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A NEW FORMULATED METHOD OF A QUASI-COMPATIBLE FINITE ELEMENT AND ITS APPLICATION IN FRACTURE MECHANICS

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Abstract—Based on the local properties of a singular field, the displacement pattern of an isoparametric element is improved and a new formulated method of a quasi-compatible finite element is proposed in this paper. This method can be used to solve various engineering problems containing singular distribution, especially, the singular field existing at the tip of cracks. The singular quasi-compatible element (SQCE) is constructed. The characteristics of the element are analysed from various angles and many examples of calculations are performed. The results show that this method has many significant advantages, by which, the numerical analysis of brittle fracture problems can be solved.

1. INTRODUCTION

THE FINITE element method is a powerful tool for numerical analysis in fracture mechanics. Centred on the simulation of singular fields at crack tips, many researchers have proposed various simulations and treatment methods of the singularity and have constructed various types of singular elements since the end of Sixties. Some important research includes: (1) compatible singular elements constructed by means of special polynomial interpolation or moving the edge-center-node of isoparametric element etc.[1–3]; (2) incompatible singular elements obtained from directly combining the main terms of Williams expansion and the displacement field of normal element[4]; (3) the singular element displacement field formed by combining some terms of the Williams expansion with proper stiff displacement terms, and the method based singular element and transition element, which takes stress intensity factors as generalized node displacements[4, 5]; (4) singular element methods based on various variational principles or sub-region mixed element methods[6–8].

Each of the existing methods has its advantages and disadvantages. There are two main problems: one is that most of them have dense networks, resulting in expensive and difficult application to 3-D fracture problems. The other is that some high accuracy elements, e.g. hybrid/mixed elements, have very high accuracy but the stability of the result is poor and difficult to control, which requires further theoretical investigations.

The isoparametric element method is most widely used in engineering owing to its many advantages. As an improvement on the isoparametric element method, this paper proposes a new and more generally formulated method of quasi-compatible elements, which can contain any singularity and easily meet the requirement of compatibility. This method can be widely used in the numerical analysis of local fields in engineering, which contain various singularities or are difficult to treat with ordinary the isoparametric element method. It is especially useful in the numerical analysis of brittle fracture problems. The establishment of displacement patterns, theoretical derivation, analysis on the characteristics of the element and the comparison between calculated results[9], show that element SQCE has the advantages of saving time, being reliable and highly accurate, and easily programmed.

2. THE ESTABLISHMENT OF THE DISPLACEMENT PATTERN OF SQCE

2.1. *The general form of a displacement pattern*

Let the local displacement field of true field in some singular zone be U_λ, divide it into two parts:

$$U_\lambda = U_\lambda^{①} + \Delta U_\lambda \tag{1}$$

take:

$$U_\lambda^{①} = Nq_\lambda \tag{2}$$

then:

$$\Delta U_\lambda = U_\lambda - Nq_\lambda \tag{3}$$

where N is the shape function matrix of the isoparametric element, q_λ is the local node displacement array formed by the value of U_λ at the nodes of the singular element boundary. Obviously, in eq. (1), $U_\lambda^{①}$ is the part, in U_λ, expressable by isoparametric interpolation. And ΔU_λ is the part not includable in the displacement field of an isoparametric element. The general form of the displacement pattern of element SQCE is derived by combining ΔU_λ in eq. (3) with the displacement field of the isoparametric element as follows:

$$U = Nq + \Delta U_\lambda = Nq + U_\lambda - Nq_\lambda \tag{4}$$

q is the actual node displacement array of the singular element.

2.2. *The application in elastic fracture mechanics*

For elastic fracture problems, of the same type, the displacement, stress and strain fields near crack tips have the same distributed pattern.

Only the strength of stress fields varies with different problems. In eq. (4), let

$$U_\lambda = M(r, \theta)\lambda \tag{5}$$

$$M(r, \theta) = \frac{1}{4G}\sqrt{\frac{r}{2\pi}}\begin{pmatrix}(2k-1)\cos\theta/2 - \cos 3\theta/2 & (2k+3)\sin\theta/2 + \sin 3\theta/2 \\ (2k+1)\sin\theta/2 - \sin 3\theta/2 & -(2k-3)\cos\theta/2 - \cos 3\theta/2\end{pmatrix} \tag{6}$$

$$k = \begin{cases}(3-v)/(1+v), & \text{(plane stress)} \\ 3-4v & \text{(plane strain)}\end{cases} \tag{7}$$

$M(r, \theta)$ is the main term of Williams expansion, λ is the array of stress intensity factor (SIF). Substituting eq. (5) into eq. (4), we have the concrete form of displacement pattern of SQCE as:

$$U = Nq + (M - N\bar{M})\lambda = Nq + M_s\lambda \tag{8}$$

where

$$\bar{M} = [M(r_1, \theta_1), M(r_2, \theta_2), \ldots, M(r_i, \theta_i), \ldots,]^T \tag{9}$$

$M(r_i, \theta_i)$ is the sub-matrix formed by $M(r, \theta)$ at node i.

