepsilon) \to 0$), *then*

$$\chi^\varepsilon u \xrightarrow{V_p^{-2}} \chi u \quad (\varepsilon \to 0),$$

where $\chi^\varepsilon u$ *is defined by*

$$\langle \chi^\varepsilon u, v\rangle_{W_p^{-2}\times W_q^2(\mathbb{R}^n)} = (u, \Delta v)_{L^p\times L^q(\Omega_\varepsilon)} - (\Delta u, v)_{L^p\times L^q(\Omega_\varepsilon)}$$

for every $v \in W_q^2(\mathbb{R}^n)$ *with* $1/p + 1/q = 1$.

Definition 2. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$ we define

$$X_q^U = X_q^U(\Omega) = \{v \mid v \in W_q^2(\mathbb{R}^n),\ \gamma v = 0\},$$

$$X_q^N = X_q^N(\Omega) = \left\{v \mid v \in W_q^2(\mathbb{R}^n),\ \gamma \frac{\partial v}{\partial n} = 0\right\}.$$

From Theorem 1, γu and $\gamma \frac{\partial v}{\partial n}$ are all well defined. It is not difficult to prove that X_q^U and X_q^N are all closed subspaces of $W_q^2(\mathbb{R}^n)$. So we can define

Definition 3. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$ we define the traces operator U and N by

$$U:\ D_\Delta(L^p(\Omega)) \to (X_q^U)'$$

$$\langle Uu, v\rangle_{(X_q^U)'\times X_q^U} = \langle \chi u, v\rangle_{W_p^{-2}\times W_q^2(\mathbb{R}^n)} \quad \forall v \in X_q^U$$

and

$$N:\ D_\Delta(L^p(\Omega)) \to (X_q^N)'$$

$$\langle Nu, v\rangle_{(X_q^N)'\times X_q^N} = \langle -\chi u, v\rangle_{W_p^{-2}\times W_q^2(\mathbb{R}^n)} \quad \forall v \in X_q^N,$$

where $1/p + 1/q = 1$. In other words,

$$Nu = -\chi u|_{X_q^N},$$

$$Uu = \chi u|_{X_q^U}.$$

From Theorem 2 we can easily obtain

Proposition 3. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with a Lipschitz boundary* $\partial\Omega$. *Assume* $u \in W_p^2(\Omega)$. *Then*

$$Nu = \gamma \frac{\partial u}{\partial n} \otimes \delta_{\partial\Omega},$$

$$Uu = \gamma u \otimes \frac{\partial \delta_{\partial\Omega}}{\partial n},$$

that is,

$$\langle Nu, v\rangle = \int_{\partial\Omega} \gamma \frac{\partial u}{\partial n} \gamma v d\sigma \quad \forall v \in X_q^N,$$

$$\langle Uu, v\rangle = \int_{\partial\Omega} \gamma u \gamma \frac{\partial v}{\partial n} d\sigma \quad \forall v \in X_q^U.$$

From the results in Ref. [2], we have

Proposition 4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary (e.g. C^∞ boundary). Assume that $u \in D_\Delta(L^p(\Omega))$. Then*

$$Nu = \gamma \frac{\partial u}{\partial n} \otimes \delta_{\partial\Omega},$$

$$Uu = \gamma u \otimes \frac{\partial \delta_{\partial\Omega}}{\partial n},$$

in the sense of

$$\langle Nu, v\rangle = \left\langle \gamma \frac{\partial u}{\partial n}, \gamma v \right\rangle_{W_p^{-1-1/p} \times W_q^{1+1/p}(\partial\Omega)} \quad \forall v \in X_q^N,$$

$$\langle Uu, v\rangle = \left\langle \gamma u, \gamma \frac{\partial v}{\partial n} \right\rangle_{W_p^{-1/p} \times W_q^{1/p}(\partial\Omega)} \quad \forall v \in X_q^U.$$

Remark 2. We may define χu, Nu, Uu in a more general way for function u such that $u \in L^p(\Omega)$, $\Delta u \in (W_q^2(\Omega))'$. This is done by replacing $(\Delta u, v)$ by $\langle \Delta u, v\rangle_{(W_q^2)' \times W_q^2(\mathbb{R}^n)}$ in the above definitions, where $1/p + 1/q = 1$.

