# A General Theory on Media with Randomly Distributed Inclusions: Part I—The Average Field Behaviors

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*In this paper, a theory is developed to calculate the average strain field in the materials with randomly distributed inclusions. Many previous researches investigating the average field behaviors were based upon Mori and Tanaka's idea. Since they were restricted to studying those materials with uniform distributions of inclusions they did not need detailed statistical information of random microstructures, and could use the volume average to replace the ensemble average. To study more general materials with randomly distributed inclusions, the number density function is introduced in formulating the average field equation in this research. Both uniform and nonuniform distributions of inclusions are taken into account in detail.*

## 1 Introduction

Since the materials with randomly distributed inclusions are widely used in engineering, many studies have been done on these materials. Here, by inclusion, we mean a rather general concept, which may refer to the transformation inclusion, inhomogeneity and defect, etc. In this paper, a theory is developed to calculate the average field value in a general material with randomly distributed inclusions.

Many researches on these materials focus on predicting their thermal, electrical, and mechanical properties. Among these are the self-consistent scheme and the variational principle proposed by Hashin and Shtrikman (1963), etc. Many approaches did not require detailed statistical information of the microstructures, since they assumed that the composites were statistically homogeneous, i.e., the inclusions distributed uniformly in the matrix. Another field of research which has received more and more attention recently is to calculate the average field created by a random distribution of inclusions, which is also essential for predicting the effective properties. A distinguished work has been done by Mori and Tanaka (1973) who calculated the average stress in the matrix created by the transformation inclusions, and then their idea has been used by many researchers, for example, Takao et al. (1982), Benveniste (1987), and Norris (1989). Tandon and Weng (1986a,b) applied the Mori-Tanaka method to calculate the average stress in the constituents of short-fiber composites. All the above studies are based on the assumption that the inclusions are uniformly distributed in an infinite matrix, and they can use the volume average to replace the ensemble average. Since the random microstructures of materials have significant effects on the average field behaviors, it naturally arises how to describe the random microstructures and incorporate the detailed statistical information in the analysis for a general class of materials with randomly distributed inclusions. To describe the random microstructures, many scientists have used the concept of the multipoint correlation function. Unfortunately, it is extremely hard to obtain the correlation function higher than three order from experiment. Therefore, in this research we use a rather different method, named for the Poisson Point Field Model, to describe the random microstructures of the materials. By introducing the number density function, which can be easily obtained from experiment, the average field equations are formulated for a general class of materials with randomly distributed inclusions.

The two cases of inclusion distributions are studied in detail. One is that the ellipsoidal inclusions distribute in the whole space uniformly. In this case we have obtained the same result with Mori and Tanaka (1973) when the inclusions are the aligned transformation inclusions, and the same result with Tandon and Weng (1986a) when the inclusions are the aligned short fibers, but we have a different result from Tandon and Weng (1986b) for randomly-oriented fibers. The other case is that the inclusions distribute in a finite region both uniformly and nonuniformly. In this paper although the higher-order moment equations of the strain field have been obtained, we

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only investigate the average field behaviors. The fluctuation of the strain field will be discussed in the following paper.

## 2 Poisson Point Field Model

The material we consider is an infinite matrix containing a finite region $V$ in which a large number of inclusions, with eigenstrain $\epsilon^*$ are randomly distributed. According to Mura (1982), the perturbation of strain field created by a single inclusion is

$$\Delta\epsilon_{ij}(\mathbf{x}) = \int_{u(r)} K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klmn}\epsilon^*_{mn}d\mathbf{x}', \tag{1}$$

where $u(\mathbf{r})$ is the region occupied by an inclusion whose center is at $\mathbf{r}$, and

$$K_{ijkl}(\mathbf{x}-\mathbf{x}') = -\frac{1}{4}\Big[G_{jk,il}(\mathbf{x}-\mathbf{x}') + G_{ik,jl}(\mathbf{x}-\mathbf{x}') + G_{jl,ik}(\mathbf{x}-\mathbf{x}') + G_{il,jk}(\mathbf{x}-\mathbf{x}')\Big], \tag{2}$$

where $G_{ij}(\mathbf{x}-\mathbf{x}')$ is the Green's function for an infinite homogeneous body with elastic constants $\mathbf{c}$.

By introducing the concept of eigenstrain, any kind of inclusion can be equivalent to a distribution of body force in the form. If there are $N_v$ inclusions in the volume $V$, the total perturbation of strain field is given by

$$\epsilon_{ij}(\mathbf{x}) = \sum_{\beta=1}^{N_v}\int_{u(r_\beta)} K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klmn}\epsilon^*_{mn}d\mathbf{x}', \tag{3}$$

where $u(\mathbf{r}_\beta)$ is the region occupied by the $\beta$th inclusion. If $\Phi$ and $\mathbf{a}$ are denoted as orientation and size of inclusion, respectively, $\epsilon_{ij}(\mathbf{x})$ is determined by $N_v$, $\Phi$, $\mathbf{a}$, $\mathbf{r}_\beta$ ($\beta = 1, \ldots, N_v$) which are all random variables. By random variables we mean that even if the material is of the same kind, they would have different values for different specimens, but they should obey some statistical laws for the same kind of material. The following assumptions, which are called the Poisson Point Field Model, are taken to describe the distribution of inclusion number $N_v$, from which the distribution of inclusion position $\mathbf{r}_\beta$ can be derived. Compared with $N_v$ and $\mathbf{r}_\beta$, inclusion orientation and size are easy to cope with.

