HIERARCHIAL STRUCTURE OF THE SIMPLIFIED NAVIER-STOKES EQUATIONS (SNSE) AND ITS MECHANICAL CONNOTATION AND APPLICATION

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Received September 1, 1986

ABSTRACT

The hierarchial structure and mathematical property of the simplified Navier-Stokes equations (SNSE) are studied for viscous flow over a sphere and a jet of compressible fluid. All kinds of the hierarchial SNSE can be divided into three types according to their mathematical property and also into five groups according to their physical content. A multilayers structure model for viscous shear flow with a main stream direction is presented. For the example of viscous incompressible flow over a flat plate there exist three layers for both the separated flow and the attached flow; the character of the transition from the three layers of attached flow to those of separated flow is elucidated. A concept of transition layer being situated between the viscous layer and inviscid layer is introduced. The transition layer features the interaction between viscous flow and inviscid flow. The inner-outer-layers-matched SNSE proposed by the present author in the past is developed into the layers matched (LsM)-SNSE.

Key words: fluid mechanics, Navier-Stokes equations, viscous flow.

I. INTRODUCTION

Varieties of approximation formulations of the Navier-Stokes (NS) equations, such as the Euler equations, the boundary layer equations and varieties of the simplified Navier-Stokes equations (SNSE), have been widely used for analyzing and computing flow fields. H. Lomax[2] and R. W. MacCormack[3] considered that the "thin-layer" approximation NS equations, i.e. SNSE, are the first logical step for computing complicated viscous flows with engineering value. This can be justified either on a physical order of magnitude analysis or on a computational accuracy argument. If Re are large, some viscous terms included in NS equations are of small order of magnitude and especially cannot be computed correctly with the available grid resolution, there is no reason to keep them. It is therefore necessary to accept or reject reasonably the viscous terms. On the other hand, in order to obtain a stable and meaningful solution of a complication viscous flow field, it is also necessary to resolve carefully many scales of the length and the time that prove a measurement of the rate of change of the variables describing the motion of the fluid and to match properly the computational grid-spacing with the scales. The numerical simulation of separated flow demonstrated that if the adopted grid does not match with the scales of the triple-deck theory[18], it is impossible to obtain a reliable exact solution[4,5].

In this paper we shall follow the idea of [1] and further study the hierarchial structure of SNSE and the mechanical connotation and mathematical character of various SNSEs. These problems are concerned at least in the following four aspects: resolving the spatial scales of the flow field, comparing the orders of magnitude between the inertial terms and viscous terms included in NS equations, subdividing the flow field and simplifying NS equations. We take these four aspects as a whole and examine them comprehensively. A multilayer structure model for viscous shear flow with a main stream direction is presented. The concept of transition layer being situated between the viscous layer and inviscid layer is introduced.

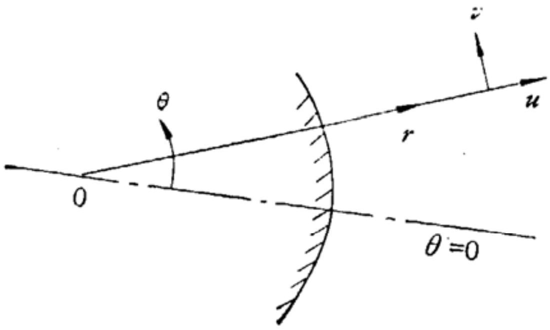


Fig. 1. Viscous flow over a sphere and spherical coordinates (r, θ, φ).

II. Spatial Scales, the Order-of-Magnitude Analysis and Hierarchial Structure of Simplified Navier-Stokes Equations

In order to study viscous compressible flow over a sphere, we take the spherical coordinates (r, θ, φ) (refer to Fig. 1). Assume that $\theta = 0$ is a symmetrical axis of the flow. The fundamental equations describing the flow are as follows:[6,7]

$$S_t \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v \sin\theta) = 0, \tag{2.1}$$

$$\mathrm{Re}^{-n_\rho+n_t} \qquad \mathrm{Re}^{-n_\rho-n_u+n_r} \qquad \mathrm{Re}^{-n_\rho-n_v+n_\theta}$$