IV. Structure of Traces on Polyhedra

In this section we will assume that Ω is a bounded polyhedral domain of $\mathbb{R}^n$ with its boundary $\partial\Omega = \bigcup_{j=1}^{J} \bar{\Gamma}_j$, where its faces Γ_j are $(n-1)$-dimensional polyhedra. In addition we assume that for every $j, k \in \{1, \cdots, J\}$, $j \neq k$, we have

$$\Gamma_j \cap \Gamma_K = \phi.$$

This implies that the possible intersection between two different faces is only the edge formed by them. In other words, there is no fissure in Ω. The case where there are fissures needs to be considered separately. Owing to the limitation ofspace we will not present them here. As in the previous sections, we shall always assume that $1/p + 1/q = 1$, $p > 1$.

Let φ_0, φ_1 be in $\mathscr{D}(\Gamma_k)$ for a fixed k in $\{1, \cdots, J\}$. Then from Lemma 1, there exists v^φ belonging to $W_q^2(\mathbb{R}^n)$ such that

$$\begin{cases} \gamma_k v^\varphi = \varphi_0, \\ \gamma_k \dfrac{\partial v^\varphi}{\partial n_k} = \varphi_1. \end{cases}$$

Now we can construct (see Appendix) a function ψ belonging to $\mathscr{D}(\mathbb{R}^n)$ such that it equals 1 in the neighborhood of $\operatorname{supp}\varphi_0 \cup \operatorname{supp}\varphi_1$ and equals zero in the neighborhood of $\bigcup_{j\neq k} \bar{\Gamma}_j$, and

$$\|\psi v^\varphi\|_{W_q^2(\mathbb{R}^n)} \leqslant C \|v^\varphi\|_{W_q^2(\mathbb{R}^n)}.$$

Here C is a constant independent of the choice of ϕ. If we put $v=\phi v^{\varphi}$, then

$$\begin{cases}\gamma_k v=\varphi_0,\ \gamma_k \dfrac{\partial v}{\partial n_k}=\varphi_1,\\ \gamma_j v=\gamma_j \dfrac{\partial v}{\partial n_j}=0\ \forall j\neq k,\end{cases} \tag{*}$$

and

$$\|v\|_{W_q^2(\mathbb{R}^n)}\leqslant C\{\|\varphi_0\|_{\widetilde{W}_q^{2-1/q}(\Gamma_k)}+\|\varphi_1\|_{\widetilde{W}_q^{1-1/q}(\Gamma_k)}\}; \tag{**}$$

where $\widetilde{W}_q^s(G)$, $G\subset\mathbb{R}^m$, is a subspace of $W_q^s(G)$, in which the extension on $\mathbb{R}^m$ by zero of functions are the elements of $W_q^s(\mathbb{R}^m)$. The norm in $\widetilde{W}_q^s(G)$ is defined by

$$\|W\|_{\widetilde{W}_q^s(G)}=\|\widetilde{W}\|_{W_q^s(\mathbb{R}^m)}.$$

We know that $\mathscr{D}(G)$ is dense in $\widetilde{W}_q^s(G)$, so we can find

Proposition 5. *For fixed k, let $\varphi=(\varphi_0,\ \varphi_1)$ belong to $\widetilde{W}_q^{2-1/q}\times\widetilde{W}_q^{1-1/q}(\Gamma_k)$. Then there exists a function v belonging to $W_q^2(\mathbb{R}^n)$ which satisfies (*) and (**).*

Theorem 3. *For fixed k, the mapping,*

$$u\to\left\{\gamma_k\frac{\partial u}{\partial n_k},\ \gamma_k u\right\}$$

defined on $W_p^2(\Omega)$, can be continuously extended to a linear continuous mapping which maps every element of $D_\Delta(L^p(\Omega))$ to an element of $(\widetilde{W}_q^{2-1/q}\times\widetilde{W}_q^{1-1/q}(\Gamma_k))'$.