*Assumptions*:

(1) In volume $v$, the number of inclusions obeys the Poisson distribution with parameter $\lambda(\mathbf{r})$, i.e., for $n = 0, 1, 2, \ldots$

$$P_r[N_v = n] = (n!)^{-1}\Big[\int_v \lambda(\mathbf{r})d\mathbf{r}\Big]^n \exp\Big[-\int_v \lambda(\mathbf{r})d\mathbf{r}\Big], \tag{4}$$

where $\Pr(N_v = n]$ is the probability of $N_v = n$, and $\lambda(\mathbf{r})$, which is called the number density function, is the ensemble average density of inclusions at the point $\mathbf{r}$.

(2) $\{Nv; v \subset V\}$ has independent increments in regions with a restriction for the intersection of them, i.e., for $v = v_1, v_2, \ldots, v_k$,

$$P_r[N_{v_1} = n_1, N_{v_2} = n_2, \ldots, N_{v_k} = n_k] = \prod_{i=1}^{k} P_r[N_{v_i} = n_i]. \tag{5}$$

According to this model, only one function $\lambda(\mathbf{r})$ is needed to describe the distribution of $N_v$ and $\mathbf{r}_\beta$ ($\beta = 1, \ldots, N_v$), and $\lambda(\mathbf{r})$ has definite physical meaning which can be obtained through experiment measurement. If the inclusions are uniformly distributed in volume $V$, $\lambda(\mathbf{r})$ keeps constant, and is equal to the average number in unit volume. With the changeable function $\lambda(\mathbf{r})$, the model represents a wide variety of materials with inclusions.

By using equations (4) and (5), the probability density function of $\mathbf{r}_\beta$ ($\beta = 1, \ldots, k$) can be derived. According to the characteristic of Poisson distribution, the probability density function of any Poisson point $\mathbf{r}_\beta$ is

$$f(\mathbf{r}_\beta) = \frac{\lambda(\mathbf{r}_\beta)}{\int_v \lambda(\mathbf{r}_\beta)d\mathbf{r}_\beta}, \tag{6}$$

and if there are $k$ inclusions in volume $V$, they should be statistically independent with each other. The joint probability density function $f(\mathbf{r}_1, \mathbf{r}_1, \ldots, \mathbf{r}_k)$ is

$$f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k) = \prod_{\beta=1}^{k}\frac{\lambda(\mathbf{r}_\beta)}{\int_v \lambda(\mathbf{r})d\mathbf{r}}. \tag{7}$$

In order to investigate the statistical properties of random strain field, the characteristic function should be introduced, which is defined as

$$M(\alpha) = <\exp(\mathbf{i}\ \alpha\ \epsilon)>, \tag{8}$$

where $\alpha$ is a constant tensor, $\mathbf{i}$ is the imaginary unit, and the symbol $< >$ means taking the average with respect to all random variables. By substituting equation (3) into equation (8) and using the properties of conditional average, equation (8) becomes

$$M(\alpha) = E^{\mathbf{a}}\Big\{E^{\Phi}\Big\{P_r(N_v=0) + \sum_{k=1}^{\infty} P_r(N_v=k)E^{\mathbf{r}\beta} \times \Big[\exp\Big(\mathbf{i}\sum_{\beta=1}^{k}\alpha\Delta\epsilon\Big)\,|(N_v=k)\Big]\Big\}\Big\}, \tag{9}$$

where $E^{\mathbf{a}}$, $E^{\Phi}$, $E^{\mathbf{r}\beta}$ mean the averages with respect to $\mathbf{a}$, $\Phi$, $\mathbf{r}_\beta$, respectively, and the symbol $|(N_v = k)$ means the condition of $N_v = k$. Using equation (7), we can obtain

$$E^{\mathbf{r}\beta}\Big[\exp\Big(\mathbf{i}\sum_{\beta=1}^{k}\mathbf{a}\ \Delta\epsilon\Big)|(N_v=k)\Big] = \Big[\Big(\int_v \lambda(\mathbf{r})\ d\mathbf{r}\Big)^{-1}\int_v \lambda(\mathbf{r})\exp(\mathbf{i}\ \alpha\ \Delta\epsilon)d\mathbf{r}\Big]^k. \tag{10}$$

By substituting equations (4) and (10) into equation (9), the characteristic function is given by

$$M(\alpha) = E^{\mathbf{a}}\Big\{E^{\Phi}\Big\{\exp\int_v \lambda(\mathbf{r})[\exp(\mathbf{i}\alpha\ \Delta\epsilon) - 1]d\mathbf{r}\Big\}\Big\}. \tag{11}$$

By using the characteristic function, the average equation of the random strain field can be obtained in the form

$$<\epsilon_{ij}> = (-\mathbf{i})\frac{\partial M(\alpha)}{\partial a_{ij}}(a_{ij}=0) = E^{\mathbf{a}}\{E^{\Phi}[\textstyle\int_v \lambda(\mathbf{r})\Delta\epsilon_{ij}d\mathbf{r}]\}. \tag{12}$$

Substitution of equation (1) into equation (12) yields

$$<\epsilon_{ij}> = E^{\mathbf{a}}\Big\{E^{\Phi}\Big[\int_v \lambda(\mathbf{r})\int_{u(r)} K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klmn}\epsilon^*_{mn}d\mathbf{x}'d\mathbf{r}\Big]\Big\}, \tag{13}$$

where $u(\mathbf{r})$ is the region occupied by an inclusion with its center at $\mathbf{r}$. The higher-order moment equations can also be obtained easily; we will discuss them in the following paper. Although we have derived equation (13), which is simple and useful under the assumption of Poisson point field, it can be proved as follows that equation (13) can be used for more general cases, even for the non-Poisson point field.

