Magnetohydrodynamics equilibrium of a self-confined elliptical plasma ball

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A variational principle is applied to the problem of magnetohydrodynamics (MHD) equilibrium of a self-contained elliptical plasma ball, such as elliptical ball lightning. The principle is appropriate for an approximate solution of partial differential equations with arbitrary boundary shape. The method reduces the partial differential equation to a series of ordinary differential equations and is especially valuable for treating boundaries with nonlinear deformations. The calculations conclude that the pressure distribution and the poloidal current are more uniform in an oblate self-confined plasma ball than that of an elongated plasma ball. The ellipticity of the plasma ball is obviously restricted by its internal pressure, magnetic field, and ambient pressure. Qualitative evidence is presented for the absence of sighting of elongated ball lightning.

I. INTRODUCTION

Ball lightning has puzzled many scientists for a long time. Reliable eyewitness accounts and recent photographs (REF) have added greatly to the supporting evidence. Efforts to produce a convincing model have met with very limited success. One of the main problems is the mechanism permitting the fireball to be stably confined for such a long time without any auxiliary magnetic field; laboratory plasma without magnetic fields are found to have very limited equilibrium lifetimes. Problems associated with the sources of the energy to support the long life fireball and scenarios for formation in the atmosphere are also under investigation by many scientists all around the world.[1-4] Wu and Chen have recently studied the confinement problem and a model for ball lightning was proposed.[5,6] This spherical plasma model addresses the equilibrium and stability properties of ball lightning, however, some sightings of ball lightning are reported to have oblate spheroid boundary shapes instead of perfect spherical shapes. In this paper, we investigate the consequences of boundary alterations to the model discussed by Wu and Chen. A variational method is introduced to study the magnetohydrodynamics (MHD) equilibrium problem of an elliptical plasma ball. The procedure changes the problem of solving a partial differential equation (Grad–Shafranov equation) to the one of solving a series of ordinary differential equations. Finally, the relations between the boundary shapes and the internal structures are computed, and reasons are presented for the infrequent occurrence of elongated self-confined fireball.

II. VARIATIONAL METHOD

Let the governing equation be

$$\hat{L}G(X,Y) = g(X,Y) \tag{1}$$

and the boundary condition

$$\Gamma(X,Y) = 0: \quad G(X,Y) = \nabla^k G(X,Y) = 0, \tag{2}$$

where $\Gamma(X,Y) = 0$ is a boundary shape and $\hat{L}$ is an elliptical differential operator and $g(X,Y)$ a given function. It is well known that if $\Gamma(X,Y)$ has other than a simple form, it is quite difficult to obtain the solution. Using a linear approximation, the boundary shape sometimes can be represented in the form of $\Gamma = \Gamma_0 + \delta\Gamma_1$, where δ has to be a small parameter.[7] Here we wish to study perturbations to the spherical boundary that require more than a linear approximation. A variational method of multiple-functional procedure is found to provide the necessary simplification for an approximate solution.

Details of the variational method are discussed in Ref. 8; the method of multiple-functional can be divided into the following three steps:

(1) The Lagrangian Λ for Eq. (1) must be defined in such a way that the governing equation can be derived from the limitation condition of the following variational equation:

$$\delta\iint \Lambda \, dX\, dY = \delta\int\left[\int \Lambda\left(X,Y,G,\frac{\partial G}{\partial X},\frac{\partial G}{\partial Y}\right)dY\right]dX$$
$$\equiv \delta\int L\, dX = 0. \tag{3}$$

(2) Assume the solution of Eq. (1) can be put in the form of

$$G(X,Y) = \Gamma^{k+1}(X,Y)\sum_{i=1}^{n} f_i(X)e_i(Y), \tag{4}$$

which satisfies boundary condition (2); here, $e_i(Y)$ is a given basis function and $f_i(X)$ is a trial function that remains to be determined by solving the Euler equation. Meanwhile, the definitions of differentiability and integrability for both $e_i(Y)$ and $f_i(X)$ are necessary in the designated area.

