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ASYMPTOTIC FIELDS NEAR THE CRACK TIP IN ELASTIC–PERFECTLY PLASTIC CRYSTALS

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Abstract—The basic equations of plane strain problem for the elastic–perfectly plastic crystals with double slip systems have been presented in the basis of three dimensional flow theory of crystal plasticity. Using these equations the stationary crack tip stress and deformation fields are analysed for tensile load. The fields involve an elastic angular sector and are fully continuous. An asymptotic solution is also obtained for the steadily growing crack that consists of five angular sectors: two plastic angular sectors in the front of the crack tip connected with the boundary on which the associated velocity field has discontinuities; a secondary plastic angular sector near the crack face; two elastically unload angular sectors connected with the boundary on which the discontinuity of the associated velocity field occurs. The asymptotoic solution is not unique. A family of solutions is obtained. Finally, the application of these solutions on both FCC and BCC crystals is discussed.

1. INTRODUCTION

RECENT YEARS crack tip stress and deformation fields for elastic–plastic crystals have attracted scientists' attention. The first reason for that may be denoted to the developments of micro fracture mechanics. The second reason is due to application of crystal element on engineering practice which shows some special advantages.

Rice and Nikolic[1] have firstly presented the analysis of elastic–perfectly plastic crack tip response of crystals in anti-plane shear. For stationary cracks they got complete field solution. For growing cracks, they obtained the asymptotic solution.

The strain hardening effect has been considered by Rice and Saeedvafa[2] and the solution of HRR singularity type has been obtained.

The nearest work given by Rice[3] shows the crack tip stress and deformation fields for tensile loaded perfectly plastic crystals. The crack tip fields are assembled by four angular sectors and shown to change discontinuously from sector to sector for a stationary crack. The asymptotic solution of growing crack consists also of four angular sectors.

But as pointed out by Rice[3], the asymptotic solution is not unique. This paper presents the analysis of crack tip stress and deformation fields for tensile loaded elastic–perfectly plastic crystals.

The basic equations of plane strain problem for the elastic–perfectly plastic crystals with double slip systems have been presented in the basis of three-dimensional flow theory of crystal plasticity.

Using these equations the stationary crack tip fields are analysed for tensile load.

The fields are assembled by three angular sectors: a plastic zone in the front of the crack; a plastic zone near the crack face; an elastic zone between two plastic zones. The stress and displacement fields are fully continuous.

In the plastic zone ahead of the crack there is a concentrated plastic shear zone between two rays: the first ray is along the active slip direction traces and the second ray is perpendicular to the active slip plane traces.

In the plastic zone near crack face there is also a concentrated plastic shear zone.

The assembly of growing crack tip field involves five angular sectors: two plastic zones ahead of crack; a secondary plastic zone near crack face; two elastic unloaded zones between them. The present solutions contain a free parameter and give a family of crack tip fields.

Finally the application of these solutions on the FCC and BCC crystals are considered and associated crack tip stress and strain fields are obtained for FCC and BCC crystals.

2. BASIC EQUATION

For the sake of simplicity, we start from the analysis of double slip plane model of crystal proposed by Asaro[4]. In the sixth section of this paper, the application of the present results on FCC and BCC crystals will be discussed.

As shown in Fig. 1, the plane model of double slip involves two slip systems: the primary slip system and the conjugate slip system.

Using this model, Asaro[4] has successfully demonstrated the complex phenomena of latent hardening and rotation of crystals.

Imagine the crack plane lies on the symmetric plane and set up a fixed Cartesian coordinate system $0XYZ$ (Fig. 2) centered at the initial crack tip. The (x_1, x_2, x_3) is the tensor description of (x, y, z).

Let $\mathbf{m}^{(2)}, \mathbf{n}^{(1)}$ be the unit vectors defining the slip direction and normal direction of slip plane of primary slip system respectively.

Let $\mathbf{m}^{(2)}, \mathbf{n}^{(1)}$ be the unit vectors along slip direction and slip plane normal of conjugate slip system, respectively.

Consider the plane strain problem of elastic–perfectly plastic crystals.

Assume the deformation gradient is small and the effects of rotation of crystal direction on the equilibrium equation and deformation are neglected. Apparently both are important for future work.

2.1. *Yield condition*

According to the Schmid rule, the yield condition for elastic–perfectly plastic crystals can be expressed

$$\tau^{(\alpha)} = \tau_c^{(\alpha)}, \tag{2.1}$$

where $\tau^{(\alpha)}$ is the resolved shear stress of the αth slip system, $\tau_c^{(\alpha)}$ is the critical shear stress of the αth slip system.

If we take the opposite slip as another slip system, there are four slip systems altogether.

Consider only isotropic yielding and neglect the Bauchinger effect. We have

$$\tau_c^{(\alpha)} = \tau_c, \quad \alpha = 1, 2, 3, 4.$$

For resolved shear stress, we have formula

$$\tau^{(\alpha)} = \boldsymbol{\sigma}\colon \mathbf{P}^{(\alpha)} = \boldsymbol{\sigma}_{ij}\mathbf{P}_{ij}^{(\alpha)}, \tag{2.2}$$

here

$$\mathbf{P}^{(\alpha)} = \tfrac{1}{2}(\mathbf{m}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)} + \mathbf{n}^{(\alpha)} \otimes \mathbf{m}^{(\alpha)}). \tag{2.3}$$

For the sake of convenience, we can express the vector with a column and the second order tensor with a matrix.

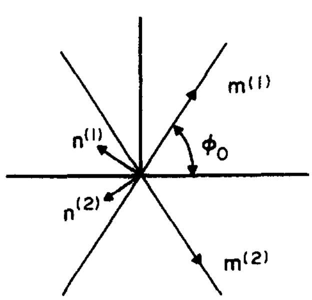

Fig. 1. Plane model of primary–conjugate slip systems.

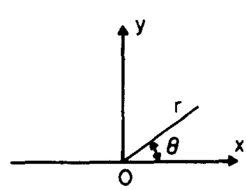

Fig. 2. Fixed cartesian coordinate system.

We have

$$
\begin{aligned}
\mathbf{n}^{(1)} &= \{-\sin\varphi_0 \quad \cos\varphi_0 \quad 0\}^{\mathrm{T}} \\
\mathbf{m}^{(2)} &= \{\ \cos\varphi_0 \quad -\sin\varphi_0 \quad 0\}^{\mathrm{T}} \\
\mathbf{n}^{(2)} &= \{-\sin\varphi_0 \quad \cos\varphi_0 \quad 0\}^{\mathrm{T}}
\end{aligned}
$$

$$
\mathbf{P}^{(1)} = \frac{1}{2}\begin{bmatrix} -\sin 2\varphi_0 & -\cos 2\varphi_0 & 0 \\ \cos 2\varphi_0 & \sin 2\varphi_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.4}
$$

$$
\mathbf{P}^{(2)} = \frac{1}{2}\begin{bmatrix} -\sin 2\varphi_0 & -\cos 2\varphi_0 & 0 \\ -\cos 2\varphi_0 & \sin 2\varphi_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5}
$$

$$
\mathbf{P}^{(3)} = -\mathbf{P}^{(1)}, \quad \mathbf{P}^{(4)} = -\mathbf{P}^{(2)}.
$$

From eqs (2.2), (2.4) and (2.5), it follows

$$
\begin{aligned}
\tau^{(1)} &= \tfrac{1}{2}(\sigma_y - \sigma_x)\sin 2\varphi_0 + \tau_{xy}\cos 2\varphi_0, \\
\tau^{(2)} &= \tfrac{1}{2}(\sigma_y - \sigma_x)\sin 2\varphi_0 + \tau_{xy}\cos 2\varphi_0.
\end{aligned} \tag{2.6}
$$

In the polar coordinates (r, θ), we have

$$
\begin{aligned}
\tau^{(1)} &= -\,(\sigma_\theta - \sigma_r)\sin 2(\theta - \varphi_0) + \tau_{\gamma\theta}\cos 2(\theta - \varphi_0), \\
\tau^{(2)} &= \tfrac{1}{2}(\sigma_\theta - \sigma_r)\sin 2(\theta + \varphi_0) - \tau_{r\theta}\cos 2(\theta + \varphi_0).
\end{aligned} \tag{2.7}
$$

2.2. *Constitutive relation*

The constitutive relation of crystals can be taken in the form,

$$
D_{ij} = \frac{(1+\nu)}{E}\dot{\sigma}_{ij} - \frac{\nu}{E}\delta_{ij}\dot{\sigma}_{kk} + D^0_{ij}, \tag{2.8}
$$

where D_{ij} is strain rate tensor. D^p_{ij} is plastic strain rate tensor,

$$
D^p = \sum_{\alpha=1}^{n} \mathbf{P}^{(\alpha)}\dot{\gamma}^{(\alpha)}, \quad n = 4, \tag{2.9}
$$

here $\dot{\gamma}^{(\alpha)}$ is the slip shear rate of the αth slip system. For continued active slip system $\dot{\gamma}^{(\alpha)} > 0$, otherwise $\dot{\gamma}^{(\alpha)} = 0$.

