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MODE III STRESS INTENSITY FACTORS FOR CONCENTRATED FORCE AT AN ARBITRARY POINT

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It is well known that problems of a concentrated force acting at an arbitrary point are of practical importance because solutions to such problems can be used as Green's functions to obtain solutions for any given distribution of tractions. For the in-plane problem as shown in Fig. 1, the stress intensity factors can be found in handbooks of stress intensity factors [1,2]. For the antiplane problem in Fig. 2, however, to our knowledge, the solution has not been given as yet. In the following an attempt is made to find mode III stress intensity factors for the problem in Fig. 2, which can be described as:

In an infinite elastic plane with a crack L, a longitudinal concentrated force P is located at an arbitrary point $z=z_0$ and stresses at infinity vanish. Let w denote the antiplane displacement, and τ_{xz} and τ_{yz} the antiplane shear stress components, the boundary condition of the problem is

$$\tau_{yz}^{+} = \tau_{yz}^{-} = 0 \qquad on \qquad L \tag{1}$$

where superscripts + and - refer to the values of the functions on the real axis as approached from the upper half-plane and from the lower half-plane, respectively.

The condition of single-valuedness of the displacements on the crack surfaces is

$$w^{+}(a) - w^{+}(-a) = w^{-}(a) - w^{-}(-a) \tag{2}$$

where a and -a are the crack tip coordinates.

To formulate the problem we use an analytic function f(z) of the complex variable z=x+iy, in terms of which w and τ_{xz}, τ_{yz} are given as [3]

$$w = R_e f(z) = \frac{1}{2}[f(z) + \overline{f(z)}] \tag{3}$$

$$\tau_{xz} - i\tau_{yz} = Gf'(z) \tag{4}$$

where G is the shear modulus of elasticity. $f'(z)$ is holomorphic in the entire region occupied by the elastic body except concentrated load points which are poles. For the problem under consideration, by analyzing the singularity of $f'(z)$, we obtain

$$F(z) = f'(z) = -\frac{P}{2\pi G} \cdot \frac{1}{z - z_0} + F_0(z) \tag{5}$$

where $F_0(z)$ is holomorphic in the entire plane cut along L and it vanishes at infinity. By applying Schwarz's reflection principle, we define a new analytic function:

$$\Omega(z) = \overline{F}(z) = \overline{F(\overline{z})} \tag{6}$$

so that from (3) and (4) we can obtain:

$$w' = \frac{\partial w}{\partial \chi} = \frac{1}{2}[F(z) + \Omega(\overline{z})] \tag{7}$$

$$\tau_{yz} = \frac{1}{2} iG[F(z) - \Omega(\overline{z})] \tag{8}$$

Substituting (8) into (1) and arranging, we obtain

$$[F(t) - \Omega(t)]^+ + [F(t) - \Omega(t)]^- = 0 \qquad on \qquad L \tag{9}$$

$$[F(t) + \Omega(t)]^+ - [F(t) + \Omega(t)]^- = 0 \qquad on \qquad L \tag{10}$$

where t denotes the coordinate on the real axis.

From (5) and (6), it is seen that

$$\Omega(z) = -\frac{P}{2\pi G} \cdot \frac{1}{z - \overline{z}_0} + \Omega_0(z) \tag{11}$$

where $\Omega_0(z)$ is also holomorphic in the entire plane cut along L and it vanishes at infinity. Substituting (5) and (11) into (10), then applying Liouville's theorem, we obtain

$$F_0(z) + \Omega_0(z) = 0 \tag{12}$$

Substituting (5) and (11) into (9), we obtain

$$[F_0(t)-\Omega_0(t)]^+ [F_0(t)-\Omega_0(t)]^- = \frac{P}{\pi G}\left(\frac{1}{z-z_o}-\frac{1}{z-\bar{z}_0}\right) \qquad on \qquad L \tag{13}$$

which has the following general solution [4]

$$[F_0(z)-\Omega_0(z)] = \frac{P}{2\pi G}\left(\frac{1}{z-z_0}-\frac{1}{z-\bar{z}_0}\right)-\frac{P}{2\pi G}\cdot\frac{1}{\sqrt{z^2-a^2}} \left(\frac{\sqrt{z_0^2-a^2}}{z-z_0}-\frac{\sqrt{\bar{z}_0^2-a^2}}{z-\bar{z}_0}\right)+\frac{C}{\sqrt{z^2-a^2}} \tag{14}$$

where $1/\sqrt{z^2-a^2}$ is a single-valued branch in the plane cut along L and for which

$$\lim_{|z|\to\infty} z/\sqrt{z^2-a^2}=1 \tag{15}$$

Constant C will be determined from (2), which can be rewritten as

$$\int_{-a}^{a} w'^{+}(t)dt = \int_{-a}^{a} w'^{-}(t)dt \tag{16}$$

Substituting (7) into (16), we obtain

$$\oint [F(\xi)-\Omega(\xi)]d\xi = 0 \tag{17}$$

where Λ is a clockwise closed contour encircling L with singular points $z=z_0$, $\bar{z}_0$ outside, ξ is the coordinate on Λ.