If we take the average of equation (3) directly, it becomes

$$<\epsilon_{ij}> = <\sum_{\beta=1}^{N_v}\int_{u(r_\beta)} K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klmn}\epsilon^*_{mn}d\mathbf{x}'>. \tag{14}$$

When $N_v$ becomes large, the fluctuation of $N_v$ can be neglected, and we can take the average value $<N_v>$ instead,

$$<N_v> = \int_v \lambda(\mathbf{r})d\mathbf{r}. \tag{15}$$

Thus, equation (14) becomes

$$<\epsilon_{ij}> = <N_v><\int_{u(r)} K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klmn}\epsilon^*_{mn}d\mathbf{x}'>, \tag{16}$$

where the second pair of brackets on the right-hand side means the average with respect to the position, orientation, and size of a single inclusion. Many inclusions distribute randomly in a specimen; their positions can be treated as trial results of a single inclusion distribution. This means that the frequency distribution is known according to a representative specimen. Therefore, according to the frequency distribution the probability density function of the inclusion position $\mathbf{r}$ can be expressed in the form

$$f(\mathbf{r})=\frac{\lambda(\mathbf{r})}{\int_v(\mathbf{r})\,d\mathbf{r}}. \tag{17}$$

By substituting equations (15) and (17) into equation (16) and using the properties of conditional expectation, we find

$$<\epsilon_{ij}>=E^{\Phi}\{E^{\Phi}[\int_v\lambda(\mathbf{r})\int_{u(r)}K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klmn}\epsilon^*_{mn}d\mathbf{x}'d\mathbf{r}]\}, \tag{18}$$

which is the same result as equation (13).

Through the above discussion, we can conclude that if the inclusion positions form Poisson point field, equation (18) is exactly correct, and if the inclusion number in $V$ is quite large, equation (18) is a good approximation for any kind of random distribution of inclusions. For simplicity, we neglect the randomness of inclusion size $\mathbf{a}$ in the following and assume that the eigenstrain $\epsilon^*$ keeps constant. Then equation (18) becomes

$$<\epsilon_{ij}>=E^{\Phi}[C_{klmn}\epsilon^*_{mn}\int_v\lambda(\mathbf{r})\int_{u(r)}K_{ijkl}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'d\mathbf{r}]. \tag{19}$$

## 3 A Uniform Distribution of Inclusions in an Infinite Matrix

In this part of the article, the spheroidal inclusions are considered to distribute uniformly in an infinite matrix, and the average strain fields both in the inclusions and in the matrix are obtained, respectively. Two kinds of inclusions are considered. One is the transformation inclusion, which has the same elastic constants with the matrix has but acquire uniform transformation strain, and the other is the inhomogeneous inclusion, which has the different elastic constants with the matrix. Some previous results have been obtained again by using the method developed in this paper.

**3.1 The Average Strain Within an Inclusion.** To calculate the average strain at $\mathbf{x}$ within an inclusion, the essential requirement is that there is an inclusion at $\mathbf{x}$. Then, due to the nonintersecting property of inclusions, the density function $\lambda(\mathbf{r})$ becomes

$$\lambda(\mathbf{r})=\begin{cases}\delta(\mathbf{r}-\mathbf{x}) & \mathbf{r}\in U(\mathbf{x})\\ n & \text{others},\end{cases} \tag{20}$$

where $U(\mathbf{x})$ is the spheroidal region whose shape is similar to $u(\mathbf{x})$ and whose center is at $\mathbf{x}$, $n$ is the average number of inclusions in unit volume, and for any continuous function $\delta(\mathbf{r}-\mathbf{x})$ is defined by

$$\int_v\delta(\mathbf{r}-\mathbf{x})\Phi(\mathbf{r})d\mathbf{r}=\Phi(\mathbf{x}). \tag{21}$$

Substitution of equation (20) into equation (19) yields

$$<\epsilon^I(\mathbf{x})>=E^{\Phi}\{[\int_{U(\mathbf{x})}\delta(\mathbf{r}-\mathbf{x})\int_{u(\mathbf{r})}\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'d\mathbf{r}+n\int_{v-U(\mathbf{x})}\times\int_{u(\mathbf{r})}\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'d\mathbf{r}]:\mathbf{C}:\epsilon^*\}, \tag{22}$$

where $<\epsilon^I(\mathbf{x})>$ is the average strain field within an inclusion. Since $U(\mathbf{x})$ and $u(\mathbf{r})$ are spheroidal inclusions, the first integral on the right-hand side can be written in the form (Eshelby, 1957)

$$\int_{U(\mathbf{x})}\delta(\mathbf{r}-\mathbf{x})\int_{u(\mathbf{r})}\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'d\mathbf{r}=\int_{u(\mathbf{x})}\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'=\mathbf{S}:\mathbf{B}, \tag{23}$$

where $\mathbf{S}$ is the Eshelby's tensor and $\mathbf{B}$ is the elastic compliance tensor of matrix. The second integral becomes

$$\int_{v-U(\mathbf{x})}\int_{u(r)}\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'d\mathbf{r}=\int_v\int_{u_0}\mathbf{K}(\mathbf{x}-\mathbf{r}-\mathbf{y})d\mathbf{y}d\mathbf{r}-\int_{U(\mathbf{x})}\int_{u_0}\mathbf{K}(\mathbf{x}-\mathbf{r}-\mathbf{y})d\mathbf{y}d\mathbf{r}, \tag{24}$$

where $\mathbf{x}'=\mathbf{r}+\mathbf{y}$, and $u_0$ is the region occupied by an inclusion with its center at zero. According to Kunin (1983), when $V$ tends to infinity, it becomes

$$\int_v\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'=\mathbf{B}. \tag{25}$$