$$S_t \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{1}{\rho r^2 \mathrm{Re}\sin\theta}\frac{\partial}{\partial\theta}\left(\mu\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$$\mathrm{Re}^{-n_u+n_t} \quad \mathrm{Re}^{-2n_u+n_r} \quad \mathrm{Re}^{-n_u-n_v+n_\theta} \qquad \mathrm{Re}^{-2n_v} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_u+2n_\theta}$$

$$+ \frac{4}{3\rho\mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial u}{\partial r}\right) - \frac{1}{3r\rho\mathrm{Re}}\left[2\left(\frac{\partial v}{\partial\theta} + v\cot\theta\right)\frac{\partial\mu}{\partial r} - \left(\mu\cot\theta + 3\frac{\partial\mu}{\partial\theta}\right)\frac{\partial v}{\partial r}\right]$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+2n_r} \qquad\qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_r+n_\theta}$$

$$- \frac{1}{\rho r^2\mathrm{Re}}\left[\mu\left(\frac{4}{3}\frac{\partial v}{\partial\theta} + \frac{7}{3}v\cot\theta\right) + \frac{\partial(\mu v)}{\partial\theta} - 4r\mu\frac{\partial u}{\partial r} + \frac{4r^2}{3}\frac{\partial}{\partial r}\left(\frac{\mu u}{r}\right)\right]$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_\theta} \qquad\qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_r}$$

$$+ 4\mu u \Big], \tag{2.2}$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-n_u}$$

$$S_t \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} + \frac{1}{\rho r} \frac{\partial p}{\partial \theta} = \frac{1}{\rho \mathrm{Re}} \left[\frac{4}{3r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\mu \sin\theta \frac{\partial v}{\partial \theta} \right) \right.$$

$$\mathrm{Re}^{-n_v+n_t} \qquad \mathrm{Re}^{-n_u-n_v+n_r} \qquad \mathrm{Re}^{-n_u-n_v} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_\theta}$$

$$\left. + \frac{\partial}{\partial r} \left(\mu \frac{\partial v}{\partial r} \right) \right] - \frac{1}{r \rho \mathrm{Re}} \left(v \frac{\partial \mu}{\partial r} - 2\mu \frac{\partial v}{\partial t} \right) - \frac{1}{\rho r^2 \mathrm{Re}} \left[\frac{\mu}{3} \left(6v + 6v\cot^2\theta - r \frac{\partial^2 u}{\partial r \partial \theta} \right. \right.$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_r} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_r} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_v} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_r+n_\theta}$$

$$\left. \left. - 8 \frac{\partial u}{\partial \theta} \right) - r \frac{\partial \mu}{\partial r} \frac{\partial u}{\partial \theta} + \frac{2}{3r} \frac{\partial \mu}{\partial \theta} \frac{\partial u}{\partial r} + \frac{2v}{3} \frac{\partial}{\partial \theta} (\mu \cot\theta) - \frac{2}{3} u \frac{\partial \mu}{\partial \theta} \right], \tag{2.3}$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_r+n_\theta} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_\theta} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_\theta}$$

$$S_t \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \theta} + \frac{p}{\rho} \mathrm{div}\vec{v} - S_t \frac{\partial p}{\partial t}$$

$$\mathrm{Re}^{-n_T+n_t} \qquad \mathrm{Re}^{-n_T-n_u+n_r} \qquad \mathrm{Re}^{-n_p+n_\rho-n_u+n_r}$$

$$= \frac{1}{\rho \mathrm{Re} P_r} \left[\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\lambda \sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial r} \left(\lambda \frac{\partial T}{\partial r} \right) \right]$$

$$\mathrm{Re}^{-1+n_\rho-n_\lambda-n_T+2n_\theta} \qquad \mathrm{Re}^{-1+n_\rho-n_\lambda-n_T+2n_r}$$

$$+ \frac{\mu}{\rho \mathrm{Re}} \left[\left(\frac{\partial v}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2 \right] - \frac{2}{r \rho \mathrm{Re}} \left(\mu v \frac{\partial v}{\partial r} - \frac{\lambda}{P_r} \frac{\partial T}{\partial r} \right)$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-2n_v+2n_r} \quad \mathrm{Re}^{-1+n_\rho-n_\mu-2n_v+n_r} \quad \mathrm{Re}^{-1+n_\rho-n_\lambda-n_T+n_r}$$