(3) Substitution of Eq. (4) into Eq. (3) results in a system of ordinary differential equations:

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$$\frac{d}{dX}\left(\frac{\partial L}{\partial \dot{f}_i}\right)-\frac{\partial L}{\partial f_i}=0, \tag{5}$$

where

$$\dot{f}_i=\frac{df_i(X)}{dX}, \quad i=1,2,\ldots,n.$$

Therefore, taking $n=N$, one should obtain N ordinary differential equations

$$\ddot{f}_i=F_i(X,f_1,f_2,\ldots,f_N,\dot{f}_1,\dot{f}_2,\ldots,\dot{f}_N), \tag{6}$$

where

$$\ddot{f}_i=\frac{d^2 f_i}{dX^2}, \quad i=1,2,\ldots,N$$

in terms of N unknown f_i's, which can be obtained by solving Eq. (6) analytically or numerically. The convergent error is estimated to be about 10^{-N}.[5]

The merits of the method are (a) the method proves to be more powerful when the boundary undergoes a nonlinear deformation; and (b) the method yields higher precision for equivalent effort than conventional methods, such as Ritz, Galerkin, etc.

III. MAGNETOHYDRODYNAMIC EQUILIBRIUM

Consider a plasma ball in a neutral atmosphere carrying a toroidal current density J_ϕ and a poloidal one J_p. Assume that the ball is imbedded in an atmosphere with a constant pressure P_0 and a boundary shape has the elliptical shape

$$\Gamma=R^2/a^2+Z^2/b^2-1$$

expressed in cylindrical coordinates (R,ϕ,Z). The gravitational force can be negligible since it is much less than the Lorentz force. The MHD equilibrium state can be described by the following equations:

$$\nabla P=\mathbf{J}\times\mathbf{B}, \quad \nabla\times\mathbf{B}=\mu\mathbf{J}, \quad \nabla\cdot\mathbf{B}=0 \tag{7}$$

and on Γ,

$$P|_\Gamma=P_0, \quad \mathbf{B}|_\Gamma=0, \tag{8}$$

where μ, P, $\mathbf{J}$, and $\mathbf{B}$ are the magnetic permeability, pressure, plasma current density, and magnetic induction, respectively.

Using the definition of the magnetic flux function

$$\psi=\oint \mathbf{B}\cdot d\,\mathbf{S}$$

and assuming the equilibrium state to be axially symmetric, i.e., $\partial/\partial\phi=0$, one can translate Eq. (7) into the Grad–Shafranov equation in the form of cylindrical coordinates as follows:

$$\hat{L}\psi(R,Z)\equiv\left(\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}+\frac{\partial^2}{\partial Z^2}\right)\psi(R,Z)$$
$$=-2\pi\mu R J_\phi, \tag{9}$$

where

$$J_\phi=2\pi R\frac{dP}{d\psi}+\frac{\mu}{4\pi R}\frac{dI^2}{d\psi}.$$

Both pressure P and poloidal current I are functionals of ψ only.

The corresponding boundary condition can be rewritten as

$$\psi|_\Gamma=0, \quad \nabla\psi|_\Gamma=0, \tag{10}$$

which implies that the ambient pressure P_0 is a constant and the magnetic field $\mathbf{B}$ vanishes on the surface of the fireball.

Assume the pressure and the poloidal current have the forms of

$$P(\psi)=a_0\psi+P_0, \quad I(\psi)=c_0\psi, \tag{11}$$

where a_0 and c_0 are constants. The toroidal current density then becomes

$$J_\phi=2\pi a_0 R+(\mu c_0^2/2\pi R)\psi.$$

Therefore, Eq. (9) can be expressed as

$$\hat{L}\psi=-\alpha R^2-\beta^2\psi, \tag{12}$$

where $\alpha\equiv 4\pi^2\mu a_0$ and $\beta\equiv\mu c_0$.

To solve Eq. (12) conveniently, the following transformation is adopted, i.e.,

$$A(R,Z)=\psi(R,Z)/R. \tag{13}$$

Substitution of Eq. (13) into Eq. (12) results in

$$\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{\partial^2}{\partial Z^2}\right)A(R,Z)$$
$$=-\alpha R+(1-\beta^2R^2)\frac{A(R,Z)}{R^2}, \tag{14}$$