By changing the order of the integrals in equation (24), it becomes

$$\int_{v-U(\mathbf{x})}\int_{u(\mathbf{r})}K(\mathbf{x}-\mathbf{x}')d\mathbf{x}'d\mathbf{r}=v_0(\mathbf{I}-\mathbf{S}):\mathbf{B}, \tag{26}$$

where $v_0$ is the average volume of an inclusion, and $\mathbf{I}$ is the identity tensor. Substitution of equations (23) and (26) into equation (22) leads to

$$<\epsilon^I>=E^{\Phi}[C_f\epsilon^*+(1-C_f)\mathbf{S}:\epsilon^*], \tag{27}$$

where $C_f\,(=nv_0)$ is the volume fraction of inclusions.

**3.2 The Average Strain in the Matrix.** In such a case, the essential requirement is that there is no inclusion at $\mathbf{x}$. The number density function takes the form

$$\lambda(\mathbf{r})=\begin{cases}0 & \mathbf{r}\epsilon u(\mathbf{x})\\ n & \text{others}.\end{cases} \tag{28}$$

By substituting equation (28) into equation (19) and following the same procedure as above, one can obtain

$$<\epsilon^M(\mathbf{x})>=E^{\Phi}[C_f(\mathbf{I}-\mathbf{S}):\epsilon^*], \tag{29}$$

where $<\epsilon^M(\mathbf{x})>$ is the average strain in the matrix. Combining equation (27) with equation (29) gives

$$C_f<\epsilon^I(\mathbf{x})>+(1-C_f)<\epsilon^M(\mathbf{x})>=C_fE^{\Phi}(\epsilon^*), \tag{30}$$

which could be obtained through the volume average (see Mura, 1982). But in this approach, $<\epsilon^I>$ and $<\epsilon^M>$ are obtained through the ensemble average process. Therefore, this coincidence means the ensemble average of the strain field equals to the volume averages for uniform distributions of inclusions.

*3.2.1 Transformation Inclusions.* In such a case, it is considered that the elastic constants are uniform throughout the whole space and every inclusion acquires uniform transformation strain $\epsilon^*$. If all the inclusions are unidirectionally distributed, equation (29) becomes

$$<\epsilon^M_{ij}(\mathbf{x})>=C_f(\mathbf{I}_{ijkl}-\mathbf{S}^L_{ijkl})\,\epsilon^*_{kl}, \tag{31}$$

which is the same result with Mori and Tanaka's result (1973), and $\mathbf{S}^L_{ijkl}$ are the components of the Eshelby's tensor in the ellipsoidal local coordinate system, which are referred to Tandon et al. (1986a) for spheroidal inclusions. For random-oriented spheroidal inclusions, the eigenstrain is axially symmetric with respect to the symmetric axis of inclusion and ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) is denoted as the fixed coordinate system. The local coordinate system can be established by ($\mathbf{e}^L_1$, $\mathbf{e}^L_2$, $\mathbf{e}^L_3$), where $\mathbf{e}^L_1$ is the symmetric axis and $\mathbf{e}^L_3$ lies in the ($\mathbf{e}_1$, $\mathbf{e}_2$) plane with no loss in generality. Thus, the strain components in the matrix, with respect to the fixed coordinate system, are given by

$$\epsilon^M_{\alpha\beta}(\mathbf{x})=C_fT_{\alpha i}T_{\beta j}(\epsilon^{*L}_{ij}-\mathbf{S}^L_{ijkl}\,\epsilon^{*L}_{kl}), \tag{32}$$

where $\epsilon^{*L}$ is the eigenstrain in the local coordinate system, and

$$T_{ai}=\begin{bmatrix}\sin\theta\cos\Phi & -\cos\theta\cos\Phi & \sin\Phi\\ \sin\theta\sin\Phi & -\cos\theta\sin\Phi & -\cos\Phi\\ \cos\theta & \sin\theta & 0\end{bmatrix}. \tag{33}$$

The average strain in the matrix is given by

$$<\epsilon^M_{\alpha\beta}>=\frac{1}{2\pi}\int_0^{\pi}\int_0^{\pi}\epsilon^M_{\alpha\beta}\sin\theta\,d\theta\,d\Phi=\frac{1}{3}\epsilon_{kk}\delta_{\alpha\beta}, \tag{34}$$