Proof. Let $(\varphi_0,\ \varphi_1)\in\mathscr{D}\times\mathscr{D}(\Gamma_k)$. Then there exists $v\in W_p^2(\mathbb{R}^n)$ such that (*) and (**) hold. Because v vanishes in the neighborhood of $\bigcup_{j\neq k}\partial\Gamma_j$, and for $u\in D_\Delta(L^p(\Omega))$ we have $u\in W_p^2(\operatorname{supp}v)$ (u is regular on smooth part of $\partial\Omega$), from Corollary 1 there is

$$\int_{\Gamma_k\cap\operatorname{supp}v}\left(\gamma_k\frac{\partial u}{\partial n_k}\varphi_0-\gamma_k u\varphi_1\right)d\sigma=\int_\Omega\Delta u\ \ v dx-\int_\Omega u\Delta v dx.$$

Then from Proposition 5, we find

$$\left|\int_{\Gamma_k\cap\operatorname{supp}v}\left(\gamma_k\frac{\partial u}{\partial n_k}\varphi_0-\gamma_k u\varphi_1\right)d\sigma\right|\leqslant C\|u\|_{D_\Delta(L^P(\Omega))}\|v\|_{W_q^2(\mathbb{R}^n)}$$
$$\leqslant C\|u\|_{D_\Delta(L^P(\Omega))}\|\varphi\|_{\widetilde{W}_q^{2-1/q}\times\widetilde{W}_q^{1-1/q}(\Gamma_k)}.$$

Since $\mathscr{D}\times\mathscr{D}(\Gamma_k)$ is dense in $\widetilde{W}_q^{2-1/q}\times\widetilde{W}_q^{1-1/q}(\Gamma_k)$, we can prove the theorem.

In the sequel we say that mapping

$$\begin{cases}u\longrightarrow\left\{\gamma_k\dfrac{\partial u}{\partial n_k},\ \gamma_k u\right\}\\ D_\Delta(L^P(\Omega))\longrightarrow(\widetilde{W}_q^{2-1/q}\times\widetilde{W}_q^{1-1/q}(\Gamma_k))\end{cases}$$

is the extended mapping in Theorem 3. It is easy to verify

Corollary 2. *Let $u\in D_\Delta(L^P(\Omega))$. Then for every $v\in W_q^2(\mathbb{R}^n)$ such that v vanishes in the neighborhood of $\bigcup_j\partial\Gamma_j$, we have*

$$\int_\Omega \Delta u\, v dx - \int_\Omega u \Delta v dx = \sum_{j=1}^{J}\Big\{\Big\langle \gamma_j \frac{\partial u}{\partial n_j}, \gamma_j v\Big\rangle_{(\widetilde{W}_q^{2-1/q})' \times \widetilde{W}_q^{2-1-q}(\Gamma_j)} - \Big\langle \gamma_j u, \gamma_j \frac{\partial v}{\partial n_j}\Big\rangle_{(\widetilde{W}_q^{1-1/q})' \times \widetilde{W}_q^{1-1/q}(\Gamma_j)}\Big\}.$$

For $p \neq 2$, we know that $\widetilde{W}_q^{2-1/q} \times \widetilde{W}_q^{1-1/q}(\Gamma_j)$ is a closed subspace of $W_q^{2-1/q} \times W_q^{1-1/q}(\Gamma_j)$, then using the Hahn-Banach's theorem[6] we can extend $\left\{\gamma_j \frac{\partial u}{\partial n}, \gamma_j u\right\}$ to a linear continuous functional on $W_q^{2-1/q} \times W_q^{1-1/q}(\Gamma_j)$. In the following we will see that the nonuniqueness of this extension can interpret the value of traces on $\partial \Gamma_j$:

Definition 4. Let $u \in D_\triangle(L^P(\Omega))$, $p \neq 2$, and let $\left(\gamma_j \overline{\frac{\partial u}{\partial n}}, \gamma_j \bar{u}\right)$ be the extension of $\left(\gamma_j \frac{\partial u}{\partial n_j}, \gamma_j u\right)$ from $\widetilde{W}_q^{2-1/q} \times \widetilde{W}_q^{1-1/q}(\Gamma_j)$ to $W_q^{2-1/q} \times W_q^{1-1/q}(\Gamma_j)$ (we can treat $p=2$, in case it is extensible, in the same way) we define distribution $\mathscr{Z}u \in W_p^2(\mathbf{R}^n)$ by

$$\langle \mathscr{Z}u, v\rangle_{W_p^{-2} \times W_q^2(\mathbf{R}^n)} = \sum_j \Big\{\Big\langle \gamma_j \overline{\frac{\partial u}{\partial n_j}}, \gamma_j v\Big\rangle_{(W_q^{2-1/q})' \times W_q^{2-1/q}(\Gamma_j)} - \Big\langle \gamma_j \bar{n}, \gamma_j \frac{\partial v}{\partial n_j}\Big\rangle_{(W_q^{1-1/q})' \times W_q^{1-1/q}(\Gamma_j)}\Big\}$$

for every $v \in W_q^2(\mathbf{R}^n)$. And we have

Theorem 4. *Let* $u \in D_\triangle(L^P(\Omega))$. *Then*

$$\chi u = \mathscr{Z}u + \mathscr{R}u,$$

where $\mathscr{R}u \in W_p^{-2}(\mathbf{R}^n)$, $\operatorname{supp}\mathscr{R}u \subset \bigcup_{j=1}^{J} \partial\Gamma_j = \partial\Omega \setminus \bigcup_{j=1}^{J} \Gamma_j$.