and the boundary conditions

$$A(R,Z)|_\Gamma=0, \quad \nabla A(R,Z)|_\Gamma=0. \tag{15}$$

The Lagrangian appropriate for Eq. (14) is found to be

$$\Lambda=(R/2)[A_R^2+A_Z^2+(1/R^2-\beta^2)A^2-2\alpha RA], \tag{16}$$

where

$$A_R=\frac{\partial A}{\partial R}, \quad A_Z=\frac{\partial A}{\partial Z}.$$

The solution of Eq. (14) can be assumed to be the form of Eq. (4). Let the basis function $e_i(Z)$ be

$$e_i(Z)=Z^{2i-2}$$

and consider $i=1$ and 2 two terms, then,

$$A(R,Z)=\Gamma^2[f_1(R)+f_2(R)Z^2], \tag{17}$$

where

$$\Gamma=R^2/a^2+Z^2/b^2-1.$$

Substitution of Eq. (17) into Eq. (16) yields for the variational of the Lagrangian functional for Eq. (14):

$$\delta\int\int\Lambda\,dZ\,dR\equiv\delta\int L\,dR=0, \tag{18}$$

where

$$L=\frac{RD^{5/2}}{2}\Bigg[\left(\frac{21bR}{a^4}+\frac{3D}{Rb}\right)Rf_1^2+\left(\frac{b^2R^2}{a^4}\right.$$
$$\left.+\frac{3D}{11}\right)D^2b^3f_2^2+bD^2\dot{f}_1^2+\frac{3b^5D^4\dot{f}_2^2}{143}$$
$$-\frac{3}{11a^2}Rb^5D^3f_2\dot{f}_2+\frac{6}{a^4}b^3R^2Df_1f_2-\frac{b^3}{a^2}RD^2\dot{f}_1f_2$$
$$+\frac{2}{11}b^3D^3\dot{f}_1\dot{f}_2-\frac{b^3}{a^2}RD^2\dot{f}_2f_1+bD^2$$

$$\times\left(\frac{1}{R^2}-\beta^2\right)\left(f_1^2+\frac{3}{143}b^4D^2f_2^2+\frac{2}{11}b^2Df_2f_1\right)$$
$$-\alpha bR\left(21f_1+\frac{3}{8}b^2Df_2\right)-\frac{9}{a^2}bRDf_1\dot{f}_1\Bigg]$$

and

$$D=1-\frac{R^2}{a^2},\quad \dot{f}_i=\frac{df_i(R)}{dR}.$$

Both $f_1(R)$ and $f_2(R)$ are determined by the following second ordinary differential equation (i.e., Euler equation):

$$A_1\ddot{f}+A_2\dot{f}+A_3F=G, \tag{19}$$

where

$$A_1=\begin{pmatrix}RD^2 & b^2RD^3/11\\ RD^2 & 3b^2RD^3/13\end{pmatrix},$$
$$A_2=\begin{pmatrix}D(1-10R^2/a^2) & b^2D^2(1-12R^2/a^2)11\\ D(1-12R^2/a^2) & 3b^2D^2(1-14R^2/a^2)/13\end{pmatrix},$$
$$A_3=\begin{pmatrix}A_{311} & A_{312}\\ A_{321} & A_{322}\end{pmatrix},$$
$$A_{311}=\frac{R}{2a^4}(39R^2-18a^2)-\frac{3RD}{b^2}+\left(\beta^2-\frac{1}{R^2}\right)RD^2,$$
$$A_{312}=-\frac{b^2RD}{2a^2}\left(2-\frac{5R^2}{a^2}\right)+\frac{b^2RD^3}{11}\left(\beta^2-\frac{1}{R^2}\right),$$
$$A_{321}=\frac{11R}{2a^2}\left(\frac{5R^2}{a^2}-2\right)+RD^2\left(\beta^2-\frac{1}{R^2}\right),$$
$$A_{322}=\frac{3b^2RD}{2a^2}\left(\frac{17R^2}{a^2}-2\right)-3RD^2$$
$$+\frac{3b^2RD^3}{13}\left(\beta^2-\frac{1}{R^2}\right),$$
$$F=\begin{pmatrix}f_1(R)\\ f_2(R)\end{pmatrix},\quad \dot{f}=\frac{d}{dR}\begin{pmatrix}f_1(R)\\ f_2(R)\end{pmatrix},$$
$$\ddot{f}=\frac{d^2}{dR^2}\begin{pmatrix}f_1(R)\\ f_2(R)\end{pmatrix},\quad G=\begin{pmatrix}-21\alpha R^2/16\\ -33\alpha R^2/16\end{pmatrix}.$$