To practical problems generally, only a few singular elements are needed at crack tip. The rest majority are normal elements. Assume that there are m singular elements, then the whole displacement field of the region with cracks can be expressed as:

$$U = \begin{cases}Nq + M_s\lambda & j \leqslant m \\ Nq & j > m\end{cases}. \tag{10}$$

2.3. *Analysis on convergence*

The convergence criterion of the displacement method requires that the displacement pattern of elements meets the conditions of being perfect and compatible. The displacement pattern of element SQCE, eq. (8), is perfect. As to the compatible condition, from eq. (10), we know that the entire region consists of two kinds of elements, namely, normal elements and singular elements. There are three types of element boundary:

(a) the boundary between singular elements;
(b) the boundary between normal elements;
(c) the boundary between a singular element and a normal one.

For boundaries (a) and (b), the compatible condition is strictly fulfilled. As for (c), the condition is met only at the nodes, for the boundary between nodes, the condition is fulfilled in the following two senses:

(i) when the element dimension h tends to zero, the second term in the displacement pattern (8) disappears and the compatible condition is guaranteed; (ii) with the boundary nodes of singular element increase, the second term, (eq. 8), ΔU_λ tends to zero. So the compatible condition is approximately fulfilled and the boundary between the two kinds of elements is quasi-compatible. The convergence of element SQCE is therefore guaranteed[10–12].

In fact, element SQCE is suggested to be based on the property of singular fields. In other words, the singularity always distributes in a local area. To a certain distance from the singular point, the difference between the displacement patterns of two kinds of elements becomes small. The interface of the two kinds of element is entirely outside of singular zone. So, even the singular element size is large, the compatible condition can be easily fulfilled simply by properly increasing the number of boundary nodes, which is proved from the results given in the following.

3. FINITE ELEMENT FORMULAE

3.1. *Strain field*

Differentiate eq. (10), and we have the strain field of element SQCE:

$$\epsilon = \begin{cases} Bq + B_s\lambda & j \leqslant m \\ Bq & j > m \end{cases} \tag{11}$$

where B is the strain matrix of isoparametric element, B_s is the differential of M_s.

3.2. *Stress field*

From eq. (11) and the relations between linear elastic stress and strain:

$$\sigma = \begin{cases} DBq + DB_s\lambda & j \leqslant m \\ DBq & j > m \end{cases} \tag{12}$$

where D is the elastic matrix.

3.3. *Global stiff matrix and the calculation of stress intensity factors*

Substituting eqs (11) and (12) into the discrete minimum potential energy principle:

$$\pi = \sum_{j=1}^{m} \left\{ q^T\left(\int_{V_e} B^T DB_s \,\mathrm{d}v\right)\lambda + \tfrac{1}{2}\lambda^T\left(\int_{V_e} B_s^T DB_s \,\mathrm{d}v\right)\lambda - \lambda^T\left(\int_{V_e} M_s^T f \,\mathrm{d}v + \int_{S_{2e}} M_s^T \bar{T} \,\mathrm{d}s\right)\right\}$$

$$+ \sum_{j=1}^{n} \left\{ \tfrac{1}{2}q^T\left(\int_{V_e} B^T DB \,\mathrm{d}v\right)q - q^T\left(\int_{V_e} N^T f \,\mathrm{d}v + \int_{S_{2e}} N^T \bar{T} \,\mathrm{d}s\right)\right\}$$

$$= \sum_{j=1}^{m} \{q^T K_{NS}^e \lambda + \tfrac{1}{2}\lambda^T K_s^e \lambda - \lambda^T F_\lambda^e\} + \sum_{j=1}^{n} \{\tfrac{1}{2}q^T K^e q - q^T F^e\} \tag{13}$$

using global matrix symbols to express:

$$\pi = \tfrac{1}{2}Q^T KQ + Q_s^T K_{NS}\lambda + \tfrac{1}{2}K_s\lambda^T\lambda - Q^T F - \lambda^T F_\lambda. \tag{13$'$}$$

To group-crack problems, many cracks exist in the region, the general form of the above eq. is:

$$\pi = \tfrac{1}{2}Q^T KQ + \sum_{i=1}^{R} Q_{si}^T K_{Nsi}\lambda i + \sum_{i=1}^{R} \tfrac{1}{2}\lambda_i^T K_{si}\lambda_i - Q^T F - \sum_{i=1}^{R} \lambda_i^T F_{\lambda i} \tag{14}$$

where λ_i is the column matrix of stress intensity factors at the ith crack tip and R is the total crack tip number in the region studied.