Proof. In fact, for $v \in \mathscr{D}\left(\mathbf{R}^n \setminus \bigcup_{j=1}^{J} \partial\Gamma_i\right)$, using Corollary 2, we find

$$\langle \chi u - \mathscr{Z}u, v\rangle = 0.$$

This means that $\operatorname{supp}(\chi u - \mathscr{Z}u) \subset \bigcup_{j=1}^{J} \partial\Gamma_j$.

Remark 3. If Ω is a polyhedral domain of $\mathbf{R}^n$, then $\bigcup_{j=1}^{J} \partial\Gamma_j$ is an $(n-2)$-dimensional manifold.

Corollary 3. *Let* Ω *be a polygon (i.e.* $n=2$*), and* $\{S_i\}$ *be the set of its vertexes. Then for* $u \in D_\triangle(L^P(\Omega))$ *we have*

$$\chi u = \begin{cases} \sum_j \gamma_j \overline{\frac{\partial u}{\partial n_j}} \otimes \delta_{\Gamma_j} + \sum_i C_i \delta(S_i) + \sum_j \gamma_j \bar{u} \otimes \frac{\partial \delta_{\Gamma_j}}{\partial n_j}, & q \leqslant 2, \\ \sum_j \gamma_j \overline{\frac{\partial u}{\partial n_j}} \otimes \delta_{\Gamma_j} + \sum_i C_i \delta(S_i) + \sum_j \gamma_j \bar{u} \otimes \frac{\partial \delta_{\Gamma_j}}{\partial n_j} + \sum_{|\alpha_j|=1} b_i^\alpha \delta^{(\alpha)}(S_i), & q > 2, \end{cases}$$

$$Nu=\sum_j \gamma_j \overline{\frac{\partial u}{\partial n_j}} \otimes \delta_{\Gamma_j}+\sum_i C_i \delta(S_i),$$

$$Uu=\begin{cases}\sum_j \gamma_j \bar{u} \otimes \dfrac{\partial \delta \Gamma_j}{\partial n_j}, & q \leqslant 2, \\ \sum_j \gamma_j \bar{u} \otimes \dfrac{\partial \delta \Gamma_j}{\partial n_j}+\sum_{|\alpha|=1} b_i^\alpha \delta^{(\alpha)}(S_i), & q>2.\end{cases}$$

Proof. From Theorem 4, there holds

$$\operatorname{supp}(\chi u-\mathscr{Z} u) \subset\{S_i\}.$$

So that

$$\mathscr{R} u=\chi u-\mathscr{Z} u=\sum_{1, \alpha} a_i^\alpha \delta^{(\alpha)}(S_i).$$

Since $\mathscr{R} u \in W_p^{-2}(\mathbf{R}^2)$, we have $a_i^\alpha=0$ for $\delta^{(\alpha)}(S_i) \notin W_p^{-2}(\mathbf{R}^2)$. So we have the corollary.

In the same way we can find

Corollary 4. *Let Ω be a polyhedral domain of $\mathbf{R}^3$, $\{A_i\}$ be the set of its edges, and $\{S_k\}$ be the set of its vertexes. Then*

$$\chi u-\mathscr{Z} u=\begin{cases}\sum_i\left(\varphi_i^0 \dfrac{\partial \delta(A_i)}{\partial n}+\varphi_i^1 \delta(A_i)\right)+\sum_k C_k \delta(s_k)+\sum_{k, \alpha} b_k^\alpha \delta^\alpha(s_k) & \text{in general}, \\ \sum_i \varphi_i \delta(A_i)+\sum_k C_k \delta(s_k) & q<2,\end{cases}$$

i.e. (*for* $q<2$)

$$Nu=\sum_j \gamma_j \overline{\frac{\partial u}{\partial n_j}} \otimes \delta_{\Gamma_j}+\sum_i \varphi_i \delta(A_i)+\sum_k C_k \delta(s_k),$$

$$Uu=\sum_j \gamma_j \bar{u} \otimes \frac{\partial \delta_{\Gamma_j}}{\partial n_j}.$$

Using the results given above, we could give a precise mathematical description of the boundary value problem for Laplace operator when there are concentrated loads on edges and at vertexes.