The relation between strain rate tensor and velocity field is

$$
D_{ij} = \tfrac{1}{2}(\partial V_i/\partial x_j + \partial V_j/\partial x_i). \tag{2.10}
$$

For plane strain, we have

$$
D_{33} = D_{13} = D_{23} = 0.
$$

Noting

$$
\begin{aligned}
& D^p_{33} = D^p_{13} = D^p_{23} = 0, \text{ we obtain} \\
& \dot{\sigma}_{13} = \dot{\sigma}_{23} = 0, \\
& \dot{\sigma}_{33} = \nu(\dot{\sigma}_{11} + \dot{\sigma}_{22}).
\end{aligned}
$$

Using the above formulas, eq. (2.8) becomes

$$
D_{\alpha\beta} = \frac{1}{2\mu}\dot{\sigma}_{\alpha\beta} - \frac{\nu}{2\mu}\delta_{\alpha\beta}\dot{\sigma}_{\rho\rho} + D^p_{\alpha\beta}, \tag{2.11}
$$

where μ is the shear modulus and $\mu = E/2(1+\nu)$. The Greek indices α, β, ρ, ran over 1 and 2 only.

The repeated indices imply summation convention.

2.3. *Stress function and stress components*

The equilibrium equation is identically satisfied through introducing the stress function ϕ,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2},\quad \sigma_y = \frac{\partial^2\phi}{\partial x^2},\quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}.$$

The stress function ϕ of the asymptotic field can be expressed as

$$\phi = r^2 F(\theta) \tag{2.12}$$

Using the stress function, the yield condition becomes

$$\begin{aligned}\tau^{(1)} &= \tfrac{1}{2}F''\sin 2(\theta-\varphi_0) - F'\cos 2(\theta-\varphi_0) = \tau_c,\\ \tau^{(2)} &= -\tfrac{1}{2}F''\sin 2(\theta+\varphi_0) + F'\cos 2(\theta+\varphi_0) = \tau_c.\end{aligned} \tag{2.13}$$

3. STRESS AND DEFORMATION FIELDS NEAR STATIONARY CRACK

As shown in Fig. 3, the crack tip zone is assembled by three angular sectors. The domains A and C are plastic zones and the domain B is an elastic zone.

Consider a pure mode I crack. Due to symmetry, we only consider the upper half plane.

In domain A, two slip systems will simultaneously attain yield. For the first equation of formula (2.13), we obtain

$$F = \frac{\tau_c}{2}\sin 2(\theta-\varphi_0)\frac{A_0}{2}\cos 2(\theta-\varphi_0) + A_1^*. \tag{3.1}$$

Hence the plastic zone is a constant stress zone. Noting $\theta = 0$, $\tau_{r\theta} = 0$, it follows

$$\begin{aligned}A_0 &= -\tau_c\cos 2\varphi_0/\sin 2\varphi_0,\\ F &= \frac{\tau_c}{2\sin 2\varphi_0}[\cos 2\theta + \bar{A}_1].\end{aligned} \tag{3.2}$$

From eq. (3.2), we arrive at

$$\tau^{(2)} = -\tfrac{1}{2}F''\sin 2(\theta+\varphi_0) + F'\cos 2(\theta+\varphi_0) = \tau_c.$$

It means that the primary slip system and the conjugate slip system are simultaneously active in domain A.

Similarly we obtain (in domain C),

$$F = \frac{k^*}{2}[1-\cos 2\theta],\quad k^* = \frac{\tau_c}{\sin 2\varphi_0}.$$

We have

$$\tau^{(1)} = \tau^{(2)} = -\tau_c,\ \text{in domain C}.$$

In elastic zone B, we have

$$F = \frac{k^*}{2}[1-\cos 2\theta] + B_0^*(1-\cos 2(\theta+\beta)) + B_1^*(\theta - \bar{\beta} - \tfrac{1}{2}\sin 2(\theta+\beta)) + B_3^*\cos 2\theta + B_4^*\sin 2\theta.$$

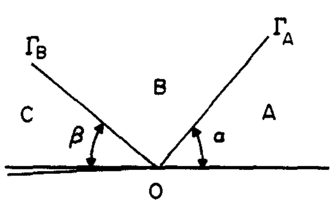

Fig. 3. Assembly of angular sectors for a stationary crack.

On boundary Γ_B, since the stress components are completely continuous, it results in $B_0^* = B_3^* = B_4^* = 0$. We obtain

$$F = \frac{k^*}{2}[1 - \cos 2\theta] + B_1^*[\theta - \bar{\beta} - \tfrac{1}{2}\sin 2(\theta + \beta)],$$

where $\bar{\beta} = \pi - \beta$, β is the angle between the boundary Γ_B and the crack face.

From the complete continuity of stress components on boundary Γ_A, we find

$$\begin{cases} \dfrac{k^*}{2}(\bar{A}_1 + \cos 2\alpha) = \dfrac{k^*}{2}(1 - \cos 2\alpha) + B_1^*[\alpha + \beta - \pi - \tfrac{1}{2}\sin 2(\alpha + \beta)] \\ -k^* \sin 2\alpha = k^* \sin 2\alpha + B_1^*[1 - \cos 2(\alpha + \beta)], \\ -2k^* \cos 2\alpha = 2k^* \cos 2\alpha + 2B_1^* \sin 2(\alpha + \beta). \end{cases} \tag{3.3}$$

The last two formulas of eq. (2.13) can be represented as

$$B_1^* \sin 2(\alpha + \beta) = -2k^* \cos 2\alpha,$$
$$B_1^*[1 - \cos 2(\alpha + \beta)] = -2k^* \sin 2\alpha.$$

Eliminating B_1^* from above formulas, we arrive at

$$\cos 2\beta - \cos 2\alpha = 0.$$

It means that $\alpha = \beta$ or $\beta = \pi - \alpha$; the latter case removes the domain B. It is not a true solution, and must be ruled out. We have

$$\begin{cases} \alpha = \beta, \quad B_1^* = -k^*/\sin 2\alpha, \\ \bar{A}_1 = 1 + 2(\pi - 2\alpha)/\sin 2\alpha, \end{cases} \tag{3.4}$$

$$F = \begin{cases} \dfrac{k^*}{2}[\cos 2\theta + \bar{A}_1], & \text{in domain A} \\ \dfrac{k^*}{2}[1 - \cos 2\theta] + B_1^* - \tfrac{1}{2}\sin 2(\theta + \beta), & \text{in domain B} \\ \dfrac{k^*}{2}[1 - \cos 2\theta]. & \text{in domain C} \end{cases} \tag{3.5}$$

The angle α is an unknown parameter which needs to be determined in the following discussion.

In domain B, we find

$$\begin{cases} \tau^{(1)} = -\tau_c + B_1^*[\cos 2(\beta + \varphi_0) - \cos 2(\theta - \varphi_0)], \\ \tau^{(2)} = -\tau_c + B_1^*[\cos 2(\beta + \varphi_0) - \cos 2(\theta - \varphi_0)], \end{cases} \tag{36}$$

$$\begin{cases} \dfrac{d\tau^{(1)}}{d\theta} = -\dfrac{k^*}{\sin 2\alpha}\sin 2(\theta - \varphi_0), \\ \dfrac{d\tau^{(2)}}{d\theta} = \dfrac{k^*}{\sin 2\alpha}\sin 2(\theta - \varphi_0). \end{cases} \tag{3.7}$$

It can be seen that the extreme points of $\tau^{(1)}$ are $\theta = \varphi_0$ and $\theta = (\pi/2) + \varphi_0$, the extreme points of $\tau^{(2)}$ are $\theta = (\pi/2) - \varphi_0$ and $\theta = \pi - \varphi_0$.

