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ON DILATATIONAL PLASTIC CONSTITUTIVE EQUATION OF DUCTILE MATERIALS AND PLASTIC LOADING PATHS AT BIFURCATION

LI GUO-CHEN (李国琛)
(*Institute of Mechanics, Academia Sinica, Beijing* 100080, *PRC*)

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ABSTRACT

The dilatational plastic constitutive equation presented in this paper is proved to be in a form of generality. Based on this equation, the constitutive behaviour of materials at the moment of bifurcation is demonstrated to follow a loading path with the response as "soft" as possible.

Keywords: ductile materials, dilatational plasticity, constitutive equation, bifurcation, strain-softening.

I. INTRODUCTION

Ductile fracture is an important topic in studying material failure. Tiny voids and shear bands initiate within materials as stress is enhanced. Their growth and coalescence eventually result in cracks[1,2]. Two immediate questions arise. First, how can we account for the damaging effect and dilatation caused by voids in the constitutive equation of continuum? Secondly, what is the loading path followed by material behaviour at the moment when the shear-band type of bifurcation occurs?

Based on the analysis of void models, Gurson[3] modified the von-Mises type of potential that is widely used in conventional plasticity and incorporated into his plastic loading surface containing a parameter called the void volume fraction f_v. This surface shrinks in stress space as the value of f_v increases, that is to say, the interior damage induces strain-softening effect. Under the assumption that the normality rule still holds, Gurson proposed his formulation of dilatational plasticity. For the shear-band type of bifurcation, Hill and Hutchinson[4] derived the governing equation in analysis. But, they still followed the assumption of "consistent loading" in buckling (bifurcation) of structures (cf. Ref. [5]).

Recent developments in researches demonstrate that: (i) Besides the larger voids (about 10μm), secondary voids in a smaller size ($1\ \mu$m) can also develop when stress is enhanced. The interaction between these two generations of voids is so strong that the material is then much deteriorated. The smoothness and convexity of the plastic loading surfaces, distinguished by different inherent values of void volume fraction f_v, no longer exist as depicted by Gurson and a simultaneous decrease in the overall Young's modulus may occur[6,7]. (ii) The tests by Anand and Spitzig[2] indicated that the solution given by Hill and Hutchinson yielded much higher values of critical strain in the shear band than those obtained from experiments.

The present paper intends to set up a theoretical frame for deriving dilatational plastic constitutive equation in a general form, and to provide a theoretical explanation for choosing a "soft" loading path in the bifurcation analysis of materials with microstructural damage.

II. Derivation of Dilatational Plastic Equation

In our previous papers[6,7], we have demonstrated that as the interaction between elasticity and plastic damage causes a drastic decrease in Young's modulus, the loading surfaces become irregular, and the conventional convexity and normality rule no longer holds. Therefore, any effort for setting up a rational constitutive equation in plasticity has to resort to a more general principle, instead of restricting to the requirement that the work done by cycling stress or strain should be positive. So, we shall base the following derivation on the equivalency in expressions for dissipated plastic energy.

Firstly, let the stress tensor σ_{ij} and the increment of plastic strain $d\varepsilon_{ij}^{(p)}$ be resolved into their deviatoric and volumetric parts. In Cartesian coordinates, they can be written as

$$\sigma_{ij} = S_{ij} + \delta_{ij}\sigma_m, \tag{1}$$

$$d\varepsilon_{ij}^{(p)} = de_{ij}^{(p)} + \delta_{ij}d\varepsilon_m^{(p)}, \tag{2}$$

where $S_{ij}\left(=\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}\right)$ and $\sigma_m\left(=\frac{1}{3}\sigma_{kk}\right)$ are the deviatoric stress and mean stress, respectively. $de_{ij}^{(p)}\left(=d\varepsilon_{ij}^{(p)} - \frac{1}{3}\delta_{ij}d\varepsilon_{kk}^{(p)}\right)$ and $d\varepsilon_m^{(p)}\left(=\frac{1}{3}d\varepsilon_{kk}^{(p)}\right)$ represent respectively the corresponding deviatoric and mean parts of the plastic strain increment.