$$+ \frac{\mu}{\rho \mathrm{Re}} \left[\frac{4}{3r^2} \left(\frac{\partial v}{\partial \theta} \right)^2 + v^2 \cot^2\theta - v \cot\theta \frac{\partial v}{\partial \theta} + u^2 + u \frac{\partial v}{\partial \theta} + uv \cot\theta \right.$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-2n_v+2n_\theta} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-2n_u} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-n_u-n_v+n_\theta}$$

$$- \frac{4}{3r} \frac{\partial u}{\partial r} \left(2u + S_t \frac{\partial v}{\partial t} + v \cot\theta \right) + \frac{v^2}{r} + \frac{4}{3} \left(\frac{\partial u}{\partial r} \right)^2$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-2n_u+n_r} \qquad \mathrm{Re}^{-1+n_\rho-n_\mu-2n_v}$$

$$\left. + \frac{2}{r} \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial r} - \frac{2v}{r^2} \frac{\partial u}{\partial \theta} \right], \tag{2.4}$$

$$\mathrm{Re}^{-1+n_\rho-n_\mu-2n_u+2n_r}$$

$$p = \frac{\gamma - 1}{\gamma} \rho T, \tag{2.5}$$

$$\mathrm{Re}^{-n_p} \qquad \mathrm{Re}^{-n_\rho-n_T}$$

where t is the time, ρ the density, p the pressure, T the temperature, μ the coefficient of viscosity, λ the coefficient of heat conductivity, u and v are the velocity components in the r- and θ-directions, respectively, r and θ are the coordinate variables. All of the above quantities are nondimensional. The reference values of these quantities are f, ρ_∞, $\rho_\infty U_\infty^2$, U_∞^2/C_p, μ_∞, λ_∞, U_∞ and L respectively. f is the characteristic vibration frequency of the body, L, the characteristic length of the body, γ the ratio of specific heats. C_p and C_V are the specific heats at constant pressure and constant volume, respectively. The subscript ∞ denotes the free stream condition.

The expressions of the four nondimensional parameters S_t, Re, P_r and γ are as follows:

$$S_t=\frac{fL}{U_\infty},\quad \mathrm{Re}=\frac{\rho_\infty U_\infty L}{\mu_\infty},\quad P_r=\frac{C_P\mu}{\lambda},\quad \gamma=\frac{C_P}{C_V}. \tag{2.6}$$

And assume that

$$\mu=\left(\frac{T}{T_\infty}\right)^{\omega_1},\quad \lambda=\left(\frac{T}{T_\infty}\right)^{\omega_2},\quad (0.5\leqslant\omega_1,\ \omega_2\leqslant 1). \tag{2.7}$$

Analysis. If the Reynold's numbers Re is larger than unity, the length scale of the viscous flow region in the θ-direction is much larger than that in the r-direction in the neighbourhood of the solid wall except the vicinity of the stagnation point and can be expressed as

$$r\theta \backsim \mathrm{Re}^{-n_\theta},\quad r\backsim \mathrm{Re}^{-n_r},\quad \mathrm{Re}^{-n_\theta}>\mathrm{Re}^{-n_r},\quad (n_r>n_\theta\geqslant 0), \tag{2.8}$$

in which we have assumed that the curvature radius of the spherical surface is of the order of unity and therefore it is much larger than the thickness of the viscous region. We point out emphatically that the length scales in the r- and θ-directions do not satisfy the relationship (2.8) in the vicinity of the stagnation point. The time scale and the order of magnitude of all the variables can be expressed similarly as

$$\begin{aligned} &t\backsim \mathrm{Re}^{-n_t},\quad u\backsim \mathrm{Re}^{-n_u},\quad v\backsim \mathrm{Re}^{-n_v},\quad p\backsim \mathrm{Re}^{-n_p},\\ &T\backsim \mathrm{Re}^{-n_T},\quad \rho\backsim \mathrm{Re}^{-n_\rho},\quad \mu\backsim \mathrm{Re}^{-n_\mu},\quad \lambda\backsim \mathrm{Re}^{-n_\lambda}. \end{aligned} \tag{2.9}$$

In Relationships (2.8) and (2.9), n_s may be called the exponent of both the scales and the order of magnitude of the variables, the subscript s denotes any of t, r, θ, u, v, p, T, ρ, μ and λ.