Using the independence between Q and λ_i, and by varying eq. (14) w.r.t. to Q and λ_i respectively, we have:

$$KQ + \sum_{i=1}^{R} K_{Nsi}\lambda_i = F \tag{15}$$

$$K_{Nsi}^T Q + K_{si}\lambda_i = F_i \tag{16}$$

From eq. (16), the array of SIF is derived:

$$\lambda_i = -K_{si}^{-1}(K_{Nsi}^T Q_{si} - F_{\lambda i}). \tag{17}$$

When volume force is not considered and there is no surface force on the crack surface, we have:

$$\lambda_i = -K_{si}^{-1} K_{Nsi}^T Q_{si}. \tag{17'}$$

Substituting it into eq. (15), we have:

$$K'Q = F \tag{18}$$

where

$$K' = K - \sum_{i=1}^{R} K_{Nsi} K_{si}^{-1} K_{Nsi}^T \tag{19}$$

which is the global stiff matrix considered singularity.

4. ANALYSIS ON ELEMENT CHARACTERISTICS

Reference [9] took a 2-D type I crack specimen as an example and analysed the characteristics of element SQCE from various angles such as the node number of element boundary (NNEB), the number of singular elements (NSE), the number of Gauss' integration points (NGIP) and the size of singular elements (SSE) etc. Some main results are listed below.

4.1. *The influence of NNEB on solution*

Standard value: $K_I = 3.058$.

4.1.1. *The results of calculation* (see Table 1).

4.1.2. *Conclusion.* Element SQCE can be combined with isoparametric elements of any type and any number of nodes. To plane problems, all the calculated examples show that satisfactory accuracy can be obtained by combining with isoparametric elements with 8 nodes.

Table 1. Results of calculation

NSE	NNEB	SSE	K_I^* computed value	Percentage difference
2	8	0.8 × 0.8	3.073	0.49%
2	12	0.8 × 0.8	3.069	0.36%

4.2. *The influence of NSE on solution*

4.2.1. *The results of calculation* (see Table 2).

4.2.2. *Conclusion.* The size of element SQCE can be larger because the singular zone at crack tip is very small. A certain distance from the tip, there is no singularity. It is unnecessary to use many singular elements. To plane problems, only two are needed when there is one axis of symmetry and four for no symmetric axes.

Table 2. Results of calculation

NSE	NNEB	SSE	K_I^*	PD†
2	8	0.8 × 0.8	3.0735	0.5%
6	8	0.8 × 0.8	3.071	0.425%

†PD = The percentage differences of results.

4.3. *The influence of NGIP on solution*

4.3.1. *The results of calculation with various Gauss' point numbers* (see Table 3).

4.3.2. *The curve of the results vs Gauss' point numbers* (Fig. 1).

4.3.3. *Conclusion.* At the internal part of singular elements, the gradient of stress and strain is great, and Gauss' integration points should be properly increased. From the results above and Fig. 1, we know that taking the mean value of 4 × 4, 5 × 5 or 4 × 4, 5 × 5, 6 × 6 points the computed results can be more ideal.

Table 3. Calculations with various Gauss' point numbers

NSE	NNEB	NGIP†	K_I^*	PD
2	8	3 × 3	3.47526	13.64%
2	8	4 × 4	3.1384	2.63%
2	8	5 × 5	3.0058	−1.70%
2	8	6 × 6	2.9497	−3.54%

†NGIP = The number of Gauss' integration points.

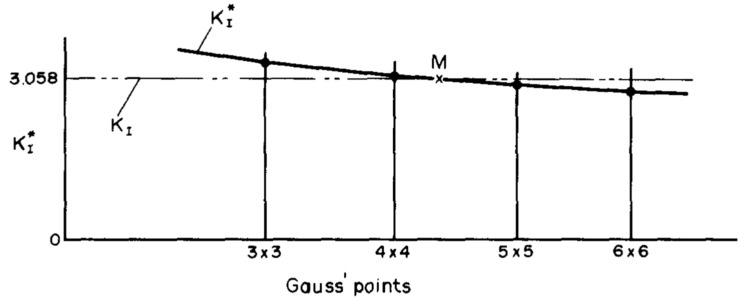

Fig. 1. Results vs Gauss' point numbers.

4.4. *The influence of SSE on solution*

4.4.1. *The computed results for different sizes* (see Table 4).

4.2.2. *The curve of computed results vs singular element sizes* ($h_x \times h_y$) (Fig. 2).