The increment of dissipated plastic energy can then take the form

$$dW^{(p)} = \sigma_{ij}d\varepsilon_{ij}^{(p)} = (S_{ij} + \delta_{ij}\sigma_m)(de_{ij}^{(p)} + \delta_{ij}d\varepsilon_m^{(p)}) = S_{ij}de_{ij}^{(p)} + 3\sigma_m d\varepsilon_m^{(p)}. \tag{3}$$

It needs to be emphasized that the stress σ_{ij} and plastic strain $\varepsilon_{ij}^{(p)}$ quoted in Eq. (3) are all general terms, applicable to both small- or large-strain condition. The requirement is that the scalar multiplication between the stress and the increment of an associated plastic strain can represent an increment of dissipated plastic energy. For example, we can express $dW^{(p)}$ with different configurations corresponding to generalized time t at 0 (initial), t (current) and t_0 (fixed). Correspondingly, $dW^{(p)}$ can be written as

(a)
$$dW_{(0)}^{(p)} = S_{ij}^{(k)}d\varepsilon_{ij}^{(p)} \tag{3a}$$

in the Lagrangian formulation. $S_{ij}^{(k)}$ is the second Piola-Kirchhoff stress, and $\varepsilon_{ij}^{(p)}$ represents the Green strain in plasticity.

(b)
$$dW_t^{(p)} = \sigma_{ij}D_{ij}^{(p)} \tag{3b}$$

in the Eulerian sense. Then, σ_{ij} is exclusively used to represent the Cauchy (true) stress, and $D_{ij}^{(p)}\left(=\frac{1}{2}(V_{i,j}+V_{j,i})^{(p)}\right)$ is the plastic part of deformation rate with V_i being the components of velocity.

(c)
$$dW_0^{(p)}=\tau_{ij}D_{ij}^{(p)} \tag{3c}$$

when the up-dated Lagrangian description is preferred. Here, τ_{ij} is called the Kirchhoff stress, which is related to the Cauchy stress as $\tau_{ij}=\frac{dV}{dV_0}\sigma_{ij}$ with dV and dV_0 representing an infinitesimal volume at time t and t_0, respectively.

On the other hand, let us define

$$dW^{(p)}=\sigma_e d\varepsilon_e^{(p)}+3\sigma_m d\varepsilon_m^{(p)}. \tag{4}$$

Here, we have the equivalent stress σ_e defined as

$$\sigma_e=\left(\frac{3}{2}S_{ij}S_{ij}\right)^{1/2}, \tag{5}$$

and call $\varepsilon_e^{(p)}$ the plastic equivalent strain.

Comparison between Eqs. (3) and (4) indicates that

$$\sigma_e d\varepsilon_e^{(p)}=S_{ij}de_{ij}^{(p)} \tag{6}$$

must be true. Since Eq. (6) always holds its equivalency under whatever stress condition with the equivalent stress being defined as (5), the following inference can be drawn:

$$de_{ij}^{(p)}=\frac{3}{2}\frac{d\varepsilon_e^{(p)}}{\sigma_e}S_{ij}. \tag{7}$$

Therefore,

$$de_{ij}^{(p)}de_{ij}^{(p)}=\frac{9}{4}\frac{d\varepsilon_e^{(p)2}}{\sigma_e^2}S_{ij}S_{ij}, \tag{8}$$

and consequently we should have

$$d\varepsilon_e^{(p)}=\left(\frac{2}{3}de_{ij}^{(p)}de_{ij}^{(p)}\right)^{1/2}. \tag{9}$$

The expression in Eq. (9) is entirely dependent on the definition for the equivalent stress in Eq. (5) and the equivalency of the deviatoric part of any increment of plastic energy being dissipated.

Substituting Eq. (7) into Eq. (2), we find

$$d\varepsilon_{ij}^{(p)}=\frac{3}{2}\frac{d\varepsilon_e^{(p)}}{\sigma_e}S_{ij}+\delta_{ij}d\varepsilon_m^{(p)}=\frac{3}{2}\frac{d\sigma_e}{E_{te}^{(p)}\sigma_e}S_{ij}+\delta_{ij}\frac{d\sigma_{kk}}{3E_{tm}^{(p)}}. \tag{10}$$

Here, $E_{te}^{(p)}=d\sigma_e/d\varepsilon_e^{(p)}$, $E_{tm}^{(p)}=d\sigma_m/d\varepsilon_m^{(p)}$ are two plastic tangent moduli along the σ_e-$\varepsilon_e^{(p)}$ curve and σ_m-$\varepsilon_m^{(p)}$ curve, respectively.

