STANDING WAVE OF FINITE AMPLITUDE IN AN CIRCULAR BASIN WITH UNEVEN BOTTOM

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Although the study of progressive waves could be traced back to the last century, it was not until 1915 when Rayleigh[1] began seriously to consider nonlinear behaviour of standing waves due to the difficulties arising from the unsteadiness. Based on the criterion that the downward acceleration of the crest cannot exceed gravity acceleration g, Penney & Price[2] discovered the slope of steepest standing wave being 0.22 with an enclosed angle of 90° around the crest. Nevertheless, the above findings are still in controversy[3,4]. In 1960, Tadjbaksh & Keller[5] calculated the approximate solution of finite depth to the third order by virtue of keller-Ting's approach. In the meanwhile, they found the nonuniqueness of the solution at several critical depths, which was dealt with by Vanden Broeck[6]. In recent decade, Rottman[7] investigated interfacial standing wave, while J.C. Li[8] examined the effects of geometric factors on the standing wave in a nearly circular basin. In this note, we attempt to further discuss the standing wave of finite amplitude in a circular basin with uneven bottom.

We assume that the fluid is incompressible, inviscid and the motion of fluid is irrotational. Also, the wave slope $\varepsilon = a/\lambda$ and the Ursell number $a\lambda^2/h^3$ are required to be small. In addition, large Rossby number $c/D\Omega$ will guarantee the negligibility of Coriolis force. Where a is the wave amplitude, λ wavalength, c phase velocity, h water depth, D the radius of the basin and Ω angular velocity of the earth.

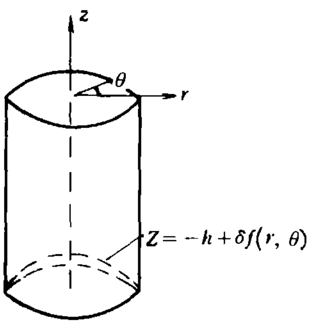

Fig. 1

For a basically axisymmetric system, we prefer to choosing cylindrical coordinate as independent variables with r being the radial coordinate, z the vertical and θ the azimuthal angle. As usual, we set the origin 0 at the surface of quiescent water, positive z direction being upwards. Then, the expression of the bottom can be written as

$$Z = -h + \delta f(r, \theta), \tag{1}$$

where δ is a small parameter with f and its derivative being of $O(1)$. If all the physical quantities are made dimensionless as follows:

$$r = k^{-1} r', \qquad z = k^{-1} z', \qquad \eta = \varepsilon k^{-1} \eta,$$

$$t = (kg)^{-1/2} t', \qquad \omega = (kg)^{1/2} \omega', \qquad t = \omega'^{-1} t'',$$

$$\Phi = \varepsilon g^{1/2} k^{-3/2} \Phi', \qquad p = (\rho g k^{-1}) p',$$

and we omit all the superscripts merely for simplicity, then, the velocity potential $\Phi(r, \theta, z, t)$ will satisfy:

$$\nabla^2 \Phi = 0; \tag{2}$$

$$\Phi_z = \omega \eta_t, \qquad \text{at } z = 0 \tag{2.1}$$

$$\omega \Phi_t + \eta = 0, \qquad \text{at } z = 0 \tag{2.2}$$

$$\Phi_r = 0, \qquad \text{at } r = R \tag{2.3}$$

$$\Phi_n = 0, \qquad \text{at } z = -h + \delta f(r, \theta), \tag{2.4}$$

where η is the wave shape, ω the circular frequency. The nonlinear effects will be considered later.

The derivatives at the bottom should be

$$\frac{\partial}{\partial n} = \frac{-\dfrac{\partial}{\partial t} + \delta f_r \dfrac{\partial}{\partial r} + \delta \dfrac{f_\theta}{r^2} \dfrac{\partial}{\partial \theta}}{\sqrt{1 + \delta^2 f_r^2 + \delta^2 f_\theta^2 / r^2}}. \tag{3}$$

If we expand Φ, η, ω as power series of δ, and transfer the boundary condition (2.4) to undisturbed surface $z = -h$, then we have

$$\Phi_{oz} = 0, \tag{4.1}$$

$$\Phi_{1z} = \Phi_{or} f_r + \frac{1}{r^2} \Phi_{o\theta} f_\theta - \Phi_{ozz} f, \tag{4.2}$$

$$\Phi_{2z} = \frac{1}{2} \Phi_{oz} \left(f_r^2 + \frac{1}{r^2} f_\theta^2\right) + \Phi_{orz} f f_r + \frac{1}{r^2} \Phi_{o\theta z} f f_\theta$$
$$- \frac{1}{2} \Phi_{ozzz} f^2 + \Phi_{1r} f_r + \frac{1}{r^2} \Phi_{1\theta} f_\theta - \Phi_{1zz} f. \tag{4.3}$$

Thus, we are able to solve the whole problem by using perturbation theory.

Evidently, the zeroth solution is nothing but the oscillation of the fluid in a circular basin with flat bottom. The influences of bottom topology will embody in the solution of higher order. If we only discuss the perturbation of fundamental wave, the first order solution turns out to be :

$$\Phi_1=[c_1^{(0,1)}\ J_0(r)(\mathrm{ch}z-\mathrm{ch}(z+h))$$
$$+\sum{}'(a_1^{(n,m)}\cos n\theta+b_1^{(n,m)}\sin n\theta)\mathrm{ch}k^{(n,m)}(z+h)J_n(k^{(n,m)}r)$$
$$+\sum{}'(c_1^{(n,m)}\cos n\theta+d_1^{(n,m)}\sin n\theta)\mathrm{ch}k^{(n,m)}zJ_n(k^{(n,m)}r)]\cos n\theta \tag{5}$$