4.3.3. *Conclusion.* Owing to the displacement pattern (8) entirely including the singularity at crack tips, the influence of element size on results is weak. Properly to increase the size of singular elements, the accuracy can be even higher, which agrees with the analysed result on this pattern[9]. This characteristic makes the network dividing of element SQCE, in practical application, need only to fulfill the requirements of the convergence for the isoparametric element itself and the gradient of normal stress, strain fields. This feature is rare compared to other existing singular elements.

Table 4. Computed results for different sizes of singular elements

NTE†	NSE	SSE	NGIP	K_I^*	Average value	PD
24	2	0.3 × 0.3	4 × 4 5 × 5	3.138 3.0058	3.072	0.46%
24	2	0.5 × 0.5	4 × 4 5 × 5	3.1442 3.0034	3.07362	0.51%
24	2	0.55 × 0.55	4 × 4 5 × 5	3.1451 3.00247	3.07375	0.52%
15	2	1.0 × 1.0	4 × 4 5 × 5 6 × 6	3.16994 3.0177 2.94268	3.04344	−0.47%
4	2	2.0 × 2.0	4 × 4 5 × 5 6 × 6	3.2102 3.0371 2.9526	3.0729	0.487%
4	2	3.0 × 2.0	4 × 4 5 × 5 6 × 6	3.34 3.0715 2.9453	3.067	0.28%
2	2	3.0 × 5.0	4 × 4 5 × 5 6 × 6	2.97658 2.85537 2.80885	2.8802	−5.8%

†NTE = Total number of elements (singular and normal).

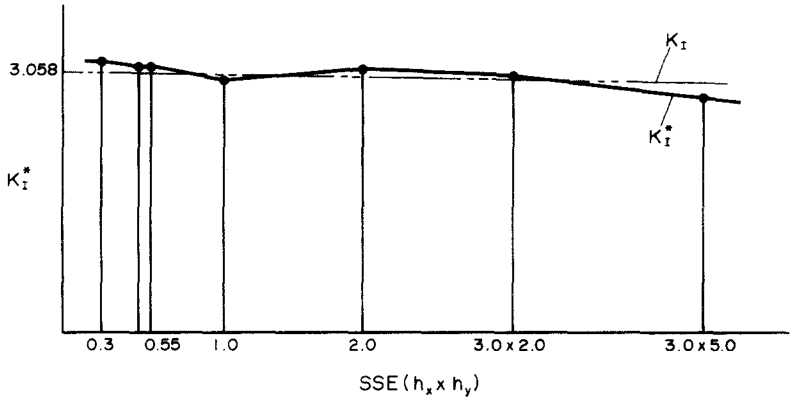

Fig. 2. Computed results vs singular element sizes.

5. VARIOUS CALCULATING EXAMPLES AND COMPUTED RESULTS

Reference [9] calculated and compared about 20 examples. Some typical ones are given in Table 5.

Table 5. Examples of calculated results

Specimens	NTE	NSE	TNN†	Standard value	Computed value	Percentage difference
Central crack	15	2	62	3.058	3.04344	−0.476%
Single edge crack	15	2	62	5.289	5.225	−0.63%
Double edge cracks	12	2	51	3.587	3.5889	0.05%
3-point bending	16	2	65	4.7494	4.7094	0.84%
Internal double cracks	12	4	51	1.9496 1.9267	1.9202 1.93306	0.332% 1.50%
Edge crack	24	2	93	4.5168	4.507	−0.33%
4-point bending	16	2	65	4.2749	4.29407	0.37%
CCPF‡	15	2	62	0.57049	0.569207	0.22%
ECET§	12	2	51	7.67884	7.7409	0.8%
ECPF‖	24	4	97	2.48	2.4805	0.022%
MMT¶	26	4	93	2.28684 1.13221	2.30788 1.141267	0.92% 0.8%

†TNN = Total number of nodes. ‡CCPF = Type I central cracks with point force on crack surface. §ECET = Type I edge crack with eccentric traction. ‖ECPF = Type II edge crack with point force. ¶MMT = mixed model of Types I and II subjected to tension.

6. CONCLUSIONS

From the establishment of the element displacement pattern, theoretical derivation, analysis on the element characteristics and all the calculated results, it is obvious that element SQCE has the following features:

(1) It can solve any type of 2-D brittle fracture problems and directly output stress intensity factors.

(2) With the different selection of singular element size, Gauss' point number etc., the computing results are stable, reliable and highly accurate.

(3) The singular element size is large, its number is small, saving computing time and giving high efficiency.

(4) There are not any new requirements from the singularity introduced into the isoparametric element. The dividing of the singular element network needs only to meet the requirements of the convergence for the isoparametric element itself. Data input and management are almost the same as that for isoparametric elements, and can be easily programmed. It can be added into large scale finite element software without any difficulty, which makes it easy to be applied in engineering.

(5) Can be combined with any other isoparametric element and easily to be extended to three-dimension fracture problems.

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