Since

$$d\sigma_e=\frac{3}{2\sigma_e}S_{ij}d\sigma_{ij}, \tag{11}$$

Eq. (10) can be rewritten as

$$d\varepsilon_{ij}^{(p)} = \frac{9}{4E_{te}^{(p)}} \frac{S_{ij}S_{kl}d\sigma_{kl}}{\sigma_e^2} + \frac{1}{3E_{tm}^{(p)}} \delta_{ij}d\sigma_{kk}. \tag{12}$$

Assuming that the total strain increment is composed of an elastic part following the Hooke's law and a plastic component, we find

$$\begin{aligned} d\varepsilon_{ij} &= d\varepsilon_{ij}^{(e)} + d\varepsilon_{ij}^{(p)} \\ &= \frac{1}{E}[(1+\nu)d\sigma_{ij} - \delta_{ij}d\sigma_{kk}] + \frac{9}{4E_{te}^{(p)}} \frac{S_{ij}S_{kl}}{\sigma_e^2} d\sigma_{kl} + \frac{1}{3E_{tm}^{(p)}} \delta_{ij}d\sigma_{kk} \end{aligned} \tag{13}$$

and its inverse form

$$\begin{aligned} d\sigma_{ij} = \frac{E}{1+\nu} \Bigg[& \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ij}\delta_{kl} \frac{\nu - E/3E_{tm}^{(p)}}{1 - 2\nu + E/E_{tm}^{(p)}} \\ & - \frac{3}{2\sigma_e^2} \frac{E}{E_{te}^{(p)}} \frac{S_{ij}S_{kl}}{\frac{2}{3}(1+\nu) + E/E_{te}^{(p)}} \Bigg] d\varepsilon_{kl}. \end{aligned} \tag{14}$$

Eqs. (13) and (14) have the same form as the dilatational plastic constitutive equations proposed by Li and Howard[8], in which, E and ν are Young's modulus and Poisson's ratio, respectively.

The process of derivation clearly demonstrates that Eqs. (13) and (14) are independent of a certain plastic potential/normality rule. They are only based on an equivalent transformation between the expressions used for the increment of plastic energy being dissipated. Obviously, this transformation form may not be unique. However, it has a nature of generality, since it is restricted neither by the prerequisites associated with conventional plasticity nor by any requirement that the energy dissipated during stress/strain cycling must be positive[9].

According to whether materials are damaged or not, we shall mainly discuss two cases in determination of the plastic tangent moduli $E_{te}^{(p)}$ and $E_{tm}^{(p)}$.

1. *Plastic Strain Without Damage*

Usually this case refers to the small-strain condition of ductile materials. If $E_{tm}^{(p)} \to \infty$, then the form of Eqs. (13) and (14) reduces to that of the Prandtl-Reuss relation. Under such circumstances, Drucker's postulate and its deduction, the convexity and normality rule, are brought into effect. Hence, the loading surfaces are assumed to have the shape as the von-Mises yield surface. Each loading surface denotes a fixed value of the equivalent stress and has a one-to-one correspondence with the plastic equivalent strain.

Following this routine, we can take the plastic tangent modulus $E_{te}^{(p)}$ as a single-valued function of $\varepsilon_{te}^{(p)}$. Then, a uniaxial stress-strain curve (σ-ε) can be employed to determine this relationship. This is owing to the fact that

$$\frac{1}{E_{te}^{(p)}} = \frac{1}{E_t} - \frac{1}{E}, \tag{15}$$

where $E_t = d\sigma/d\varepsilon$, $\sigma_e = \sigma$ (the uniaxial stress), and $\varepsilon_e^{(p)} = \varepsilon^{(p)}$ (the plastic part of uniaxial strain).

The necessary regulations for distinguishing plastic loading from elastic unloading are well stated in the conventional plasticity and no recapitulation is needed.

2. *Plastic Strain With Damage*

When strain enlarges, ductile damage develops and makes its contribution to plastic strain. In this case, we consider $E_{te}^{(p)}$ and $E_{tm}^{(p)}$ as two scalar functions of stress state $\boldsymbol{\sigma}$, plastic strain state $\boldsymbol{e}_e^{(p)}$ and internal parameters $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ which are in the vectorial form characterizing damage. Hence we assume

$$E_{te}^{(p)} = F_1(\boldsymbol{\sigma},\boldsymbol{\epsilon}^{(p)},\boldsymbol{K}_1),\ E_{tm}^{(p)} = F_2(\boldsymbol{\sigma},\boldsymbol{\epsilon}^{(p)},\boldsymbol{K}_2). \tag{16}$$

Microstructural modelling and experimental studies[1,6,10-12] on void damage show that strain-softening effect and plastic dilatancy are the two main factors reflecting damaging behaviour. Owing to the fact that some damage in shearing may not be accompanied by any obvious change in volume, it is then preferable to treat the deviatoric part and the volumetric part of responses separately. The crucial point then becomes how to determine the turning from strain hardening to strain softening in the deviatoric and the volumetric spaces. For the volumetric part, dilatation before the occurrence of strain softening is often negligible.