$$\eta_1=\{\omega_1/\omega_0J_0(r)+\omega_0c_1^{(0,1)}(1-\mathrm{ch}\ h)J_0(r)$$
$$+\omega_0\sum{}'[(c_1^{(n,m)}+a_1^{(n,m)}\mathrm{ch}\ k^{(n,m)}h)\cos n\theta+(d_1^{(n,m)}+b_1^{(n,m)}$$
$$\times\mathrm{ch}k^{(n,m)}\sin n\theta)]\ J_n(k^{(n,m)}r)\}\ \sin t, \tag{6}$$

where $J_n(r)$ is the n-th order Bessel function, $k^{(n,m)}=z^{(n,m)}/z^{(0,1)}$($z^{(n,m)}$ is the m-th zero of $J_n'(z)$). $\sum'$ indicates that the summation is carried out for $n\neq 0$ or $m\neq 1$. $a_1^{(n,m)}$, $b_1^{(n,m)}$, $c_1^{(n,m)}$, $d_1^{(n,m)}$ are related to the coefficients of Fourier-Bessel expansion for the function

$$B_1(r,\theta)=\omega_0/\mathrm{sh}h(f_rJ_0'(r)-fJ_0(r)). \tag{7}$$

The revised circular frequency of the first order looks like

$$\omega_1=-\frac{\omega_0^2}{2}c^{(0,1)}=-\frac{\omega_0^2}{2\pi R^2\,\mathrm{sh}hJ_0^2(R)}\int_0^{2\pi}\int_0^R rJ_0(r)B_1(r,\theta)\,dr\,d\theta. \tag{8}$$

Following (6), we are able to derive the solution of the first order due to nonlinearity.

$$\Phi_1=\beta_0t+[a_0+\sum a_mJ_0(k^{(0,m)}r)\mathrm{ch}\ k^{(0,m)}(z+h)\sin 2t, \tag{9}$$

$$\eta_1=-\beta_0\omega_0+1/4[\omega_0^2J_0^2(r)-1/\omega_0^2J_0'^2(r)]$$
$$-[2a_0\omega_0+3/4\omega_0^2J_0^2(r)+1/4\omega_0^2J_0'^2(r)+\sum 2\omega_0a_m\ \mathrm{ch}\ k^{(0,m)}h\ J_0(k^{(0,m)}r)]\cos 2t, \tag{10}$$

where β_0, a_m are related to the Bessel expansion coefficients of $J_0^2(r)$ and $J_0'^2(r)$. The revised circular frequency ω_1 happens to vanish.

The results obtained above lead us to draw the following conclusions:

(i) The sign of the revised frequency depends basically on that of the integral

$$\int_0^{2\pi}\int_0^R rJ_0(r)B_1(r,\theta)\,dr\,d\theta. \tag{11}$$

Generally speaking, the convex bottom with smaller total volume of water will result in higher frequency. On the other hand, the concave bottom will show the contrary effects. As a comparison, ω_1 due to nonlinearity happens to be zero.

(ii) The uneven bottom will also deform the shape of standing wave so that it will become unaxisymmetric. However, there still exists a moment when the free surface is completely flat. The node line of wave is removed to $r_n=r_0+\delta r_1(\theta)$, in which r_0 is the first zero point of the zeroth order Bessel function and

$$r_1(\theta) = \omega_0 / J_0'(r_0) \sum{}' [(c_1^{(n,m)} + a_1^{(n,m)} \mathrm{ch}\, k^{(n,m)} h) \cos n\theta + (d_1^{(n,m)} + b_1^{(n,m)} \mathrm{ch}\, k^{(n,m)} h) \sin n\theta) J_n (k^{(n,m)} r_0)] . \qquad (12)$$

As for nonlinear wave, the surface will never be flat though the wave shape remains axisymmetric.

If $\delta = 0(\varepsilon)$, the effects of both kinds are capable of being superposed. Their coupling will manifest itself in the higher order solution.

(iii) According to Bernoulli equation, the revised pressure of the first order includes the static one due to both factors and the dynamic one due merely to nonlinearity.

(iv) An example with convex paraboloid bottom is illustrated in Figs. 2 — 4, i. e. $f(r, \theta) = a^2 - r^2$.

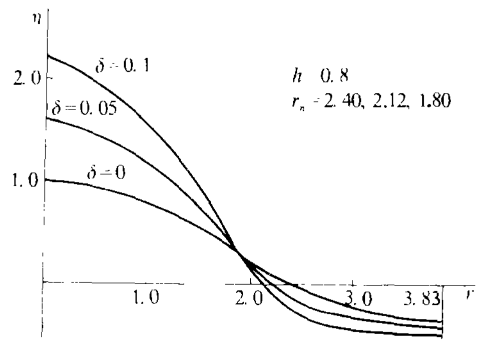

Fig. 2.

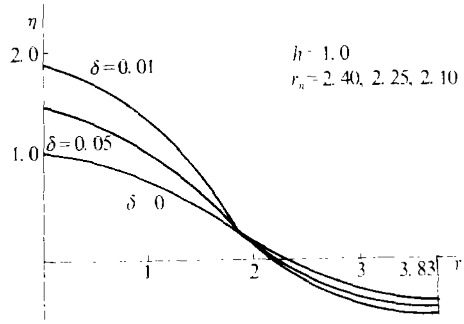

Fig. 3.

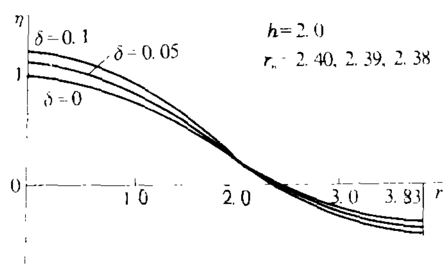

Fig. 4.

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