From the equation of continuity (2.1), equation of state (2.5) and the relationship of $\frac{v}{r}\frac{\partial v}{\partial\theta}\backsim\frac{1}{\rho r}\frac{\partial p}{\partial\theta}$, we can deduce the following results

$$n_t=-n_u+n_r=-n_v+n_\theta,\quad n_p=n_\rho+n_T,\quad 2n_v=n_p-n_\rho. \tag{2.10}$$

Using Relationships (2.8)—(2.10), we can easily find the orders of magnitude of each term in Eqs. (2.1)—(2.5), which are also indicated under the corresponding terms. The order-of-magnitude exponents of all the inertial terms in the NS equations (2.2) and (2.3) satisfy the following relationships

$$\begin{aligned} &(-n_v+n_t),\ (-2n_v+n_\theta),\ (-n_u-n_v+n_r),\ (-n_p+n_\rho+n_\theta)\geqslant(-2n_v)>\\ &\quad(-n_u+n_t),\ (-2n_u+n_r),\ (-n_u-n_v+n_\theta)\geqslant(-n_u-n_v). \end{aligned} \tag{2.11}$$

The order-of-magnitude exponents of all the diffusion terms including the viscous and heat conduction terms satisfy the following relationships

$$\begin{aligned} &(-n_v+2n_r)>(-n_u+2n_r),\ (-n_v+n_r+n_\theta)>(-n_v+n_r)>(-n_v+2n_\theta)>\\ &\quad(-n_v+n_\theta),\ (-n_u+n_r)>(-n_v)>(-n_u+2n_\theta)>(-n_u+n_\theta)>(-n_u), \end{aligned} \tag{2.12}$$

in which $(-1+n_\rho-n_\mu)$ included in all the order-of-magnitude exponents has been omitted for brevity and it is assumed that $n_r\geqslant 3n_\theta$. For the case of $n_\theta<n_r<3n_\theta$, an inequality similar to Relationship (2.12) can be easily found.

For the case where the maximum inertial term and the maximum viscous term have the same order of magnitude, i.e. $u\frac{\partial v}{\partial r} \backsim \frac{1}{\rho \mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right)$, we neglect the small order of magnitude terms from Eqs. (2.2) to (2.4) in sequence of Inequalities (2.11) and (2.12) and then obtain nine kinds of SNSE. These nine kinds of SNSE are as follows.

1) The boundary layer equations (BLE). When all the terms in Eqs. (2.1)—(2.5) are retained up to the order of magnitude of $\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_r}$, we obtain the boundary layer equations (BLE). BLE is as follows:

$$\frac{\partial p}{\partial r} = 0, \tag{2.13}$$

$$S_t\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{1}{\rho r}\frac{\partial p}{\partial \theta} = \frac{1}{\rho \mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right), \tag{2.14}$$

$$S_t\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} + \frac{p}{\rho}\operatorname{div}\boldsymbol{v} - S_t\frac{\partial p}{\partial t}$$
$$= \frac{1}{\rho \mathrm{Re} P_r}\frac{\partial}{\partial r}\left(\lambda\frac{\partial T}{\partial r}\right) + \frac{\mu}{\rho \mathrm{Re}}\left(\frac{\partial v}{\partial r}\right)^2. \tag{2.15}$$

The equation of continuity and equation of state are always Eqs. (2.1) and (2.5) for any of SNSE; therefore, we will not write them.