Based on the knowledge and criteria obtained from the studies of void damage, the following proposals can be made:

(1) If either

$$\sigma_m + \lambda_e\sigma_e = \sigma_{ce} \quad \text{or} \quad e_e = e_{ce} \tag{17}$$

is reached, the σ_e-e_e curve turns to the strain-softening stage and $E_{te}^{(p)}$ becomes negative.

(2) If

$$\sigma_m + \lambda_m\sigma_m = \sigma_{cm} \tag{18}$$

is met, then the σ_m-e_m curve turns downwards with negative $E_{tm}^{(p)}$.

(3) The condition

$$e_m + \lambda e_e = e_c \tag{19}$$

is taken as the failure criterion. It means that after undergoing a certain extent of strain softening the material is so severely damaged that it can no longer withstand any stress.

In Eqs. (17)—(19), λ_e, σ_{ce}, e_{ce}, λ_m, σ_{cm}, λ, e_c are the material constants to be determined together with the values of $E_{te}^{(p)}$ and $E_{tm}^{(p)}$ in the strain-softening stage. They constitute all the parameters needed to characterize ductile damage and failure in material and provide a full description for the symbolical representation in Eq. (16). The three parameters λ_e,λ_m and λ play the role of balancing the influence between deviatoric and volumetric stresses or strains. Since the difference between the total strain and the plastic strain is negligible under the large-strain condition, here and hereafter we delete the index (p) that denotes plasticity.

A full exemplification for determining the parameters listed above has been given in Ref. [12], using the computer-simulation technique together with both macroscopic and microstructural data of tests. As stated previously in the text, microstructural studies do help us understand the damage mechanism in setting up a frame line and selecting the most necessary parameters for constitutive description of materials. But, the internal mechanism is so complex[6] that no solid ground for quantitative estimation of material damage can be provided without making use of the data obtained from tests. That is why we lay emphasis on both the computer simulation and its incorporation with experimental informations[12].

III. Plastic Loading Paths at Bifurcation

As is well known, the bifurcation condition will be reached when the second variation, called Q, of functional π approaches zero, i. e.

$$\delta^2\pi = Q = 0. \tag{20}$$

Although the details of Eq. (20) may differ with the choice of referential coordinates, essentially, an incremental formulation within body volume v can always be expressed as

$$Q = \int_V [\delta(d\sigma_{ij})\delta(d\varepsilon_{ij}) + F(\boldsymbol{\sigma},\delta\boldsymbol{V})]dv. \tag{21}$$

Actual examples of Q with regard to Lagrangian and up-dated Lagrangian formulation can be seen in Refs. [13,14]. The first term in the integral of Eq. (21) is a contraction multiplication of the first variations of stress and strain increments, while the second term is a scalar function F of stress $\boldsymbol{\sigma}$ and the pattern of velocity variation $\delta\boldsymbol{V}$. The actual forms of the stress and strain increments and the function F depend upon the referential coordinates chosen for analysis. However the following procedure will show that the results of proof are irrelevant *of* this difference.

Let a solid be subjected to any kinematically possible disturbance $\delta\boldsymbol{V}$. At the bifurcation point there are several loading paths to be discussed.

(a) Elastic unloading occurs at bifurcation within part of the plastic region developed by pre-bifurcation loading.

(b) All the plastic region remains plastic.

(c) Strain softening appears in the plastic region.

(d) Not only strain softening but also plastic dilatation takes place in the plastic region.

Comparison between paths (a) and (b) has been discussed[5] when the strain is small and no damage occurs, as these are the only two possibilities allowed by the material condition then. However, when damage is developed as strain enlarges, paths (c) and (d) might happen. An example for this statement can be seen in Ref. [1], where shear band is shown to be composed of tiny voids linking up into a line. If we use a mechanically equivalent continuum to substitute this band region, it may then have not only strain-softening effect but also plastic dilatancy.