At $\theta = \varphi_0$,

$$\tau^{(1)} = \tau^{(1)}_{\max} = \frac{2\tau_c \sin^2(\beta + \varphi_0)}{\sin 2\alpha \sin 2\varphi_0} - \tau_c.$$

At $\theta = \dfrac{\pi}{2} + \varphi_0$,

$$\tau^{(1)} = \tau^{(1)}_{\min} = -\frac{2\tau_c \cos^2(\beta + \varphi_0)}{\sin 2\alpha \sin 2\varphi_0} - \tau_c.$$

At $\theta = \pi - \varphi_0$,

$$\tau^{(2)} = \tau^{(2)}_{\min} = \frac{-2\tau_c \sin^2(\beta + \varphi_0)}{\sin 2\alpha \sin 2\varphi_0} - \tau_c.$$

At $\theta = \dfrac{\pi}{2} - \varphi_0$,

$$\tau^{(2)} = \tau^{(1)}_{\max} = \frac{2\tau_c \cos^2(\beta + \varphi_0)}{\sin 2\varphi_0 \sin 2\alpha} - \tau_c.$$

Since the yield constraint condition should be satisfied in domain B, hence we have $\alpha = \beta \geqslant \varphi_0, \alpha = \beta \geqslant (\pi/2) - \varphi_0$.

In fact, if $\alpha < \varphi_0$, then the angle $\theta = \pi - \varphi_0$ must lie in domain B, and $\tau^{(2)} = \tau^{(2)}_{\min} < -\tau_c$.

On the other hand, if $\alpha < (\pi/2) - \varphi_0$, the angle $\theta = (\pi/2) + \varphi_0$ is within domain B and $\tau^{(1)} = \tau^{(1)}_{\min} < -\tau_c$.

Thus we confirm that

$$\alpha = \beta \geqslant \max\left\{\varphi_0, \frac{\pi}{2} - \varphi_0\right\}. \tag{3.8}$$

Consider now the deformation field in domain A. From eq. (2.9), we find

$$\begin{cases} D^{\mathrm{P}}_{11} = -\frac{1}{2}(\dot{\gamma}_1 + \dot{\gamma}_2)\sin 2\varphi_0, \\ D^{\mathrm{P}}_{22} = \frac{1}{2}(\dot{\gamma}_1 + \dot{\gamma}_2)\sin 2\varphi_0, \\ D^{\mathrm{P}}_{12} = \frac{1}{2}(\dot{\gamma}_1 - \dot{\gamma}_2)\cos 2\varphi_0, \end{cases} \tag{3.9}$$

$$\begin{cases} D^{\mathrm{P}}_r = -D^{\mathrm{P}}_\theta = \frac{1}{2}\{\dot{\gamma}_1 \sin 2(\theta - \varphi_0) - \dot{\gamma}_2 \sin 2(\theta + \varphi_0)\}, \\ D^{\mathrm{P}}_{r\theta} = \frac{1}{2}\{\dot{\gamma}_1 \cos 2(\theta - \varphi_0) - \dot{\gamma}_2 \cos 2(\theta + \varphi_0)\}, \end{cases} \tag{3.10}$$

where D^{P}_r, D^{P}_θ, $D^{\mathrm{P}}_{r\theta}$ are strain rate components in polar coordinates and $\dot{\gamma}_1, \dot{\gamma}_2$ are the slip shear rate of primary and conjugate slip systems, respectively.

Because domain A is a constant stress zone, hence the elastic strain rate vanishes in the asymptotic sense. The singularity part of strain rate is due to plastic strain rate.

Let

$$\dot{\gamma}_1 = \frac{1}{r}\tilde{\gamma}_1(\theta), \quad \dot{\gamma}_2 = \frac{1}{r}\tilde{\gamma}_2(\theta),$$

eq. (2.10) has the form (in polar coordinates),

$$\begin{cases} D_r = \dfrac{\partial V_r}{\partial r}, \\[2ex] D_\theta = \dfrac{1}{r}\dfrac{\partial V_\theta}{r\partial\theta} + \dfrac{V_r}{r}, \\[2ex] 2D_{r\theta} = \dfrac{1}{r}\dfrac{\partial V_r}{\partial\theta} + \dfrac{\partial V_\theta}{\partial r} - \dfrac{V_\theta}{r}. \end{cases} \tag{3.11}$$

Using the first two formulas of eq. (3.11), we find

$$\begin{cases} V_r = -h(\theta)\ln\left(\dfrac{R}{r}\right) - f_0'(\theta), \\ V_\theta = H(\theta)\ln\left(\dfrac{R}{r}\right) - H(\theta) + f_0(\theta), \end{cases} \tag{3.12}$$

here

$$\begin{cases} h(\theta) = \frac{1}{2}[\tilde{\gamma}_1 \sin 2(\theta = \varphi_0) - \tilde{\gamma}_2 \sin 2(\theta + \varphi_0)], \\ H(\theta) = \displaystyle\int h(\theta)\,\mathrm{d}\theta. \end{cases} \tag{3.13}$$

Substituting eq. (3.12) into the third formula of eq. (3.11), it follows

$$-\frac{1}{r}\ln\left(\frac{R}{r}\right)[h'(\theta) + H(\theta)] - \frac{1}{r}(f_0'' + f_0) = \frac{1}{r}[\tilde{\gamma}_1 \cos 2(\theta - \varphi_0) - \tilde{\gamma}_2 \cos 2(\theta + \varphi_0)].$$

Thus we obtain following compatibility equations:

$$H''(\theta) + H(\theta) = 0, \tag{3.14}$$

$$f_0'' + f_0 = \tilde{\gamma}_2 \cos 2(\theta + \varphi_0) - \tilde{\gamma}_1 \cos 2(\theta - \varphi_0). \tag{3.15}$$

From eq. (3.14), we find

$$\begin{cases} H(\theta) = A_1 \sin\theta - A_2 \cos\theta, \\ h(\theta) = A_1 \cos\theta + A_2 \sin\theta. \end{cases} \tag{3.16}$$

Due to eqs (3.13) and (3.15), we arrive at

$$\begin{cases} \tilde{\gamma} = -\{\sin 2(\theta + \varphi_0)[f_0'' + f_0] + \cos 2(\theta + \varphi_0)\cdot 2h(\theta)\}/\sin 4\varphi_0, \\ \tilde{\gamma} = -\{\sin 2(\theta - \varphi_0)[f_0'' + f_0] + \cos 2(\theta - \varphi_0)\cdot 2h(\theta)\}/\sin 4\varphi_0. \end{cases} \tag{3.17}$$

At $\theta = 0$, $V_\theta = 0$, hence $A_2 = 0$. Domain B is an elastic zone, so that the strain rate and velocity fields have nonsingularity.

For stationary crack, the normal component V_θ of velocity should be continuous across the boundary Γ_A.

Thus we have $A_1 = 0$. It results in

$$h(\theta) = H(\theta) = 0. \tag{3.18}$$

$$\begin{cases} \tilde{\gamma}_1 = -\sin 2(\theta + \varphi_0)[f_0'' + f_0]/\sin 4\varphi_0, \\ \tilde{\gamma}_2 = -\sin 2(\theta - \varphi_0)[f_0'' + f_0]/\sin 4\varphi_0. \end{cases} \tag{3.19}$$

If $\varphi_0 > (\pi/4)$, thus $\alpha \geqslant \varphi_0$. In domain $0 < \theta < (\pi/2) - \varphi_0$, the functions $\sin 2(\theta - \varphi_0)$ and $\sin 2(\theta + \varphi_0)$ have different signs. In order to confirm that

$$\tilde{\gamma}_1 \geqslant 0, \quad \tilde{\gamma}_2 \geqslant 0, \tag{3.20}$$

we must have

$$[f_0'' + f_0] = 0, \quad 0 \leqslant \theta \frac{\pi}{2} - \varphi_0, \quad \tilde{\gamma}_1 = \tilde{\gamma}_2 = 0.$$

In domain $(\pi/2) - \varphi_0\theta < \varphi_0$, the functions $\sin 2(\theta + \varphi_0)$ and $\sin 2(\theta - \varphi_0)$ are minus. Hence the constraint condition will be met if and only if

$$f_0'' + f_0 \leqslant 0.$$

In a similar way, we find

$$\begin{cases} [f_0'' + f_0] = 0, \\ \tilde{\gamma}_1 = \tilde{\gamma}_2 = 0. \end{cases} \quad \varphi_0 \leqslant \theta \leqslant \alpha$$

Anyway there is a concentrated strain zone in domain A. The strain rate has singularity of type $1/r$ in domain

$$\frac{\pi}{2} - \varphi_0\theta < \varphi_0.$$

Similarly, there is a concentrated strain zone in domain C.

It is worth noting that the angle α is a free parameter which needs to be determined from the analysis of complete solution. But the following constraint condition should be met

$$\mathrm{Max}\left\{\varphi_0, \frac{\pi}{2} - \varphi_0\right\} \leqslant \alpha\frac{\pi}{2}.$$

4. STRESS AND DEFORMATION FIELDS NEAR GROWING CRACK TIP

As shown in Fig. 4, the polar coordinate system (r, θ) is moving with crack tip. According to McClintock[5], Slepyan[6], Rice *et al.*[7] and Gao[8], the tip zone for a steadily growing crack involves an elastic unloading zone and a secondary plastic zone.