2) Modified boundary layer equations (MBLE). When all the terms in Eqs. (2.1)—(2.5) are retained up to the order of magnitude of Re^{-2n_v}, we obtain a modified boundary layer equations (MBLE). The difference between MBLE and BLE is only in the momentum equation in the r-direction. The normal momentum equation of MBLE is

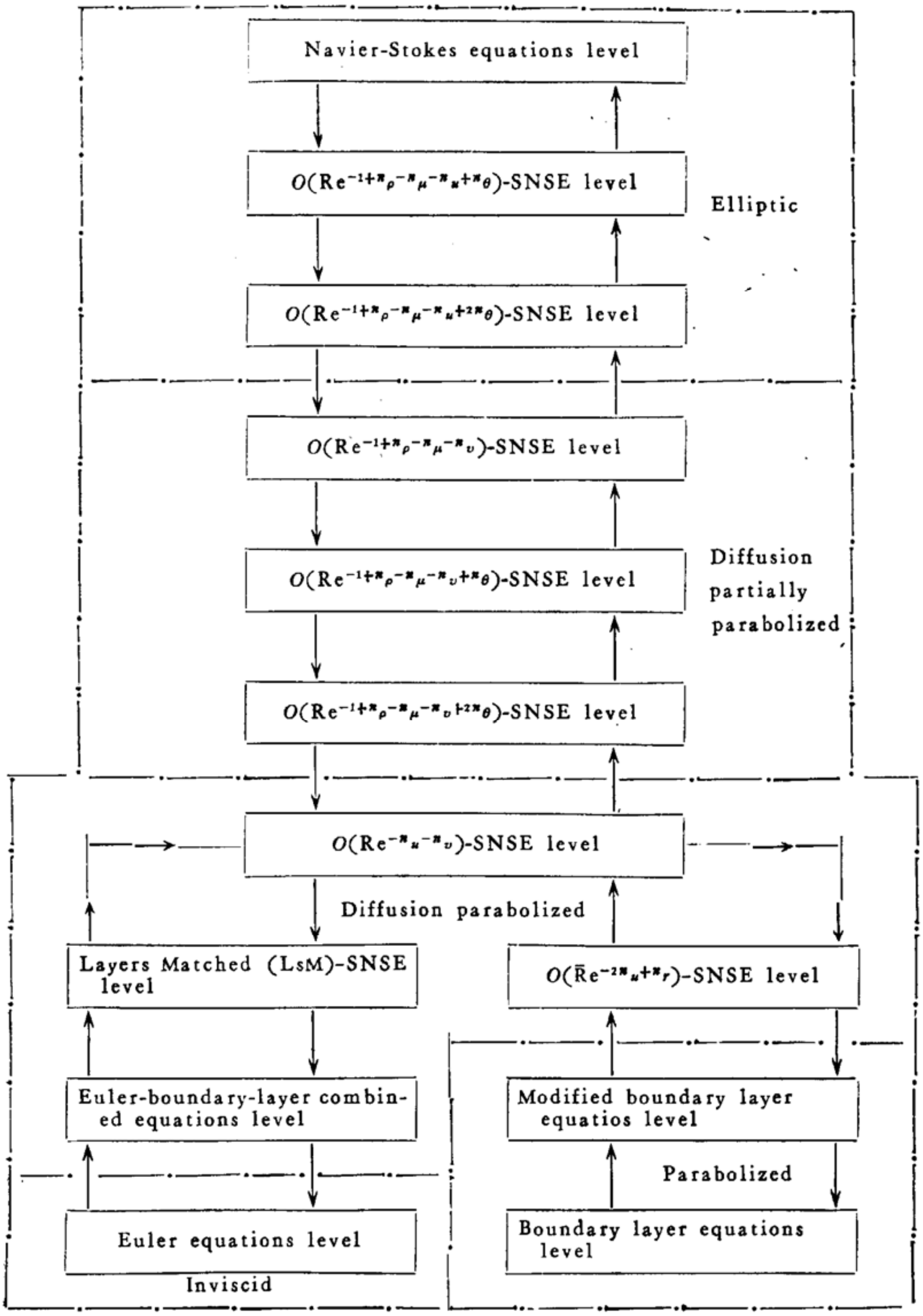
$$-\frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0. \tag{2.16}$$

3) $O(\mathrm{Re}^{-2n_u+n_r})$-SNSE. This SNSE is obtained by retaining all the terms in Eqs. (2.1)—(2.5) up to the order of magnitude of $\mathrm{Re}^{-2n_u+n_r}$. We omit them for brevity.