Employing Eq. (14), we can express the variation of stress increment in the Eq. (21) at bifurcation as

$$\delta(d\sigma_{ij}) = L^*_{ijkl}\delta(d\varepsilon_{kl}) \tag{22a}$$

and

$$L^*_{ijkl} = \frac{E}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ij}\delta_{kl}\frac{\gamma - E/3E^*_{tm}}{1 - 2\nu + E/E^*_{tm}} - \frac{3}{2\sigma_e^2}\frac{E}{E^*_{te}}\frac{S_{ij}S_{kl}}{\frac{2}{3}(1+\nu) + E/E^*_{te}}\right]. \tag{22b}$$

In order to distinguish the tangent moduli at bifurcation from those of the pre-bifurcation, an asterisk $*$ is added to the corresponding symbol.

Substituting Eqs. (22a) and (22b) into Eq. (21) we obtain Q_a, Q_b, Q_c and Q_d corresponding to the four possible loading paths. Then we can prove

$$Q_a \geqslant Q_b \geqslant Q_c \geqslant Q_d. \tag{23}$$

Firstly, let the total body be subdivided into two parts of volume,

$$v = v_s + v_c, \tag{24}$$

in which, v_s is the common elastic/plastic region in comparison, while v_c denotes the part of volume subjected to different loading paths. Then we can write

$$\begin{aligned} &Q_a = Q_s + \Delta Q_a, \ Q_b = Q_s + \Delta Q_b, \\ &Q_c = Q_s + \Delta Q_c, \ Q_d = Q_s + \Delta Q_d, \end{aligned} \tag{25}$$

where

$$Q_s = \int_{v_s} L_{ijkl}\delta(d\varepsilon_{ij})\,\delta(d\varepsilon_{kl})dv_s + \int_v F(\boldsymbol{\sigma}, \delta\boldsymbol{V})dv. \tag{26}$$

Since the loading path is defined to be common in the region v_s, there is then neither softening nor dramatic change of volume. Therefore, the stiffness tensor L_{ijkl} does not need to have the asterisk $*$. It is also regulated that the comparison should be made on the same (although arbitrarily kinematically possible) basis of velocity variation $\delta\boldsymbol{V}$ and the same stress state $\boldsymbol{\sigma}$ until bifurcation occurs. The integration of function F is not affected by the difference in loading cases at bifurcation, and therefore it is irrelevant of the comparison made within the whole volume v concerned. Consequently, Q_s becomes a common part that can be removed from the comparison and we only need to focus our attention on ΔQ_a, ΔQ_b, ΔQ_c and ΔQ_d expressed as

$$\Delta Q_a = \int_{v_c} L^{*(a)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl})dv_c, \tag{27a}$$

$$\Delta Q_b = \int_{v_c} L^{*(b)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl})dv_c, \tag{27b}$$

$$\Delta Q_c = \int_{v_c} L^{*(c)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl})dv_c, \tag{27c}$$

$$\Delta Q_d = \int_{v_c} L^{*(d)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl})dv_c. \tag{27d}$$

According to the four possible loading paths originated from Eq. (22b), we find

(a) $$L^{*(a)}_{ijkl} = \frac{E}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right]; \tag{28a}$$

(b) $$L^{*(b)}_{ijkl} = L^{*(a)}_{ijkl} - \frac{3}{2\sigma_e^2}\frac{S_{ij}S_{kl}}{1+\frac{2}{3}(1+\nu)E_{te}/E}, \quad E > E_{te} \geqslant 0; \tag{28b}$$

(c) $$L^{*(c)}_{ijkl} = L^{*(a)}_{ijkl} - \frac{3}{2\sigma_e^2}\frac{S_{ij}S_{kl}}{1+\frac{2}{3}(1+\nu)E_{te}/E}, \quad -E < E^*_{te} \leqslant 0; \tag{28c}$$

(d) $L^{*(d)}_{ijkl} = L^*_{ijkl}$ (in Eq. (22b)), and E^*_{tm} has a finite value. (28d)

By substituting (28a) and (28b) into (27a) and (27b) respectively, it is easy to see that

$$S_{ij}S_{kl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}) = [S_{ij}\delta(d\varepsilon_{ij})]^2 > 0$$

and

$$1 + \frac{2}{3}(1+\nu)E_{te}/E > 0,$$

such that

$$\begin{aligned} L^{*(b)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}) &= L^{*(a)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}) - (\text{positive quantity}) \\ &\leqslant L^{*(a)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}). \end{aligned} \tag{29}$$