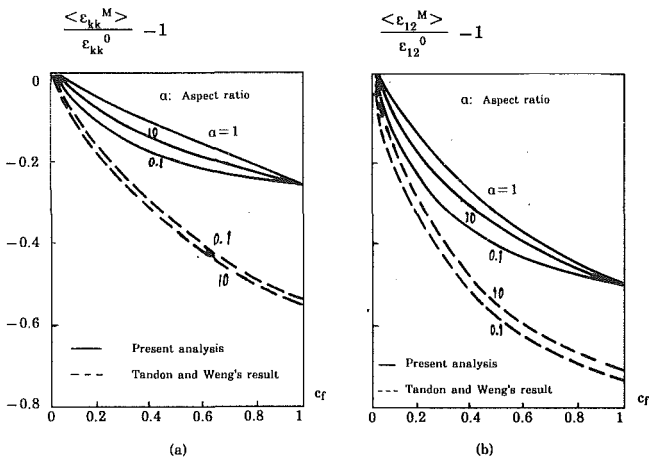


**Fig. 1 The variation of (*a*) hydrostatic and (*b*) deviatoric perturbed stress components**

where

$$\epsilon_{kk} = C_f[(1 - S^L_{1111} - 2S^L_{2211})\epsilon^{*L}_{11} + 2(1 + S^L_{1122} - S^L_{2222} - S^L_{2233})\epsilon^{*L}_{22}]. \quad (35)$$

*3.2.2 Inhomogeneous Inclusions.* In this case, the elastic constants of inclusions are different from those of the matrix. The inclusions and matrix are assumed to be linearly isotropic and perfectly bonded with the elastic moduli $C^1$ and $C^0$, respectively. A surface traction now prescribed on the boundary is a uniform stress $\sigma^0$. By using Eshelby's equivalent inclusion principle (1957), the relation between the eigenstrain $\epsilon^*$ and the internal strain $\epsilon^I$ can be established by

$$\mathbf{C}^1:\epsilon^I = \mathbf{C}^0:(\epsilon^I - \epsilon^*). \quad (36)$$

From equation (36) one obtains

$$\epsilon^* = -\mathbf{B}^0:(\mathbf{C}^1 - \mathbf{C}^0):\epsilon^I, \quad (37)$$

where $\mathbf{B}^0 = (\mathbf{C}^0)^{-1}$. To obtain the average strain in the constituents of composite with randomly-oriented inclusions, the conditional average strain $\mathbf{E}(\epsilon|\Phi)$ should first be obtained for a given orientation $\Phi$. Combining equations (29), (30), and (37) gives

$$E(\epsilon^M|\Phi) = \epsilon^0 + C_f(\mathbf{I} - \mathbf{S}):[(\mathbf{C}^1 - \mathbf{C}^0)^{-1}:\mathbf{C}^0 - \mathbf{S}]^{-1}:E(\epsilon^M|\Phi), \quad (38)$$

where $\epsilon^0 = \mathbf{B}^0:\sigma^0$. For aligned inclusions $\Phi$ can be taken as zero and the components of $\mathbf{S}$ are those in the local coordinate system with no loss in generality. Equation (38) gives the same result as Tandon and Weng (1986a). But for randomly-oriented inclusions, their result is given by

$$<\epsilon^M> = \epsilon^0 + C_f E^\Phi\{(\mathbf{I} - \mathbf{S}):[(\mathbf{C}^1 - \mathbf{C}^0)^{-1}:\mathbf{C}^0 - \mathbf{S}]^{-1}\}:<\epsilon^M>, \quad (39)$$

where

$$<\epsilon^M> = E^\Phi[(\epsilon^M|\Phi)], \quad (39a)$$

and the assumption is contained in equation (39) that $\mathrm{E}(\epsilon^M|\Phi)$ is statistically independent of $\Phi$. Through equation (38), the result is given by

$$<\epsilon^M> = E^\Phi\{\{\mathbf{1} - C_f(\mathbf{I} - \mathbf{S}):[(\mathbf{c}^1 - \mathbf{c}^0)^{-1}:\mathbf{c}^0 - \mathbf{S}]^{-1}\}^{-1}\}:\epsilon^0. \quad (40)$$

In order to shed some light on the difference between equations (39) and (40), we present some numerical results for the average strain in the matrix. The material constants used in numerical computations are those of glass/epoxy, i.e.,

$$k_0 = 3.07 \text{ GPa}, \quad \mu_0 = 1.02 \text{ GPa}$$
$$k_1 = 40.2 \text{ GPa}, \quad \mu_1 = 30.2 \text{ GPa} \quad (41)$$

where $k_0$, $k_1$ are bulk moduli of matrix and inclusions, and $\mu_0$, $\mu_1$ are shear moduli of matrix and inclusions. The variations of average perturbed strain ($<\epsilon^M> - \epsilon^0$) in the matrix are shown in Fig. 1, where $\alpha$ is the aspect ratio of spheroidal inclusion.

## 4 A Random Distribution of Inclusions in a Finite Region

The material considered in this part is an infinite matrix containing an ellipsoidal region in which a large number of inclusions distribute randomly. The aim is to investigate the average strain field both inside the ellipsoidal region and outside it.