Figure 5 shows an optional assembly of sectors for a steadily growing crack: the angular sectors A and B are plastic zones; the sector D is a secondary plastic zone and the sector C is a elastic unloading zone.

The stress function of asymptotic field can be expressed[7]

$$\phi = r^2F(\theta)$$

$$F(\theta) = \begin{cases} \dfrac{k^*}{2}[\cos 2\theta + \bar{A}_1], & \text{in domains A and B} \\ \dfrac{C_1^*}{4}F_s(\theta) + \dfrac{C_2^*}{4} + C_3 + C_4[1 - \cos 2(\theta - \beta)] + C_5 \sin 2(\theta - \beta), & \text{in domain C} \\ \dfrac{k^*}{2}[1 - \cos 2\theta], & \text{in domain D} \end{cases} \tag{4.1}$$

$$k^* = \tau_c/\sin 2\varphi_0 \tag{4.2}$$

where

$$F_s(\theta) = (1 - \cos 2\theta)\ln \sin\theta - \theta\cdot\sin 2\theta - \cos^2\theta. \tag{4.3}$$

As pointed out by Drugan and Rice[9], for a quasistatically growing crack, any moving surface whose normal is inclined with the moving direction, is not a discontinuity surface. Hence on boundary Γ_B and Γ_C, full stress components are continuous. We find that

$$\begin{cases} \dfrac{C_1^*}{4}F_s(\beta) + \dfrac{C_2^*}{4}\beta + C_3 = \dfrac{k^*}{2}(\bar{A}_1 + \cos 2\beta), \\ \dfrac{C_1^*}{4}F_s'(\beta) + \dfrac{C_2^*}{4} + 2C_5 = -k^*\sin 2\beta, \\ C_1^*(\ln\sin\beta - F_s(\beta)) + 4C_4 = -2k^*\cos 2\beta, \end{cases} \tag{4.4}$$

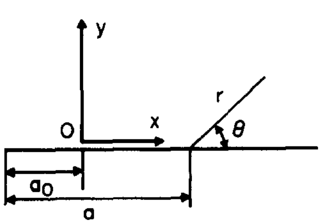

Fig. 4. Polar coordinates (r, θ) centered at the moving crack tip.

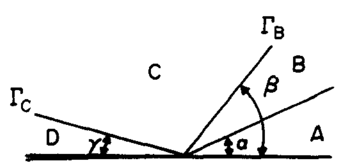

Fig. 5. An optional assembly of four angular sectors for a steady growing crack.

$$\begin{cases} \dfrac{C_1^*}{4}F_s(\bar\gamma)+\dfrac{C_2^*}{4}\bar\gamma+C_3+C_4[1-\cos 2(\bar\gamma-\beta)]+C_5\sin 2(\bar\gamma-\beta)=\dfrac{k^*}{2}[1-\cos 2\gamma], \\ \dfrac{C_1^*}{4}F_s'(\bar\gamma)+\dfrac{C_2^*}{4}+2C_4\sin 2(\bar\gamma-\beta)+2C_5\cos 2(\bar\gamma-\beta)=-k^*\sin 2\gamma, \\ C_1^*(\ln\sin\bar\gamma-F_s(\bar\gamma))+4C_4\cos 2(\bar\gamma-\beta)-4C_5\sin 2(\bar\gamma-\beta)=2k^*\cos 2\gamma, \end{cases} \tag{4.5}$$

here γ is the angle between Γ_c and the crack face. $\bar\gamma=\pi-\gamma$; the inclined angle of Γ_B with respect to x axis.

Let

$$x_1^*=C_1^*/4,\quad x_2^*=C_2^*/4.$$

From eqs. (4.4)–(4.5), it follows

$$\begin{cases} a_{11}x_1+a_{12}x_2=b_1, \\ a_{21}x_1+a_{22}x_2=b_2, \\ a_{31}x_1+a_{32}x_2=b_3, \end{cases} \tag{4.6}$$

where

$$\begin{cases} a_{11}=F_s'(\bar\gamma)+2\sin 2(\bar\gamma-\beta)[F_s(\beta)-l\text{n}\sin\beta]-\cos 2(\bar\gamma-\beta)F_s'(\beta), \\ a_{12}=1-\cos 2(\bar\gamma-\beta), \\ b_1=-2k^*\sin 2\gamma, \end{cases} \tag{4.7}$$

$$\begin{cases} a_{21}=\ln\sin\bar\gamma-F_s(\bar\gamma)-[\ln\sin\beta-F_s(\beta)]\cos 2(\bar\gamma-\beta)+\frac{1}{2}F_s'(\bar\gamma)\sin 2(\bar\gamma-\beta), \\ a_{22}=\frac{1}{2}\sin 2(\bar\gamma-\beta), \\ b_2=k^*\cos 2 \end{cases} \tag{4.8}$$

$$\begin{cases} a_{31}=F_s(\bar\gamma)-F_s(\beta)-[\ln\sin\beta-F_a(\beta)][1-\cos 2(\bar\gamma-\beta)]-\frac{1}{2}F_s'(\beta)\sin 2(\gamma-\beta), \\ a_{32}=\bar\gamma-\beta-\frac{1}{2}\sin 2(\bar\gamma-\beta), \\ b_3=\frac{1}{2}k^*[1-\bar A_1-2\cos 2\gamma]. \end{cases} \tag{4.9}$$

For a given γ, one can get the solution of $x_1, x_2, \bar A_1$ from eq. (4.6), then all coefficients $c_1^*, c_2^*, c_3, c_4, c_5$ and stress field near crack tip. But we need to check the solution in order to confirm all constraint conditions to be satisfied.

Firstly the yield constraint condition should be met in domains C and D:

$$\begin{cases} -\tau_c\leqslant\tau^{(1)}\leqslant\tau_c \\ -\tau_c\leqslant\tau^{(2)}\leqslant\tau_c. \end{cases} \tag{4.10}$$

Secondly the velocity discontinuity may occur on boundary Γ_A or Γ_B, Γ_C, but they must obey some additional constraint conditions. In the next section we will prove that

$$C_1\,\mathrm{tg}\,\beta-C_2\leqslant 0, \tag{4.11}$$

$$\beta=\mathrm{Max}\left(\varphi_0,\frac{\pi}{2}-\varphi_0\right). \tag{4.12}$$

The calculation is carried out for $\varphi_0=54.74°$. We do not find any suitable angle γ, at which the eq. (4.6) and the constraint conditions (4.10), (4.11) are simultaneously satisfied. Hence this option is ruled out.

The other option is shown in Fig. 6. The assembly of sectors involves five angular sectors. The domains A, B are constant stress zones; domains C, D are elastic unloading zones; domain E is a secondary plastic zone. The velocity discontinuity may occur on boundary Γ_A, Γ_B and Γ_C.

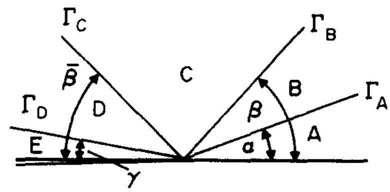

Fig. 6. Another optional assembly of five angular sectors for a steady growing crack.

As pointed out by Rice[3], the velocity discontinuity ray must have the direction of slip plane traces or normal traces of slip plane. Hence we have

$$\alpha = \mathrm{Min}\left(\varphi_0, \frac{\pi}{2} - \varphi_0\right),$$

$$\beta = \mathrm{Max}\left(\varphi_0, \frac{\pi}{2} - \varphi_0\right).$$

When $\varphi_0 \geqslant (\pi/4)$, we have

$$\alpha = \frac{\pi}{2} - \varphi_0, \quad \beta = \varphi_0.$$

Thus the velocity jump is slip-type shear on Γ_B but kinking-type shear on Γ_A.