Only when $E_{te} \to \infty$, it means that there is no plastic strain, do both sides of Eq. (29) become identical. Hence, after integration in volume v_c, we come to the conclusion

$$\Delta Q_a \geqslant \Delta Q_b. \tag{30}$$

In the same manner, the comparison between (28b) and (28c) shows

$$1 + \frac{2}{3}(1+\nu)E_{te}/E \geqslant 1 + \frac{2}{3}(1+\nu)E^*_{te}/E.$$

Then, we find

$$L^{*(c)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}) \leqslant L^{*(b)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}). \tag{31}$$

They are equal when $E^*_{te} = E_{te} = 0$. Therefore,

$$\Delta Q_b \geqslant \Delta Q_c. \tag{32}$$

The difference between paths (d) and (c) is caused by the plastic dilatation formed at bifurcation, and then E^*_{tm} becomes a finite quantity. Generally speaking, if the condition $|E/E^*_{tm}| \gg 1$ is true, then

$$\frac{\nu}{1-2\nu} = \frac{\nu - E/3E^*_{tm}}{1-2\nu+E/E^*_{tm}} \to -\frac{1}{3}. \tag{33}$$

The identity holds only when there is no plastic dilatation, that is $E^*_{tm} \to \infty$. Since

$$\delta_{ij}\delta_{kl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}) = [\delta_{ij}\delta(d\varepsilon_{ij})]^2 > 0,$$

and E^*_{te} is the same, we find

$$L^{*(d)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}) \leqslant L^{*(c)}_{ijkl}\delta(d\varepsilon_{ij})\delta(d\varepsilon_{kl}). \tag{34}$$

Thus,

$$\Delta Q_c \geqslant \Delta Q_d, \tag{35}$$

with condition (33) being its prerequisite. This requirement is actually not a restriction, since a very small negative value of E/E^*_{tm} is usually meaningless.

Now we shall prove the comparison relation in Eq. (23). Let us first take Q_a and Q_b as an example, then we can find the lowest critical value in bifurcation within the scope of Q_b associated with loading path (b). This is because there is no possibility of allowing any pattern of velocity variation to yield a situation such that Q_a first turns from a positive of stable state to zero at bifurcation, in a stationary manner associated with a smaller critical value, while Q_b is still in a positive state larger than zero. If that could be true, it would violate the sequence regulated in Eq. (23). Similarly, if path (c) becomes realistic, we can discard Q_b and concentrate our attention on Q_c to find the lowest critical value. When path (d) occurs, the priority should be given to Q_d instead of Q_c.

The "elastic comparison model" or "consistent loading" concept[5] is virtually based on the comparison between Q_a and Q_b and finds its use in treating plastic buckling problem. Although buckling induces large rotations, the strain in materials is still too small to cause any internal damage. Hence, paths (c) and (d) are not likely to occur and there is no need to consider these two cases in the buckling analysis.

On the other hand, recent studies indicate that strain-softening conditions, as prescribed in Eqs. (17) and (18), do occur in materials undergoing large-strain deformations so that geometrical disturbance and dramatic change of material can simultaneously occur. Nowadays, the terminology "material instability" appears frequently in references, but the concept of its loading path should be deemed as different from that of the structural buckling/instability.

Employing the loading-path concept, we are able to explain why the therotical critical strains[4] are far larger than the experimental ones[2]. Furthermore, distributions of velocity variation are shown to be corresponding and related to the sudden change of material behaviour[14], and the occurrence of curve shear bands[15] can be realized in the computer simulation. The difference between curve shear band and straight shear band lies in their material conditions at bifurcation. The material behaviour of the curve shear band not only changes abruptly as that in the straight shear band, but also has its tangent modulus varying continuously all along the tangential direction. These examples demonstrate that, when material conditions permit the local bifurcation of the velocity distribution occurs simultaneously with the sudden change of material behaviour resulting in the diversion of loading paths.

IV. Conclusions

1. Dilatational plastic equation has been derived in a general form. This equation is not confined to the following requirements: (i) the energy dissipated within a cycle of stress/strain must be positive, (ii) elasticity and plasticity are irrelevant of each other, and (iii) the loading surface is in a convex shape. In the case of small strain without damage, the equation satisfies the above requirements and takes the simplified form of the Prandtl-Reuss relation in conventional plasticity. When strain enlarges and material damage develops internally, the parameters involved in the equation are to be determined by microstructural studies, material tests, and

computer simulation.

2. The possible loading paths that may occur at bifurcations are discussed in the light of this equation. It is proved that material without damage follows the "consistent loading" path, and is free from elastic unloading. Once any "softer" response of material suddenly occurs, the velocity bifurcation and abrupt change of material behaviour become correlated with each other.

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