**4.1 The Inclusions Distribute Uniformly in an Ellipsoidal Region V.** By uniform distribution we mean the number density function $\lambda(\mathbf{r})$ keeps constant inside the region $V$, i.e.,

$$\lambda(\mathbf{r}) = n \text{ (constant)}. \quad (42)$$

Substitution of equation (42) into equation (19) yields

$$<\epsilon_{ij}> = nE^\Phi[C_{klmn}\epsilon^*_{mn}\int_V\int_{u(\mathbf{r})} K_{ijkl}(\mathbf{x} - \mathbf{x}')d\mathbf{x}'\,d\mathbf{r}]. \quad (43)$$

By taking $\mathbf{x}' = \mathbf{r} + \mathbf{y}$ and changing the order of integration in equation (43), it becomes

$$<\epsilon_{ij}> = nE^\Phi[C_{klmn}\epsilon^*_{mn}\int_{u_0}\int_V K_{ijkl}(\mathbf{x} - \mathbf{y} - \mathbf{r})d\mathbf{r}d\mathbf{y}]. \quad (44)$$

Since $V$ is an ellipsoidal region and if $\mathbf{x} - \mathbf{y}$ is inside the region $V$, the average strain inside the region $V$ can be obtained in the form

$$<\epsilon^I_{ij}> = C_f S_{ijkl}(V)E^\Phi(\epsilon^*_{kl}), \quad (45)$$

where $S_{ijkl}(V)$ is the Eshelby's tensor in connection with the ellipsoidal region $V$. Equation (45) is true regardless of the shape of inclusion, and it leads to the conclusion that to calculate the average strain field inside an ellipsoidal region containing a uniform distribution of inclusions, one can treat the ellipsoidal region as a big inclusion with the eigenstrain $\mathbf{C}_1\mathbf{E}^\Phi(\epsilon^*)$.

From the foregoing discussion, the question arises naturally: To calculate the average strain field outside the ellipsoidal region, could one treat the ellipsoidal region as a big inclusion with the eigenstrain $\mathbf{C}_f\mathbf{E}^\Phi(\epsilon^*)$? Now, we try to answer this question. For simplicity, a spherical region containing spherical inclusions is considered in detail. From equation (44), one obtains

$$<\epsilon^M_{ij}(\mathbf{x})> = n\epsilon^*_{kl}\int_{u(x)} D_{ijkl}(\mathbf{z})d\mathbf{z}, \quad (46)$$

where $\mathbf{z} = \mathbf{x} - \mathbf{y}$, and $\mathbf{D}$ is expressed in terms of potentials $\Phi$ and $\Psi$ (Eshelby, 1959) as

$$D_{ijkl}(\mathbf{x}) = \frac{1}{8\pi(1-\gamma)}\{\Psi_{,klij} - 2\gamma\delta_{kl}\Phi_{,ij} - (1-\gamma)[\Phi_{,kj}\delta_{il} + \Phi_{,ki}\delta_{jl} + \Phi_{,lj}\delta_{ik} + \Phi_{,li}\delta_{jk}]\}, \quad (47)$$

with

$$\Phi = \frac{4\pi a^3}{3}\times\frac{1}{|\mathbf{x}|} \text{ and } \Psi = \frac{4\pi a^3}{3}|\mathbf{x}| + \frac{4\pi a^5}{15}\times\frac{1}{|\mathbf{x}|}, \quad (48)$$

where $a$ is the radius of the region $V$. The integral in equation (46) can be done directly by integration of expression (47) (Willis and Acton, 1976). Rodin and Hwang (1989) obtained the same result as a consequence of a more general formula. By using their result, equation (46) becomes

$$<\epsilon^M_{ij}(x> = C_f\left[D_{ijkl}(\mathbf{x})\epsilon^*_{kl} + \frac{3b^2a^3}{10(1-\gamma)|\mathbf{x}|^5}f_{ijkl}(\mathbf{n})\epsilon^*_{kl}\right], \quad (49)$$

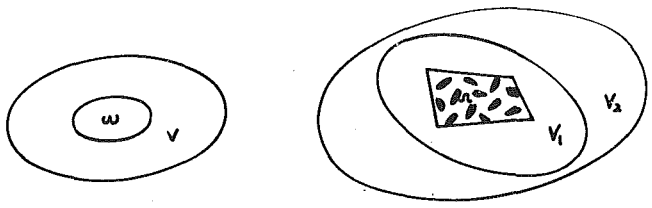


**Fig. 2 Schematic of the integral domain between two ellipsoidal regions**

where $b$ is the radius of inclusion, $n_i = x_i / |\mathbf{x}|$ and

$$f_{ijkl}(\mathbf{n}) = 5n_in_jn_kn_l - 5(\delta_{ij}n_kn_l + \delta_{ik}n_jn_l + \delta_{jk}n_in_l - \delta_{il}n_jn_k - \delta_{kl}n_in_j - \delta_{jl}n_in_k) + \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}. \quad (50)$$

Equation (49) leads to an interesting conclusion: If the second term in equation (49) is neglected, equation (49) is the just result obtained by treating $V$ as a big inclusion with the eigenstrain $\mathbf{C}_f\epsilon^*$. However, the second term in equation (49) depends on $a^2b^3/|\mathbf{x}|^5$ only when $|\mathbf{x}|$ becomes rather large, and it can be approximately neglected. That means that in a general case, one cannot treat the region as a big inclusion with the eigenstrain $C_f\ \epsilon^*$ in calculating the average strain field at $\mathbf{x}$ outside the region containing inclusions, but if $\mathbf{x}$ is far way from the region, the region can be replaced by a big inclusion approximately.