The stress function of asymptotic field can be expressed as

$$\phi = r^2 F(\theta), \tag{4.13}$$

$$F(\theta) = \begin{cases} \dfrac{k^*}{2}[\bar{A}_1 + \cos 2\theta], & \text{in domain A and B} \\[2ex] \dfrac{C_1^*}{4} F_s(\theta) + \dfrac{C_2^*}{4}\theta + C_3 + C_4[1 - \cos(\theta - \beta)] + C_5 \sin 2(\theta - \beta), & \text{in domain C} \\[2ex] \dfrac{D_1^*}{4} F_s(\theta) + \dfrac{D_2^*}{4}\theta + D_3 + D_4[(- \cos 2(\theta - \bar{\gamma})] + D_5 \sin 2(\theta - \bar{\gamma}), & \text{in domain D} \\[2ex] \dfrac{k^*}{2}[1 - \cos 2\theta], & \text{in domain E.} \end{cases} \tag{4.14}$$

Due to continuity of stress components on Γ_B, we find

$$\begin{cases} C_3 = \dfrac{k^*}{2}(A_1 + \cos 2\beta) - \dfrac{C_1^*}{4} Fs(\beta) - \dfrac{C_2^*}{4}\beta, \\[2ex] C_4 = -\dfrac{k^*}{2}\cos 2\beta - \dfrac{C_1^*}{4}(\ln \sin \beta) - F_s(\beta), \\[2ex] C_5 = -\dfrac{k^*}{2}\sin 2\beta - \dfrac{C_1^*}{4}\cdot\dfrac{F_s'(\beta)}{2} \quad - \dfrac{C_2^*}{8}. \end{cases} \tag{4.15}$$

The continuity of stress components on Γ_D results in

$$\begin{cases} D_3 = \dfrac{k^*}{2}(1 - \cos 2\bar{\gamma}) - \dfrac{D_1^*}{4} F_s(\bar{\gamma}) - \dfrac{D_2^*}{4}\bar{\gamma}, \\[2ex] D_4 = \dfrac{k^*}{2}\cos(2\bar{\gamma}) - \dfrac{D_1^*}{4}[\ln \sin \bar{\gamma} - F_s(\bar{\gamma})], \\[2ex] D_5 = \dfrac{k^*}{2}\sin 2\bar{\gamma} - \dfrac{D_1^*}{4}\cdot\dfrac{F_s'(\gamma)}{2} - \frac{1}{8}D_2^*. \end{cases} \tag{4.16}$$

From the stress continuity on Γ_C, it follows that

$$\frac{(C_1^* - D_1^*)}{4} F_s(\bar{\beta}) + \frac{(C_2^* - D_2^*)}{4} \bar{\beta} + (C_3 - D_3) + C_4[1 - \cos 2(\bar{\beta} - \beta)] + C_5 \sin 2(\bar{\beta} - \beta)$$
$$- D_4[1 - \cos 2(\bar{\beta} - \bar{\gamma})] + D_5 \sin 2(\bar{\beta} - \bar{\gamma}) = 0, \tag{4.17}$$

$$\tfrac{1}{4}(C_1^* - D_1^*)F_s'(\bar{\beta}) + \tfrac{1}{4}(C_2^* - D_2^*) + 2C_4 \sin 2(\bar{\beta} - \beta) + 2C_5 \cos 2(\bar{\beta} - \beta)$$
$$- 2D_4 \sin 2(\bar{\beta} - \gamma) - 2D_5 \cos 2(\bar{\beta} - \gamma) = 0, \tag{4.18}$$

$$(C_1^* - D_1^*)(\ln \sin \bar{\beta} - F_s(\bar{\beta})) + 4C_4 \cos 2(\bar{\beta} - \beta) - 4C_5 \sin 2(\bar{\beta} - \beta)$$
$$- 4D_4 \cos 2(\bar{\beta} - \bar{\gamma}) - 4D_5 \sin 2(\bar{\beta} - \bar{\gamma}) = 0. \tag{4.19}$$

On Γ_c, velocity jump occurs, we have

$$\tau^{(2)} = -\tau_c, \quad (\text{or } \tau^{(1)} = -\tau_c). \tag{4.20}$$

On the other hand, the normal velocity component must be continuous across Γ_c, it yields (the proof will be given in the next section)

$$C_1^* \sin \bar{\beta} - C_2^* \cos \bar{\beta} = D_1^* \sin \bar{\beta} - D_2^* \cos \bar{\beta}. \tag{4.21}$$

Hence we have

$$\begin{cases} D_1^* = D_0^* \cos \bar{\beta} + C_1^*, \\ D_2^* = D_0^* \sin \bar{\beta} + C_2^*. \end{cases} \tag{4.22}$$

In the next section we will find the following constraint condition:

$$D_0^* \leqslant 0. \tag{4.23}$$

Let

$$x_1 = \frac{C_1^*}{4\tau_c}, \quad x_2 = \frac{C_2^*}{4\tau_c}, \quad x_3 = \frac{D_0^*}{4\tau_c}.$$

From eqs (4.17)–(4.19), (4.20) and (4.23), it follows

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{cases} \tag{4.24}$$

here

$$\begin{cases} a_{11} = F_s'(\bar{\beta}) + 2[\ln \sin \beta - F_s(\beta)]\sin 4\beta - F_s'(\beta) \cos 4\beta, \\ a_{12} = 1 - \cos 4\beta, \quad b_1 = -2, \\ a_{21} = F_s'(\bar{\beta}) + 2[\ln \sin \bar{\gamma} - F_s(\bar{\gamma})]\sin 2(\beta - \gamma) - F_s'(\bar{\gamma})\cos 2(\beta - \gamma), \\ a_{22} = 1 - \cos 2(\beta - \gamma), \quad b_2 = 0, \\ a_{31} = d_{11} - d_{21}, \quad a_{32} = d_{22} - d_{12}, \\ a_{33} = d_{22} \sin \beta + d_{21} \cos \beta, \\ a_{23} = a_{22} \sin \beta - a_{21} \cos \beta, \\ b_3 = \cos 2\beta / \sin 2\varphi_0. \end{cases} \tag{4.25}$$

$$\begin{cases} d_{11} = [\ln \sin \bar{\beta} - F_s(\bar{\beta})] - [\ln \sin \beta - F_s(\beta)]\cos 4\beta - \frac{1}{2}F_s'(\beta)\sin 4\beta, \\ d_{12} = \frac{1}{2}\sin 4\beta, \qquad d_{22} = \frac{1}{2}\sin 2(\beta - \gamma), \\ d_{21} = [\ln \sin \bar{\beta} - F_s(\bar{\beta})] - [\ln \sin \bar{\gamma} - F_s(\bar{\gamma})]\cos 2(\beta - \gamma) - \frac{1}{2}F_s'(\bar{\gamma})\sin 2(\beta - \gamma). \end{cases} \tag{4.26}$$

For a given angle γ, the solution of x_1, x_2, x_3 is obtained from eq. (4.24). Then get all coefficients. Check now the following constraint conditions:

On domains C, D, we have

$$\begin{cases} |\tau^{(1)}| \leqslant \tau_c, \\ |\tau^{(2)}| \leqslant \tau_c. \end{cases} \tag{4.27}$$

For coefficients C_1^*, C_2^*, D_0^*, we have

$$\begin{cases} C_1^* \operatorname{tg}\beta - C_2^* \leqslant 0, \\ D_0^* \leqslant 0. \end{cases} \tag{4.28}$$

The calculation is carried out for $\varphi_0 = 54.74°$. Thus $\alpha = 35.26°$, $\beta = \varphi_0 = 54.74°$.

The calculation shows that the correct solution is obtained for each γ when $9.009° \leqslant \gamma \leqslant 12.7°$. The solutions satisfy all asymptotic equations and full constraint conditions.

For $\gamma = 12.7°$, the present result is completely coincident with the result given by Rice[3].

The stress distribution along circumferential direction is shown in Fig. 7. The velocity field is continuous across the boundary Γ_A.

Hence the difference between domains A and B is only the strain rate field. There is a concentrated shear strain on domain B.

The resolved shear stresses $\tau^{(1)}$ and $\tau^{(2)}$ are shown in Fig. 8, for $\gamma = 12.7°$. On the front crack, $\tau^{(1)} = \tau^{(2)} = \tau_c$; near the crack face, $\tau^{(1)} = \tau^{(2)} = -\tau_c$. It means that, the primary and conjugate slip systems are simultaneously active on domains A and B, but sliding in opposite directions on domain E.

Figure 9 shows the stress distribution along circumferential direction for $\gamma = 9.009°$. Figure 10 shows the resolved shear stress for $\gamma = 9.009°$.

The calculation confirms that the boundaries Γ_A, Γ_B and Γ_C have a velocity jump.

5. VELOCITY AND DEFORMATION FIELDS NEAR GROWING CRACK TIP

The analysis of velocity field is an essential part for asymptotic solution of a steadily growing crack.