4) $O(\mathrm{Re}^{-n_u-n_v})$-SNSE. This SNSE is obtained by retaining all the terms in Eqs. (2.1)—(2.5) up to the order of magnitude of $\mathrm{Re}^{-n_u-n_v}$, i.e. to the order of $\mathrm{Re}^{-1+n_\rho-n_\mu-n_v-n_r}$. The left-hand sides of $O(\mathrm{Re}^{-n_u-n_v})$-SNSE are consistent with those of Eqs. (2.2)—(2.4), that is to say, all of the terms in the Euler equations are retained. The right-hand sides (RHS) of this SNSE are as follows:

$$\mathrm{RHS}_r = \frac{4}{3\rho \mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial u}{\partial r}\right) - \frac{1}{3r\rho \mathrm{Re}}\left[2\left(\frac{\partial v}{\partial \theta} + v\cot\theta\right)\frac{\partial \mu}{\partial r}\right.$$
$$\left. - \left(\mu\cot\theta + 3\frac{\partial \mu}{\partial \theta}\right)\frac{\partial v}{\partial r}\right], \tag{2.17}$$

$$\mathrm{RHS}_\theta = \frac{1}{\rho \mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right) - \frac{1}{r\rho \mathrm{Re}}\left(v\frac{\partial \mu}{\partial r} - 2\mu\frac{\partial v}{\partial r}\right), \tag{2.18}$$



$$\frac{uv}{r} \geqslant \frac{1}{\rho \mathrm{Re}} \frac{\partial}{\partial r}\left(\mu \frac{\partial v}{\partial r}\right) \qquad u \frac{\partial v}{\partial r} \sim \frac{1}{\rho \mathrm{Re}} \frac{\partial}{\partial r}\left(\mu \frac{\partial v}{\partial r}\right)$$

Fig. 2. Hierarchial structure of the Simplified Navier-Stokes Equations for a viscous compressible flow over a sphere (length scales in the r- and θ-directions: $r \sim \mathrm{Re}^{-n_r}$, $r\theta \sim \mathrm{Re}^{-n_\theta}$; $n_r > n_\theta \geqslant 0$).

$$\mathrm{RHS}_T = \frac{1}{\rho \mathrm{Re} P_r} \frac{\partial}{\partial r}\left(\lambda \frac{\partial T}{\partial r}\right) + \frac{\mu}{\rho \mathrm{Re}}\left(\frac{\partial v}{\partial r}\right)^2 - \frac{2}{r \rho \mathrm{Re}}\left(\mu v \frac{\partial v}{\partial r} - \frac{\lambda}{P_r} \frac{\partial T}{\partial r}\right), \tag{2.19}$$

where the subscripts r, θ and T denote the momentum equations in the r- and θ-

directions and the equation of energy, respectively. Such marks will be used below.

5) $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_\theta})$-SNSE and other four kinds of SNSE with much higher accuracy, i.e. $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_\theta})$-SNSE, $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v})$-SNSE, $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+2n_{\theta\theta}})$-SNSE and $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_\theta})$-SNSE, contain all of the terms in the Euler equations. Their right-hand sides are omitted here for the sake of brevity.

6) $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u})$-SNSE. When all the terms in Eqs. (2.1)—(2.5) are retained up to the order of magnitude of $\mathrm{Re}^{-l+n_\rho-n_\mu-n_u}$, we obtain the NS equations. Thus we see that the inner hierarchy from BLE to NS equations includes ten levels and nine kinds of SNSE (refer to Fig. 2).

For the case where the maximum order of magnitude of viscous terms is comparable to the minimum order of magnitude of inertial term, i. e.

$$\frac{uv}{r} \geqslant \frac{1}{\rho\mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right),$$

we neglect the small order of magnitude terms from Eqs. (2.1) to (2.5) in sequence of the inequalities (2.11) and (2.12) and then obtain nine kinds of SNSE, which are as follows:

1) The Euler equations. When all the terms in Eqs. (2.1)—(2.5) are retained up to the order of magnitude of $\mathrm{Re}^{-n_u-n_v}$, we obtain the Euler equations which are not written down for the sake of brevity.