Example. Consider the boundary value problem for Laplace operator on a 2-dimensional sector, and a concentrated load at the vertex of the sector. In this case the very weak form of the solution is given by[7,8]

$$\begin{cases}u \in L^2(\Omega), \\ \int_{\Omega} u \Delta v=-v(0), \quad \forall v \in E_0(\Omega),\end{cases}$$

where $\Omega=\{(r, \vartheta) \mid 0<r<R,\ 0<\vartheta<\omega<2 \pi\}$, and $E_0(\Omega)$ is the set of all variational solutions corresponding to $\Delta u \in L^2(\Omega)$. It is a closed subspace in $H^1(\Omega)$, i.e.

$$E_0(\Omega)=\left\{u \mid u \in H^1(\Omega),\left.\frac{\partial u}{\partial n}\right|_{\partial \Omega}=0,\ \Delta u \in L^2(\Omega)\right\},$$

as $H^2(\Omega)\subset E_0(\Omega)$. Thus, we have

$$Nu=\delta(0).$$

Noticing the situation on the boundary we find (for $r>0$)

$$\left.\frac{\partial u}{\partial n}\right|_{\vartheta=0}=0,\quad \left.\frac{\partial u}{\partial n}\right|_{\vartheta=\omega}=0.$$

Their extension can be written as

$$\left.\overline{\frac{\partial u}{\partial n}}\right|_{\sigma=0}=a_0\delta(0),\ \left.\overline{\frac{\partial u}{\partial n}}\right|_{\theta=\omega}=a_\omega\delta(0),$$

a_0, a_ω being arbitrary constants. So that

$$Nu=a_0\delta(0)+a_\omega\delta(0)+a\delta(0),$$

that is,

$$a_0+a_\omega+a=1.$$

From the above discussion we see that the concentrated load at the vertex could be regarded as the summation of the loads acting on the ends of each face of the boundary. The global effect only depends on the summation of these loads and is independent of the partition.

Finally we would like to point out that the results of this paper can be easily generalized to elliptic equations or systems of general form[8].

Appendix

Now let us construct the function ϕ to ensure that the constant C in (**) is independent of the choice of ϕ.

Using the theorem of partition of unity, we can transform the problem to the neighborhood of a vertex, i.e. to study the situation of a polyhedral cone G under the condition that the function u has compact support. According to the hypothesis that there is no fissure in Ω, there exists a conical domain Q such that

$$\begin{cases}\Gamma_k\subset Q,\\ \Gamma_j\cap Q=\phi,\ j\neq k.\end{cases}$$

In order to have suitable choice of ϕ, we need the following lemmas[9] (they can be obtained by using Hardy inequalities).

Lemma 1. *Let* $u\in W_q^1(\mathbf{R}^n)$, *and assume that* supp(u) *is bounded. If there exists* $\varepsilon>0$ *such that*

$$u|_{\Gamma_k\cap\{r<\varepsilon\}}=\left.\frac{\partial u}{\partial n_k}\right|_{\Gamma_k\cap\{r<\varepsilon\}}=0,$$

then we have

$$\left\|\frac{u}{r}\right\|_{L^q(\mathbf{R}^n)}\leqslant C\|u\|_{\dot W_q^1(\mathbf{R}^n)},$$

$$\left\|\frac{u}{r^2}\right\|_{L^q(\mathbf{R}^n)} \leqslant C\|u\|_{W_q^2(\mathbf{R}^n)},$$

where C is a constant independent of ε and u.

Lemma 2. *Let* $u \in W_q^2(\mathrm{R}^n)$ *and* $\mathrm{supp}(u)$ *be bounded. Assume that* γ *is a subset in* R^n*. If u vanishes in the neighborhood of* γ*, then there exists a constant C independent of u such that*

$$\left\|\frac{u}{\rho}\right\|_{L^q(\mathbf{R}^n)} \leqslant C\|u\|_{W_q^1(\mathbf{R}^n)},$$

$$\left\|\frac{u}{\rho^2}\right\|_{L^q(\mathbf{R}^n)} \leqslant C\|u\|_{W_q^2(\mathbf{R}^n)},$$

where ρ *is the distance to* γ.