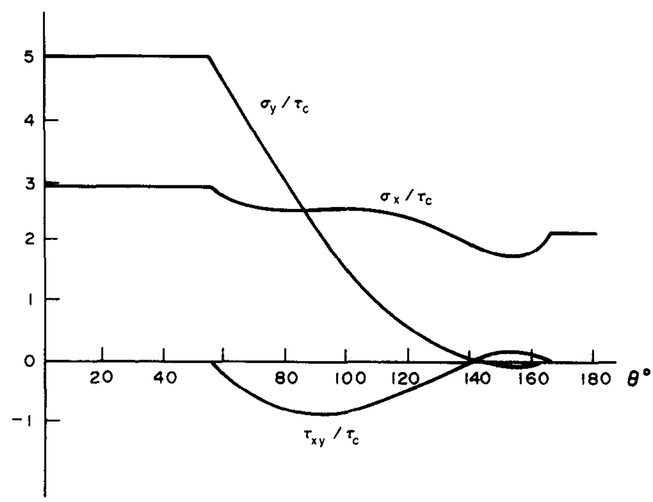

Fig. 7. Stress distribution at the crack tip along the circumferential direction for the elastic–perfectly plastic crystals undergoing double slips in the case $\gamma = 12.7°$.

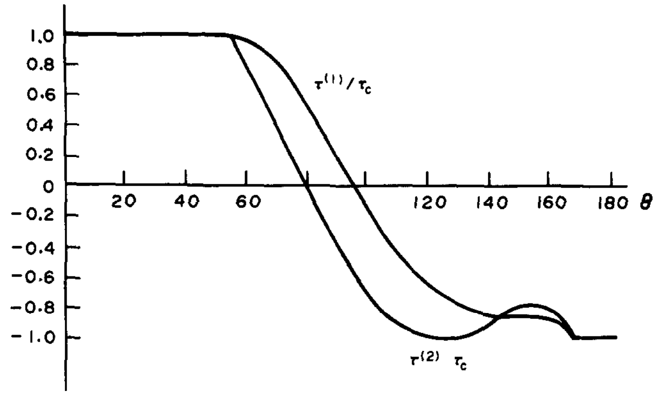

Fig. 8. Resolved shear stresses at the crack tip along the circumferential direction for the elastic–perfectly plastic crystals the same as Fig. 7.

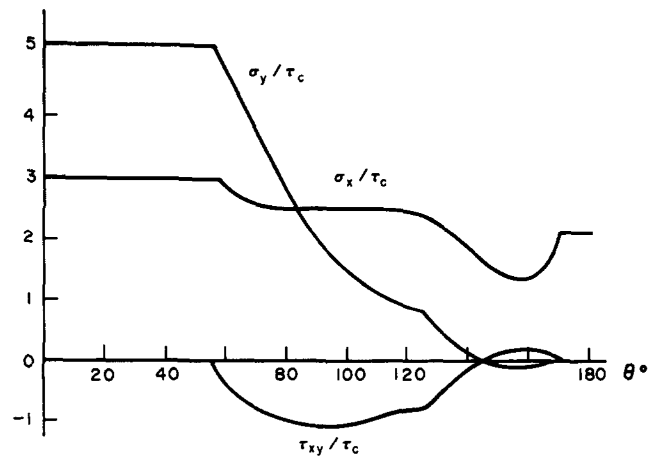

Fig. 9. Stress distribution at the crack tip for the crystals the same as Fig. 7 in the case $\gamma = 9.007°$.

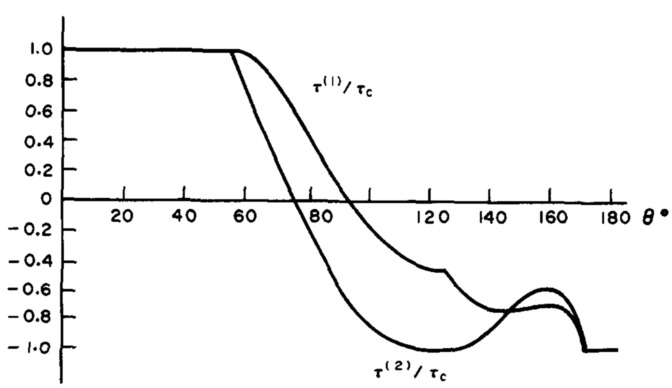

Fig. 10. Resolved shear stresses at the crack tip for the crystals the same as Fig. 9 in the case $\gamma = 9.007°$.

For a fixed material point, the polar coordinates (r, θ) are the functions of the crack length a and the material coordinates x_1, x_2. We have

$$\begin{cases} x_1 = \Delta a + r\cos\theta, \\ x_2 = \Delta a + r\sin\theta, \end{cases} \tag{5.1}$$

$$\Delta a = a - a_0$$

$$\begin{cases} \left(\dfrac{\partial r}{\partial a}\right)_M = -\cos\theta, \\ \left(\dfrac{\partial \theta}{\partial a}\right)_M = \frac{1}{2}\sin\theta. \end{cases} \tag{5.2}$$

5.1. *Velocity fields in domain A and B*

We have

$$\begin{cases} V_r = H(\theta)\ln\left(\dfrac{R}{r}\right) - f_0'(\theta), \\ V_\theta = H(\theta)\ln\left(\dfrac{R}{r}\right) - H(\theta) + f_0(\theta), \end{cases} \tag{5.3}$$

$$\begin{cases} \tilde{\gamma}_1 = -\{\sin 2(\theta + \varphi_0)(f_0'' + f_0) + \cos 2(\theta + \varphi_0)\cdot 2h(\theta)\}/\sin 4\varphi_0, \\ \tilde{\gamma}_2 = -\{\sin 2(\theta + \varphi_0)(f_0'' + f_0) + \cos 2(\theta + \varphi_0)\cdot 2h(\theta)\}/\sin 4\varphi_0, \end{cases}$$

$$\begin{cases} h(\theta) = A_1\cos\theta, \\ H(\theta) = A_1\sin\theta, \end{cases} \quad \text{in domain A} \tag{5.5}$$

$$\begin{cases} h(\theta) = B_1\cos\theta + B_2\sin\theta, \\ H(\theta) = B_1\sin\theta - B_2\cos\theta. \end{cases} \quad \text{in domain B} \tag{5.6}$$

At $\theta = 0$,

$$\tilde{\gamma}_1 = -\{\sin 2\varphi_0(f_0'' + f_0) + \cos 2\varphi_0\cdot 2h(0)\}/\sin 4\varphi_0,$$
$$\tilde{\gamma} = -\{-\sin 2\varphi_0(f_0'' + f_0) + \cos 2\varphi_0\cdot 2h(0)\}/\sin 4\varphi_0.$$

Due to $\tilde{\gamma}_1 \geqslant 0, \tilde{\gamma} \geqslant 0$, it follows

$$2A_1 = 2h(0) \leqslant -|\mathrm{tg}\, 2\varphi_0(f_0'' + f_0)_{\theta=0}| \leqslant 0. \tag{5.7}$$

The normal velocity component is continuous across the boundary Γ_A, it results in

$$\begin{cases} B_1 = B_0\cos\alpha + A_1, \qquad B_2 = B_0\sin\alpha, \\ h(\theta) = B_0\cos(\theta - \alpha) + A_1\cos\theta, \\ H(\theta) = B_0\sin(\theta - \alpha) + A_1\sin\theta. \end{cases} \tag{5.8}$$

The plastic deformation work rate is non-negative when a material element passes through the boundary Γ_A. Thus we have

$$\tau_{ns}[V_s]\Gamma_A \geqslant 0.$$

It results in $B_0 \leqslant 0$.

When $\varphi_0 \geqslant (\pi/4)$, we have $\alpha = (\pi/2) - \varphi_0, \beta = \varphi_0$.

In domain A, $0 \leqslant \theta \leqslant (\pi/2) - \varphi_0$

$$\sin 2(\theta - \varphi_0) \leqslant 0, \quad \sin 2(\theta + \varphi_0) \geqslant 0.$$

Let

$$f_0'' + f_0 = 2h(\theta)\left[\delta - \frac{\cos 2(\theta - \varphi_0)}{\sin 2(\theta - \varphi_0)}\right]. \tag{5.9}$$

Substitute into eq. (5.4), we find

$$\tilde{\gamma}_2 = -\delta \cdot 2h(\theta) \cdot \sin 2(\theta - \varphi_0)/\sin 4\varphi_0.$$

Noting $h(\theta) = A_1 \cos\theta \leqslant 0$, while $A_1 \neq 0$, we obtain $\delta \geqslant 0$

$$\begin{aligned} \tilde{\gamma}_1 &= -\left\{\frac{-\sin 4\varphi_0}{\sin 2(\theta - \varphi_0)} + \delta \sin 2(\theta + \varphi_0)\right\} 2h(\theta)/\sin 4\varphi_0 \\ &= 2\,h(\theta)\left\{\frac{1}{\sin 2(\theta - \varphi_0)} + \delta\frac{\sin 2(\theta + \varphi_0)}{\sin 4\varphi_0}\right\} \geqslant 0. \end{aligned} \tag{5.10}$$

Hence if $A_1 \neq 0, \delta \geqslant 0$ and eq. (5.9) holds true, we confirm that $\tilde{\gamma}_1 \geqslant 0, \tilde{\gamma}_2 \geqslant 0$.