Taking the radius of the fireball in the R direction a as a dimensional factor, one can make Eq. (19) dimensionless:

$$C_1\ddot{Y}+C_2\dot{Y}+C_3Y=H, \tag{20}$$

where

$$r=R/a,\quad Y_1(r)=f_1(ar),\quad Y_2(r)=f_2(ar),$$
$$\gamma=a/b,\quad \alpha_0=a^3\alpha,\quad \beta_0=a\beta,$$
$$C_1=\begin{pmatrix}r^2D^2 & r^2D^3/11\gamma^2\\ r^2D^2 & 3r^2D^3/13\gamma^2\end{pmatrix},$$
$$C_2=\begin{pmatrix}rD(1-10r^2) & rD^2(1-12r^2)/11\gamma^2\\ rD(1-12r^2) & 3rD^2(1-14r^2)/13\gamma^2\end{pmatrix},$$
$$C_3=\begin{pmatrix}C_{311} & C_{312}\\ C_{321} & C_{322}\end{pmatrix},\quad H=\begin{pmatrix}-21\alpha_0r^3/16\\ -33\alpha_0r^3/16\end{pmatrix},$$
$$C_{311}=(r^2/2)(39r^2-18)-3r^2\gamma^2D+D^2(\beta_0^2r^2-1),$$
$$C_{312}=(r^2/2\gamma^2)D(5r^2-2)+(D^3/11\gamma^2)(\beta_0^2r^2-1),$$
$$C_{321}=(11r^2/2)(5r^2-2)+D^2(\beta_0^2r^2-1),$$
$$C_{322}=(3r^2D/2\gamma^2)(17r^2-2)-3r^2D^2$$
$$+(3D^3/13\gamma^2)(\beta_0^2r^2-1).$$

After the analysis at the singularity $r=1$, the analytical solution of the Eq. (20) can be obtained, in principle, in the form of a power series near the singularity $r=1$

$$Y_1(r)=b_{10}+b_{11}(1-r)+b_{12}(1-r)^2+\cdots$$
$$=\sum_i b_{1i}(1-r)^i \tag{21a}$$
$$Y_2(r)=b_{20}+b_{21}(1-r)+b_{22}(1-r)^2+\cdots$$
$$=\sum_i b_{2i}(1-r)^i. \tag{21b}$$

By means of tedious calculations, comparison of coefficients, the first coefficients are found, such as

$$b_{10}=-0.125\alpha_0,\quad b_{20}=0.023\,054\alpha_0\gamma^4,$$
$$b_{11}=-\alpha_0(0.125+0.028\,743\gamma^2),$$
$$b_{21}=(\alpha_0\gamma^2/7)(0.121\,68+0.468\,57\gamma^2$$
$$+0.076\,014\gamma^4+0.175\,73\beta_0^2),$$
$$b_{12}=-0.022\,09\alpha_0(5.760\,96+0.5614\beta_0^2$$
$$+2.1522\gamma^2+0.206\,43\gamma^4),$$
$$b_{22}=-(\gamma^2/57.705)(\epsilon_1b_{10}+\epsilon_2b_{11}+\epsilon_3b_{12}+\epsilon_4b_{20}$$
$$+\epsilon_5b_{21}-1.912\,08\alpha_0),$$

where

$$\epsilon_1=-16.9484+14.4408\gamma^2-26.4408\beta_0^2,$$
$$\epsilon_2=24.7438-18.051\gamma^2+8.8136\beta_0^2,$$
$$\epsilon_3=69.3412+7.2204\gamma^2,$$
$$\epsilon_4=(4572.19+24\beta_0^2)/13\gamma^2+36,$$
$$\epsilon_5=7955.3/26\gamma^2-12.$$

It seems too difficult to continue the calculation for all other coefficients in such a way, but these first coefficients are enough to provide detailed information on the boundary. In other words, the boundary condition at $r=1$ can be obtained from these coefficients, i.e.,

$$Y_1(1)=b_{10},\quad Y_2(1)=b_{20},$$
$$\dot{Y}_1(1)=-b_{11},\quad \dot{Y}(1)=-b_{21}. \tag{22a}$$

Besides, the symmetric assumption requires

$$Y_1(0)=Y_2(0)=0, \tag{22b}$$

which means $\mathbf{B}\cdot\mathbf{r}=\mathbf{0}$ on Z axis.