Now we construct ϕ by recurrence. First we assume $n=2$. Let a be the intersection of Q with unit circle in $\mathbf{R}^2$, i.e. $a = Q \cap S^1$. Take $\Phi \in \mathfrak{D}(S^1) \cap \mathfrak{D}(a)$ such that

$$\Phi = 1$$

in the neighborhood of $\Gamma_k \cap S^1$. At the same time we take $\varphi \in \mathfrak{D}(\mathbf{R}_+)$ such that

$$\varphi(r) = 0,\ r < 1/2,$$

$$\varphi(r) = 1,\ r > 1.$$

Put

$$\phi_m(r, \omega) = \varphi(mr)\Phi(\omega).$$

Then for every $m > 0$, $\phi_m \in \mathfrak{D}(Q) \cap \mathfrak{D}(\mathbf{R}^n)$ and

$$\Gamma^m = \{x \mid x \in \Gamma_k,\ \phi_m(x) = 1\} \longrightarrow \Gamma_k,\ (m - \infty),$$

so that for any compact set Γ in Γ_k, there exists an M_Γ such that for $m > M_\Gamma$, we have $\Gamma \subset \Gamma^m$. In the following we shall prove that such ϕ_m satisfies (**) with a constant C independent of m.

In fact, by using Lemma 1 (because $u \in \mathfrak{D}(\Gamma_k)$), we can conclude

$$\sum_{1 \leqslant |\alpha| + |\beta| \leqslant 2} \int_Q |(D^{\alpha+\beta}\phi_m)u|^2 dx \leqslant cM^2 \int_{B_m}\left(\left|\frac{u}{r}\right|^2 + \left|\frac{u}{r^2}\right|^2\right)dx \xrightarrow[(m\to\infty)]{} 0,$$

where

$$M = \mathrm{Max}\{|\varphi''\Phi|,\ |\varphi'\Phi'|,\ |\varphi\Phi''|,\ |\varphi'\Phi|,\ |\varphi\Phi'|,\ |\varphi\Phi|\},$$

$$B_m = \{x \mid |x| < 1/m\},$$

so that for $|\alpha| \leqslant 2$

$$D^\alpha(\phi_m u) \xrightarrow{L^q(\mathbf{R}^n)} D^\alpha u\ (m \to \infty),$$

and then there exists a constant C such that

$$\|\phi_m u\|_{W_q^2(\mathbf{R}^n)} \leqslant C\|u\|_{W_q^2(\mathbf{R}^n)}.$$

The case of $n=2$ is proved. Let us assume that the assertion is true for $n-1$, and we try to prove the assertion for n.

Taking ψ_m^{n-1} as ψ_m defined in $(n-1)$-dimensional case, putting

$$\psi_m^n(r, \omega) = \varphi(mr)\psi_m^{n-1}(\omega),$$

we can verify that supp ψ_m^n is in Q and

$$\text{supp } \psi_m^n \cap \Gamma_k \longrightarrow \Gamma_k \ (m \to \infty)$$

so, what is left to be proved is

$$\|\psi_m^n u\|_{W_q^2(\mathbf{R}^n)} \leqslant C\|u\|_{W_q^2(\mathbf{R}^n)}, \qquad (\#)$$

with C independent of m. For this purpose, it suffices to prove that the multiplication of u by the derivatives (up to order 2) of ψ_m^n converges to zero in L^q $(\mathbf{R}^n)$ when m tends to infinite. In fact $(\varphi_m(r) = \varphi(mr))$

$$\int_{\mathbf{R}^n} |(D^\alpha \varphi_m D^\beta \psi_m^{n-1})u|^2 dx \leqslant C \int_{r<1/m}\int_{\rho<1/m} \left|\frac{u}{r}\right| \left|\frac{u}{\rho}\right| dx, \quad |\alpha| = |\beta| = 1, \quad (\#\#)$$

where ρ is the distance to $\partial Q \cup \Gamma_k$. By the Cauchy inequality and the lemmas above we find that the right-hand side converges to zero when $m \to \infty$. It should be noted that (##) needs $|\alpha| = |\beta| = 1$, and for the other cases we only need to use the lemmas directly. And therefore we have (#).

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