When $A_1 = 0$, we arrive at

$$\begin{cases} f_0'' + f_0 = 0, \\ \tilde{\gamma}_1 = \tilde{\gamma}_2 = 0. \end{cases} \quad \text{in domain A}$$

Consider now the velocity field in domain B. We have

$$B_0 \leqslant 0, \quad h(\theta) \leqslant 0.$$

When B_0 or A_1 is not equal to zero, assume that

$$f_0'' + f_0 = 2h(\theta)\left[\delta - \frac{\cos 2(\theta - \varphi_0)}{\sin 2(\theta - \varphi_0)}\right].$$

If $\delta \geqslant 0$, thus $\tilde{\gamma}_2 \geqslant 0$. We find

$$\tilde{\gamma}_1 = 2h(\theta)\left\{\frac{1}{\sin 2(\theta - \varphi_0)} + \delta\frac{\sin 2(\theta + \varphi_0)}{\sin 4\varphi_0}\right\}.$$

In order to keep $\tilde{\gamma} \geqslant 0, \delta$ must be small enough.

5.2. *Velocity field in elastic unloading zone*

The stress rate function in domain C is

$$\dot{\phi} = \dot{a} r f(\theta), \tag{5.11}$$

$$f(\theta) = F'(\theta)\sin\theta - 2F\cos\theta. \tag{5.12}$$

It results in

$$\begin{cases} \dot{\sigma}_\theta = \dot{\tau}_{r\theta} = 0, \\ \dot{\sigma}_r = \dfrac{\dot{a}}{r}[f'' + f]. \end{cases} \tag{5.13}$$

Using above equations and formula (4.1), we find

$$\dot{\sigma}_r = \frac{\dot{a}}{r}[C_1^* \cos\theta + C_2^* \sin\theta]. \tag{5.14}$$

The elastic constitutive equations have the form

$$\begin{cases} D_r = \dfrac{(1-\nu)}{2\mu}\dot{\sigma}_r = \dfrac{(1-\nu)}{2\mu}\dfrac{\dot{a}}{r}[C_1^* \cos\theta + C_2^* \sin\theta], \\ D_\theta = -\dfrac{\nu}{2\mu}\dot{\sigma}_r = -\dfrac{\nu}{(1-\nu)}\dfrac{h(\theta)}{r}, \\ D_{r\theta} = 0, \\ h(\theta) = \dfrac{(1-\nu)}{2\mu}\dot{a}[C_1^* \cos\theta + C_2^* \sin\theta]. \end{cases} \tag{5.15}$$

From the first two formulas of eq. (5.15), it follows

$$\begin{cases} V_r = -h(\theta)\ln\left(\dfrac{R}{r}\right) - f_0'(\theta) \\ V_\theta = H(\theta)\ln\left(\dfrac{R}{r}\right) - \dfrac{v}{(1-v)}H(\theta) + f_0(\theta). \end{cases} \tag{5.16}$$

Substituting the above formula onto the third formula of eq. (5.15), we obtain

$$\frac{1}{r}\ln\left(\frac{R}{r}\right)[H'' + H] - \frac{1}{r}[f_0'' + f_0] + \frac{1}{r}\left[\frac{v}{(1-v)} - 1\right]H(\theta) = 0.$$

Hence we have

$$H'' + H = 0, \tag{5.17}$$

$$f_0'' + f_0 = \left(\frac{v}{(1-v)} - 1\right)H(\theta). \tag{5.18}$$

Apparently eq. (5.17) is met, because

$$H(\theta) = \int h(\theta)\,\mathrm{d}\theta = \int \frac{(1-v)}{2\mu}\dot{a}[C_1^*\cos\theta + C_2^*\sin\theta]\,\mathrm{d}\theta$$

$$= \frac{(1-v)}{2\mu}\dot{a}[C_1^*\sin\theta - C_2^*\cos\theta] + C_0^*.$$

Let $C_0^* = 0$, eq. (5.17) is satisfied.
Define

$$C_1 = \frac{(1-v)}{2\mu}\dot{a}C_1^*, \quad C_2 = \frac{(1-v)}{2\mu}\dot{a}C_2^*. \tag{5.19}$$

We have

$$\begin{cases} h(\theta) = C_1\cos\theta - C_2\sin\theta, \\ H(\theta) = C_1\sin\theta - C_2\cos\theta. \end{cases} \tag{5.20}$$

Comparing eq. (5.16) with eq. (5.3), one can see that the major singularity term of velocity is the same on both the elastic unloading zone and plastic zones A and B, but the nonsingularity term of velocity is a little bit different. If and only if $v = 1/2$, the nonsingularity term is the same.

Consider now the constraint condition for coefficients C_1 and C_2.

The normal velocity component has non-jump on the boundary Γ_B. Hence we have

$$C_1\sin\beta - C_2\cos\beta = B_1\sin\beta - B_2\cos\beta. \tag{5.21}$$

It follows

$$C_1\,\mathrm{tg}\,\beta - C_2 = (B_0\cos\alpha + A_1)\mathrm{tg}\,\beta - B_0\sin\alpha = B_0\sin(\beta-\alpha)/\cos\beta + A_1\,\mathrm{tg}\,\beta \leqslant 0 \tag{5.22}$$

It is the constraint condition (4.11) and (4.18).

In a similar way, we obtain the velocity field on domain D:

$$\begin{cases} h(\theta) = D_1\cos\theta + D_2\sin\theta, \\ H(\theta) = D_2\sin\theta - D_2\cos\theta, \end{cases} \tag{5.23}$$

$$\begin{cases} D_1 = \dfrac{(1-v)}{2\mu}\dot{a}D_1^*, \\ D_2 = \dfrac{(1-v)}{2\mu}\dot{a}D_2^*. \end{cases} \tag{5.24}$$

The continuity of normal velocity on Γ_c yields

$$C_1\sin\bar{\beta} - C_2\cos\bar{\beta} = D_1\sin\bar{\beta} - D_2\cos\bar{\beta}. \tag{5.25}$$

Thus

$$C_1^* \sin \bar{\beta} - C_2^* \cos \bar{\beta} = D_1^* \sin \bar{\beta} D_2^* \cos \bar{\beta}. \tag{5.26}$$

It is the formula (4.21). From eq. (5.26), we arrive at

$$\begin{cases} D_1^* = D_0^* \cos \bar{\beta} + C_1^*, \\ D_2^* = D_0^* \sin \bar{\beta} + C_2^*. \end{cases} \tag{5.27}$$

When a material element transverses across the Γ_c, the plastic work rate should be non-negative. Thus

$$D_0^* \leqslant 0. \tag{5.28}$$

6. APPLICATION ON FCC AND BCC CRYSTALS

The crystal axis coordinates for FCC and BCC crystals are shown in Fig. 11. The x_c, y_c, z_c axes are along the [100], [010], [001] directions, respectively. The fixed Cartesian coordinates $0XYZ$ used in the previous sections are also shown in Fig. 11.

The crack plane lies on (010). Crack tip is along $[10\bar{1}]$. Obviously such cracked crystals will undergo the plane strain deformation when external load is parallel to the plane $0XY$ and distributed uniformly along the Z axis direction.

The second orientation has the crack line rotated at 90° anticlockwise from the above orientation which provides also a plane strain solution.

The four slip planes of FCC crystals are (111), $(\bar{1}1\bar{1})$, $(\bar{1}11)$, $(11\bar{1})$.

There are three slip directions $[10\bar{1}]$, $[1\bar{1}0]$ and $[0\bar{1}1]$ on the slip plane (111). The equal slip along the $[1\bar{1}0]$ and $[0\bar{1}1]$ directions on the slip plane (111) will result in slip along $[1\bar{2}1]$, and yield plane strain deformation. In fact designating (111) $[1\bar{1}0]$ as a 2A slip system which have slip direction $\mathbf{m}_A^{(2)}$ and unit normal $\mathbf{n}^{(2)}$ of slip plane.