Tanaka and Mori (1972) have calculated the volume integral of the elastic field inside an ellipsoidal region around an ellipsoidal inclusion (Fig. 2(*a*)) and obtained

$$\int_{v-w}\epsilon d\mathbf{x} = 0. \quad (51)$$

Now, let us consider the volume integral of the elastic field around a region containing a random distribution of inclusions (Fig. 2(*b*)), i.e.,

$$\int_{V_2-V_1} <\epsilon(\mathbf{x})> d\mathbf{x} = \int_{V_2} <\epsilon(\mathbf{x})> d\mathbf{x} - \int_{V_1} <\epsilon(\mathbf{x})> d\mathbf{x}, \quad (52)$$

where $V_1$, $V_2$ are ellipsoidal regions. Substitution of equation (19) into equation (52) gives

$$\int_{V_2-V_1} <\epsilon(\mathbf{x})> d\mathbf{x} = E^{\Phi}\{[\int_{V_2} d\mathbf{x}\int_{\Omega}\lambda(\mathbf{r})d\mathbf{r}\int_{u(\mathbf{r})}\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}' - \int_{V_1} d\mathbf{x}\int_{\Omega}\lambda(\mathbf{r})d\mathbf{r}\int_{u(\mathbf{r})}\mathbf{K}(\mathbf{x}-\mathbf{x}')d\mathbf{x}']:C:\epsilon^*\}. \quad (53)$$

If we let $\mathbf{x}' = \mathbf{y} + \mathbf{r}$ and change the order of integration in equation (53), we find

$$\int_{V_2-V_1} <\epsilon(\mathbf{x})> d\mathbf{x} = V_F E^{\Phi}\{[\mathbf{S}(V_2) - \mathbf{S}(V_1)]:\epsilon^*\}, \quad (54)$$

where $V_F$ is the average total volume occupied by inclusions in $\Omega$, i.e.,

$$V_F = v_0\int_{\Omega}\lambda(\mathbf{r})d\mathbf{r}, \quad (55)$$

and $\mathbf{S}(V_2)$ and $\mathbf{S}(V_1)$ are Eshelby's tensors in connection with $V_2$ and $V_1$, respectively. Equation (54) leads to the interesting conclusions: If $V_1$, $V_2$ are ellipsoidal regions, the volume integral of $<\epsilon>$ over $V_2 - V_1$ is proportional to the average volume occupied by all inclusions. It is independent of the absolute position and size of $V_1$ and $V_2$ as long as $V_2 \supset V_1$, independent of the position distribution of inclusions, and independent of the shape of $\Omega$. It depends only on the shape of $V_1$ and $V_2$ and it vanishes when $V_1$ and $V_2$ are similar in shape and have the same orientation. The above properties of the integration in the domain are extensions of results obtained by Tanaka and Mori (1972) in studying the single inclusion problem.

**4.2 Nonuniform Distribution of Inclusions.** In this part, we consider that inclusions distribute in an ellipsoidal region $\Omega$. The number density function $\lambda(\mathbf{r})$ is considered as an arbitrary smooth function and can be expanded in the Maclauren series in the form

$$\lambda(\mathbf{r}) = \sum_{N=0}^{\infty}\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{1}{\alpha!\beta!\gamma!}\frac{\partial^N f(0)}{\partial r_1^{\alpha}\partial r_2^{\beta}\partial r_3^{\gamma}}r_1^{\alpha}r_2^{\beta}r_3^{\gamma}, \quad (56)$$

where $\mathbf{r} = \mathbf{r}_1\mathbf{i} + \mathbf{r}_2\mathbf{j} + \mathbf{r}_3\mathbf{k}$, and the summation convention in this formula implies that $\alpha + \beta + \gamma = N$.

*4.2.1 The Average Strain Field Outside the Ellipsoidal Region* $\Omega$. Substitution of equation (56) into equation (19) yields

$$<\epsilon_{ij}(\mathbf{x})> = \sum_{N=0}^{\infty}\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{1}{\alpha!\beta!\gamma!}\frac{\partial^N f(0)}{\partial r_1^{\alpha}\partial r_2^{\beta}\partial r_3^{\gamma}}E^{\Phi}[C_{klmn}\epsilon^*_{mn} \times \int_{\Omega}r_1^{\alpha}r_2^{\beta}r_3^{\gamma}\,d\mathbf{r}\int_{u(\mathbf{r})}K_{ijkl}(\mathbf{x}-\mathbf{x}')d\mathbf{x}']. \quad (57)$$

If the inclusion size is rather small, one can obtain approximately

$$\int_{u(\mathbf{r})}K_{ijkl}(\mathbf{x}-\mathbf{x}')d\mathbf{x}' = v_0K_{ijkl}(\mathbf{x}-\mathbf{r}), \quad (58)$$

where $v_0$ is the volume of an inclusion. By substituting equation (58) into equation (57), it follows

$$<\epsilon_{ij}(\mathbf{x})> = \sum_{N=0}^{\infty}\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{1}{\alpha!\beta!\gamma!}\frac{\partial^N F(0)}{\partial r_1^{\alpha}\partial r_2^{\beta}\partial r_3^{\gamma}}E^{\Phi}\,[D_{ijmn}(\mathbf{x})\epsilon^*_{mn}], \quad (59)$$

where

$$D_{ijmn}(\mathbf{x}) = \frac{1}{8\pi(1-\gamma)}[\Psi_{,mnij} - 2\gamma\delta_{mn}\Phi_{,ij} - (1-\gamma)(\Phi_{,mj}\delta_{ni} + \Phi_{,mi}\delta_{jn} + \Phi_{,nj}\delta_{im} + \Phi_{,ni}\delta_{jm})], \quad (60)$$

with

$$\Phi(\mathbf{x}) = \int_{\Omega}r_1^{\alpha}r_2^{\beta}r_3^{\gamma}\,|\mathbf{x}-\mathbf{r}|\,d\mathbf{r},\ \Psi(\mathbf{x}) = \int_{\Omega}\frac{r_1^{\alpha}r_2^{\beta}r_3^{\gamma}}{|\mathbf{x}-\mathbf{r}|}\,d\mathbf{r}. \quad (61)$$