An existence theorem for an ordinary differential equation with two boundary conditions requires for compatibility that there be additional parameters in the differential equation. For example, only because there are three parameters α_0, β_0, and γ in Eq. (20), can the existence theorem for the equation associated with the boundary condition Eq. (22) be tenable.

IV. NUMERICAL RESULTS

Meaningful comparisons between different elliptical fireballs require that the volume $V=4\pi a^2b/3$ and the total magnetic flux $\psi_{\max}$ of the plasma balls be the same for various values of $\gamma(=a/b)$.

Using the method of pivot element elimination and adjusting the parameters β_0, α_0, and γ, we find a series of β_0

TABLE I. β_0 and α_0 depend on elongation γ.

$\gamma = a/b$	0.25	0.5	0.75	0.8	1	1.25	1.5
β_0	8.75	7.16	6.11	6.03	5.72	5.46	5.16
α_0	13.4	5.4	1.9	1.8	1	0.77	0.45

and α_0 corresponding to different γ. The numerical results are shown in Table I as well as in Figs. 1 and 2.

The error estimation of β_0 is made by means of comparison between the approximate $\beta_{0,ap}$ of Table I and the exact $\beta_{0,ex}$ in the references.[5] The corresponding error, when $\gamma = 1$, is about

$$|(\beta_{0,ex} - \beta_{0,ap})/\beta_{0,ex}| \sim 10^{-3.}$$

From Fig. 1 and Table I, one finds that the β_0 grows rapidly when the ellipticity ($\gamma = a/b$) becomes smaller, i.e., when the boundary shape approaches an elongate one, the absolute value of the current $|I(\psi)|$ increases greatly. Whereas, the current distribution inside the oblate plasma ball would have a flatter profile than that in the elongated one. It is also seen that the spatial pressure distribution inside the oblate plasma ball has a flatter configuration than that inside the elongated one because α_0 gets smaller when γ becomes greater.

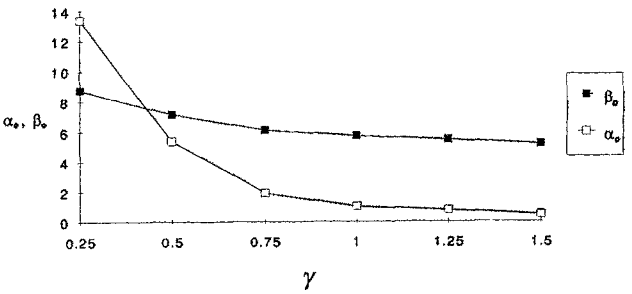

FIG. 1. Current and pressure distribution versus elongation γ.

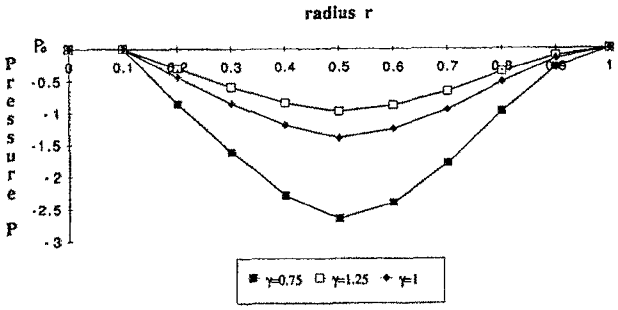

FIG. 2. Pressure distribution versus radius r for different elongations.

In the Fig. 2, it is demonstrated that the more elongated the plasma ball shape becomes the lower pressure there will be inside the ball. The gradient of the pressure gets larger when the plasma ball shape becomes elongated. However, from kinetic theory, a negative pressure would be impossible. Therefore, one can conclude that the ellipticity of an elongated self-confined plasma ball must be restricted by some physics parameters, such as the ambient pressure P_0 and the magnetic field **B** inside the ball. For example, consider a self-confined plasma ball in which $\alpha_0 = 10^3$, $\psi_{max} = 100$ G, and $P_0 = 10$ N/cm^2, then the ellipticity $\gamma = a/b$ would never be less than 0.8. From this example, one may get some knowledge about the existence of self-confined plasma ball.

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