The slip system (111) [011] is designated as a 2B slip system which have slip direction $\mathbf{m}_B^{(2)}$ and unit normal $\mathbf{n}^{(2)}$ of slip plane. Both slip systems are simultaneous and have equal amounts of slip and result in the same shear rate $\dot{\gamma}^{(2)}$. Thus the associated plastic strain rate $\mathbf{D}^P$ is

$$\begin{aligned} \mathbf{D}^P &= (\mathbf{P}_A^{(2)} + \mathbf{P}_B^{(2)})\dot{\gamma}^{(2)} \\ &= \tfrac{1}{2}\{(\mathbf{m}_A^{(2)} + \mathbf{m}_B^{(2)}) \otimes \mathbf{n}^{(2)} + \mathbf{n}^{(2)} \otimes (\mathbf{m}_A^{(2)} + \mathbf{m}_B^{(2)})\}\dot{\gamma}^{(2)} \\ &= \frac{\sqrt{3}}{2}\{\mathbf{m}^{(2)} \otimes \mathbf{n}^{(2)} + \mathbf{n}^{(2)} \otimes \mathbf{m}^{(2)}\}\dot{\gamma}^{(2)} \\ &= \sqrt{3}\,\mathbf{P}^{(2)}\dot{\gamma}^{(2)}, \end{aligned} \tag{6.1}$$

fcc Crystal

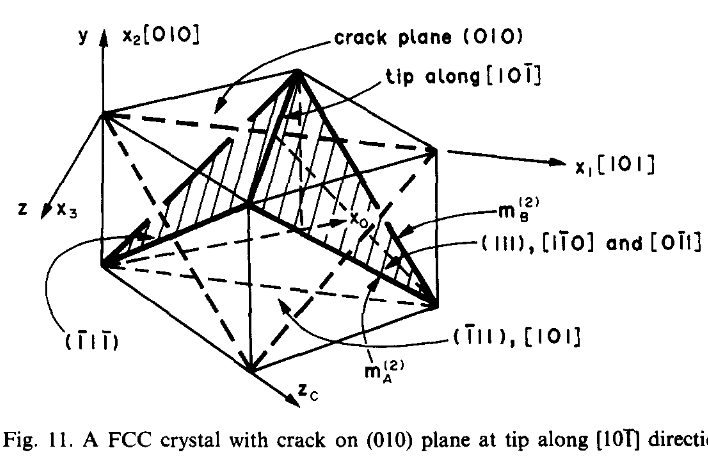

Fig. 11. A FCC crystal with crack on (010) plane at tip along $[10\bar{1}]$ direction.

where

$$\mathbf{P}^{(2)} = \tfrac{1}{2}(\mathbf{m}^{(2)} \otimes \mathbf{n}^{(2)} + \mathbf{n}^{(2)} \otimes \mathbf{m}^{(2)})$$

$$\mathbf{m}_{A}^{(2)} + \mathbf{m}_{B}^{(2)} = \sqrt{3}\,\mathbf{m}^{(2)}. \tag{6.2}$$

Similarly, for slip systems $(\bar{1}1\bar{1})$ [011] and $(\bar{1}1\bar{1})$ [110], we have

$$\mathbf{D}^{(P)} = \sqrt{3}\,\mathbf{P}^{(1)}\dot{\gamma}^{(1)}, \tag{6.3}$$

where

$$\mathbf{P}^{(1)} = \tfrac{1}{2}(\mathbf{m}^{(1)} \otimes \mathbf{n}^{(1)} + \mathbf{n}^{(1)} \otimes \mathbf{m}^{(1)}). \tag{6.4}$$

From eqs (6.1) and (6.3), we find

$$\mathbf{D}^{P} = \sqrt{3}\,(\mathbf{P}^{(1)}\dot{\gamma}^{(1)} + \mathbf{P}^{(2)}\dot{\gamma}^{(2)}). \tag{6.5}$$

Comparing eq. (6.4) with eq. (2.9), it can be obtained that the problem of FCC crystals discussed here is equivalent to the corresponding problem of double slip crystals, if we take $\sqrt{3}\,\dot{\gamma}^{(1)}$ and $\sqrt{3}\,\dot{\gamma}^{(2)}$ considered here to be equal to $\dot{\gamma}^{(1)}$ and $\dot{\gamma}^{(2)}$ of that considered on double slip crystals.

Exactly speaking, the equal amounts of slip on both slip systems $(\bar{1}11)$ [101] and $(11\bar{1})$ [101] will also result in plane plastic strain:

$$\mathrm{D}^{P} = \tfrac{1}{2}\{\overset{*}{\mathbf{m}}^{(3)} \otimes (\overset{*}{\mathbf{n}}_{A}^{(3)} + \mathbf{n}_{B}^{(3)}) + (\overset{*}{\mathbf{n}}_{A}^{(3)} + \mathbf{n}_{B}^{(3)}) \otimes \overset{*}{\mathbf{m}}^{(3)}\} \cdot \gamma^{*(3)} = (2/\sqrt{3})\overset{*}{\mathbf{P}}^{(3)}\dot{\gamma}^{*(3)} \tag{6.6}$$

where

$$\overset{*}{\mathbf{P}}^{(3)} = \tfrac{1}{2}\{\overset{*}{\mathbf{m}}^{(3)} \otimes \overset{*}{\mathbf{n}}^{(3)} + \overset{*}{\mathbf{n}}^{(3)} \otimes \overset{*}{\mathbf{m}}^{(3)}\}.$$

However, if the stress state does not attend yielding on those slip systems, then the crack problem of FCC crystals considered here is actually equivalent to the corresponding problem of double slip crystals.

The yield condition is now discussed. The resolved shear stresses of slip systems 2_A and 2_B are

$$\tau_{A}^{(2)} = \boldsymbol{\sigma}: \mathbf{P}_{A}^{(2)}, \quad \tau_{B}^{(2)} = \boldsymbol{\sigma}: \mathbf{P}_{B}^{(2)}.$$

When both slip systems are simultaneously active, we have

$$\boldsymbol{\sigma}: \mathbf{P}_{A}^{(2)} = \boldsymbol{\sigma}: \mathbf{P}_{B}^{(2)} = \tau,$$

where τ is the critical shear stress of slip systems 2_A and 2_B. We find

$$\boldsymbol{\sigma}: \mathbf{P}^{(2)} = \tau\frac{2}{\sqrt{3}} = \tau_c. \tag{6.7}$$

It means that the critical shear stress τ_c of double slip crystals is equal to $2\sqrt{3}\,\tau$.

On the other hand, for slip systems $(\bar{1}11)$ [101] and $(11\bar{1})$ [101], we have

$$\boldsymbol{\sigma}: \overset{*}{\mathbf{P}}_{A}^{(3)} = \boldsymbol{\sigma}: \overset{*}{\mathbf{P}}_{B}^{(3)} = \tau, \tag{6.8}$$

$$\boldsymbol{\sigma}: \overset{*}{\mathbf{P}}^{(3)} = \sqrt{3}\,\tau = \tfrac{3}{2}\tau_c. \tag{6.9}$$

The eqs (6.7) and (6.9) can be represented as

$$\tau_c = \sin 2\varphi_0 \cdot \tfrac{1}{2}(\sigma_y - \sigma_x) - \gamma_{xy}\cos 2\varphi_0 = \frac{2}{\sqrt{3}}\tau, \tag{6.10}$$

$$\tau_{xy} = \sqrt{3}\,\tau = \tfrac{3}{2}\tau_c. \tag{6.11}$$

7. CONCLUSION AND DISCUSSION

Based on the above analysis, one can draw the following conclusions:

(1) The plane double slip model of crystals[4] is adequate for explaining the crack tip fields of FCC and BCC crystals in the case of plane strain.

The asymptotic solution is not unique for a stationary crack. In order to confirm the continuity of full stress components, the crack tip zone must involve an elastic angular sector. This paper presents an asymptotic solution which consists of three angular sectors and satisfies all governing equations of asymptotic fields and is full of constraint conditions. It is confirmed in this paper that there is a shear strain concentrated zone at the front of the crack, which is coincident with the experimental observation[10, 11].

(2) The asymptotic fields for growing crack of double slip crystals involve five angular sectors: two plastic zones ahead of the crack connected with the boundary across which the velocity jump usually occurs; a secondary plastic zone near crack faces; two unloading elastic zones connected with discontinuity boundary.

(3) The above results can be immediately applicated on plane strain crack problems of the FCC and BCC crystals.

For the case of the crack on the (010) plane with growth in the [101] direction and the case of a crack on the (101) plane with growth in the [010] direction, the above analyses are applicable. Thus one can get the crack tip stress and deformation fields for FCC and BCC crystals under tensile loading. This paper neglects the effect of finite deformation. The rotation, especially the rotation of crystal axes, is important which needs to be taken account in the future work.

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