According to Ferrers (1877) and Dyson (1981) (see Mura, 1982, p. 82), the above integrals can be expressed in terms of the elliptic integrals. After some mathematical calculation, it is obtained that

$$\Psi(\mathbf{x}) = I(\alpha,\beta,\gamma) = a_1^{\alpha}\alpha_2^{\beta}a_3^{\gamma}\left\{\sum_{n=0}^{\alpha/2}\frac{\alpha!x_1^{\alpha-2n}x_2^{\beta}x_3^{\gamma}}{2^{2n}n!(n+1)!(\alpha-2n+2)!a_1^2} \times\int_c^{\infty}\frac{s^nU^{n+1}(s)(a_1^2+s)}{(a_1^2+s)^{\alpha}(a_2^2+s)^{\beta}(a_3^2+s)^{\gamma}\Delta(s)}\,ds + \sum_{n=0}^{\beta/2}\frac{\beta!x_1^{\alpha}x_2^{\beta-2n}x_3^{\gamma}}{2^{2n}n!(n+1)!(\beta-2n+2)!a_2^2}\int_c^{\infty} \times\frac{s^nU^{n+1}(s)(a_2^2+s)}{(a_1^2+s)^{\alpha}(a_2^2+s)^{\beta}(a_3^2+s)^{\gamma}\Delta(s)} + \sum_{n=0}^{\gamma/2}\frac{\gamma!x_1^{\alpha}x_2^{\beta}x_3^{\gamma-2n}}{2^{2n}n!(n+1)!(\gamma-2n+2)!a_3^2} \times\int_c^{\infty}\frac{s^nU^{n+1}(s)(a_3^2+s)}{(a_1^2+s)^{\alpha}(a_2^2+s)^{\beta}(a_3^2+s)^{\gamma}\Delta(s)}\,ds\right\}, \quad (62)$$

and

$$\Phi(\mathbf{x}) = x_ix_jI(\alpha,\beta,\gamma) - 2[x_1I(\alpha+1,\ \beta,\ \gamma) + x_2I(\alpha,\ \beta+1,\ \gamma) + x_3I(\alpha,\ \beta,\ \gamma+1)] + I(\alpha+2,\ \beta,\ \gamma) + I(\alpha,\ \beta+2,\gamma) + I(\alpha,\ \beta\gamma+2), \quad (63)$$

where

$$U(s) = 1 - \left(\frac{x_1^2}{a_1^2+s} + \frac{x_2^2}{a_2^2+s} + \frac{x_3^2}{a_3^2+s}\right), \quad \Delta(s) = \sqrt{(a_1^2+s)(a_2^2+s)(a_3^2+s)}, \tag{64}$$

and $c$ is the largest positive root of $U(c) = 0$.

*4.2.2 The Average Strain Field Inside the Ellipsoidal Region* $\Omega$. If we let $\mathbf{x}' = \mathbf{y} + \mathbf{r}$ and change the order of integration in equation (19), it becomes

$$<\epsilon_{ij}(\mathbf{x})> = E^{\Phi}[C_{klmn}\epsilon^*_{mn} \int_{u_0} d\mathbf{y} \int_{\Omega} \lambda(\mathbf{r}) K_{ijkl}(\mathbf{x}-\mathbf{y}-\mathbf{r}) d\mathbf{r}]. \tag{65}$$

If it is assumed that $\lambda(\mathbf{r})$ is a polynomial of $\mathbf{r}$ with degree $n$. According to Eshelby (1961), the first-order integration will give a polynomial function of $\mathbf{x} - \mathbf{y}$ with degree $n$, since $\mathbf{x} - \mathbf{y}$ is inside the ellipsoidal region $\Omega$, and $u_0$ is also an ellipsoidal inclusion. The average strain field $< \epsilon_{ij}(\mathbf{x}) >$ inside the ellipsoidal region $\Omega$ can be proved to be a polynomial function of $\mathbf{x}$ with degree $n$. This conclusion is similar to that obtained by Eshelby (1961) in calculating the internal strain field for a single inclusion with a polynomial eigenstrain.

## 5 Conclusions and Discussions

Our major purpose is to seek an efficient model for general materials with randomly distributed inclusions and to incorporate detailed statistical information of the random microstructures in deriving the average strain field in such materials. The main results are summarized as follows.

1 By introducing the number density function of inclusions, the present analysis can easily evaluate the average strain field in more complicated materials, such as those with nonuniform distribution of inclusions.

2 One must be cautious in taking the orientation average for randomly oriented inclusions. The suggestion of the author is that one should first evaluate the aligned inclusion problem, then the average value can be obtained by taking orientation average, which is same with Christensen (1979).

3 When the inclusions distribute randomly in a finite ellipsoidal region, we have obtained the average strain fields inside the region and outside the region. The properties of the integration of the average strain in the domain between two ellipsoidal regions around an inclusion concentrated region are similar with those obtained by Tanaka and Mori (1972) for a single inclusion problem.

Although the theory can be extended to evaluate the polynomial eigenstrain problem, one must bear in mind that the present analysis is restricted to the constant eigenstrain problem. Therefore, it would only give approximate results for inhomogeneous inclusions.

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