Substituting (5), (11) and (14) into (17), then applying the residue theorem, we obtain

$$C=0 \tag{18}$$

From (5), (12) (14) and (18), it is seen that

$$F(z) = -\frac{P}{4\pi G}\left(\frac{1}{z-z_0}+\frac{1}{z-\bar{z}_0}\right)-\frac{P}{4\pi G}\cdot\frac{1}{\sqrt{z^2-a^2}}\left(\frac{\sqrt{z_0^2-a^2}}{z-z_0}-\frac{\sqrt{\bar{z}_0^2-a^2}}{z-\bar{z}_0}\right) \tag{19}$$

According to the definition given in [3], the stress intensity factor $k_3(a)$ in

Fig. 2 is

$$k_3(a) = i\sqrt{2}G\lim_{z\to a}(z-a)^{\frac{1}{2}}F(z) = \frac{\text{P}i}{4\pi\sqrt{a}}\left(\sqrt{\frac{z_0+a}{z_0-a}}-\sqrt{\frac{\bar{z}_0+a}{\bar{z}_0-a}}\right) \tag{20}$$

Noting that as z_0 approaches to b on the upper crack surface,

$$\sqrt{\frac{z_0+a}{z_0-a}} = -i\sqrt{\frac{a+b}{a}-b} \qquad \sqrt{\frac{\bar{z}_0+a}{\bar{z}_0-a}} = i\sqrt{\frac{a+b}{a-b}} \tag{21}$$

we can obtain the stress intensity factors for various special cases, several of which are listed in Table 1.

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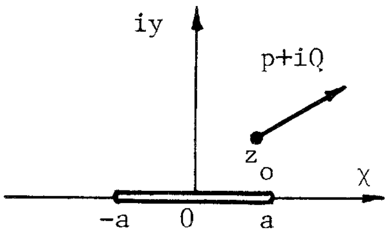

Figure 1

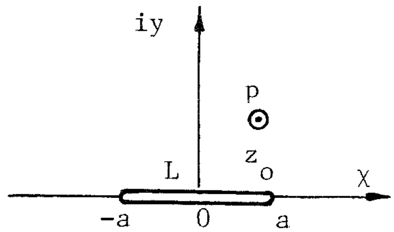

Figure 2

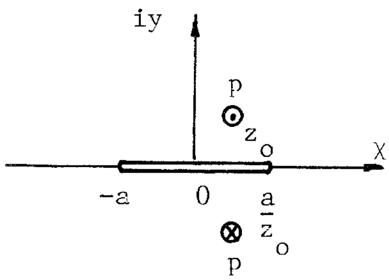

Figure 3

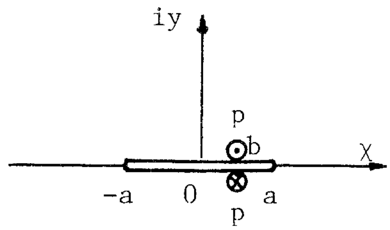

Figure 4

Table 1. Stress intensity factor k_3

Fig.	$k_3(-a)$	$k_3(a)$
2	$-\frac{Pi}{4\pi\sqrt{a}}\left(\sqrt{\frac{\zeta_0-a}{\zeta_0+a}}-\sqrt{\frac{\bar{\zeta}_0-a}{\bar{\zeta}_0+a}}\right)$	$\frac{Pi}{4\pi\sqrt{a}}\left(\sqrt{\frac{\zeta_0+a}{\zeta_0-a}}-\sqrt{\frac{\bar{\zeta}_0+a}{\bar{\zeta}_0-a}}\right)$
3	$-\frac{Pi}{2\pi\sqrt{a}}\left(\sqrt{\frac{\zeta_0-a}{\zeta_0+a}}-\sqrt{\frac{\bar{\zeta}_0-a}{\bar{\zeta}_0+a}}\right)$	$\frac{Pi}{2\pi\sqrt{a}}\left(\sqrt{\frac{\zeta_0+a}{\zeta_0-a}}-\sqrt{\frac{\bar{\zeta}_0+a}{\bar{\zeta}_0-a}}\right)$
4*	$\frac{P}{\pi\sqrt{a}}\sqrt{\frac{a-b}{a+b}}$	$\frac{P}{\pi\sqrt{a}}\sqrt{\frac{a+b}{a-b}}$

* which are in agreement with the classical results [1,2,3].