2) Euler-boundary-layer combined equations. This SNSE is obtained by retaining all the terms in Eqs. (2.1)—(2.5) up to the order of magnitude of $\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_r}$, which has the following right-hand sides

$$\mathrm{RHS}_r = 0, \tag{2.20}$$

$$\mathrm{RHS}_\theta = \frac{1}{\rho\mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right), \tag{2.21}$$

$$\mathrm{RHS}_T = \frac{1}{\rho\mathrm{Re}P_r}\frac{\partial}{\partial r}\left(\lambda\frac{\partial T}{\partial r}\right) + \frac{\mu}{\rho\mathrm{Re}}\left(\frac{\partial v}{\partial r}\right)^2. \tag{2.22}$$

3) Layers matched (LsM)-SNSE. This SNSE is obtained by retaining all the terms in Eqs. (2.1)—(2.5) up to the order of magnitude of $\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+2n_r}$. The LsM-SNSE has the following right-hand sides

$$\mathrm{RHS}_r = \frac{4}{3\rho\mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial u}{\partial r}\right) - \frac{1}{3\rho r\,\mathrm{Re}}\left[2\left(\frac{\partial v}{\partial \theta} + v\cot\theta\right)\frac{\partial \mu}{\partial r} - \left(\mu\cot\theta + 3\frac{\partial\mu}{\partial\theta}\right)\frac{\partial v}{\partial r}\right], \tag{2.23}$$

$$\mathrm{RHS}_\theta = \frac{1}{\rho\mathrm{Re}}\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right), \tag{2.24}$$

$$\mathrm{RHS}_T = \frac{1}{\rho\mathrm{Re}P_r}\frac{\partial}{\partial r}\left(\lambda\frac{\partial T}{\partial r}\right) + \frac{\mu}{\rho\mathrm{Re}}\left(\frac{\partial v}{\partial r}\right)^2. \tag{2.25}$$

4) $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_r})$-SNSE. The inertial terms and viscous terms are retained

up to the order of magnitude of $\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_r}$ and the terms smaller than them are neglected from Eqs. (2.2) to (2.4). $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_r})$-SNSE is consistent with $O(\mathrm{Re}^{-n_u-n_v})$-SNSE that belongs to the inner hierarchy is from BLE to NS equations (refer to Eqs. (2.17)—(2.19) and Fig. 2). However, it should be noted that this same set of equations is deduced according to some different estimations of the order of magnitude. Other five kinds of SNSE with much higher accuracy are respectively $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_\theta})$-SNSE, $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_\theta})$-SNSE, $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v})$-SNSE, $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+2n_\theta})$-SNSE and $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_\theta})$-SNSE. $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u})$-SNSE that is located at the tenth level of the outer hierarchy is just NS equations. From here we see that the two branches of the hierarchial structure, i.e. the inner hierarchy from BLE to NS equations and the outer hierarchy from the Euler equations to NS equations begin to intersect (or say, begin to branch) at the level of $O(\mathrm{Re}^{-n_u-n_v})$ — SNSE (refer to Fig. 2).

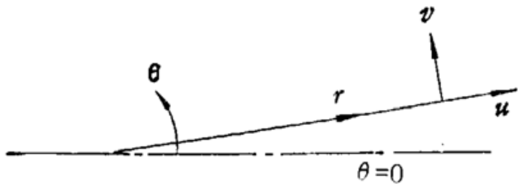


Fig. 3. Viscous jet and spherical coordinates (r, θ, φ).

Now we consider the round laminar jet of a viscous compressible fluid. The basic equations describing the jet in the spherical coordinates (r, θ, φ) are still Eqs. (2.1)—(2.5). However, the length scales for the jet are completely different from those for the viscous compressible flow over a sphere. This is why we study simultaneously these two flows. In the vicinity of symmetrical axis of the jet, i.e. the vicinity of $\theta = 0$ (refer to Fig. 3), the length scale in the r-direction is much larger than that in the θ-direction and has

$$r \backsim \mathrm{Re}^{-n_r},\ r\theta \backsim \mathrm{Re}^{-n_\theta},\ \mathrm{Re}^{-n_r} > \mathrm{Re}^{-n_\theta},\ n_\theta > n_r \geqslant 0. \tag{2.26}$$

The expressions of the time scale and the orders of magnitude of all the variables are still the relationship (2.9); the relationship (2.10) ought to be changed into

$$n_t = -n_u + n_r = -n_v + n_\theta,\ n_p = n_\rho + n_T,\ 2n_u = n_p - n_\rho. \tag{2.27}$$

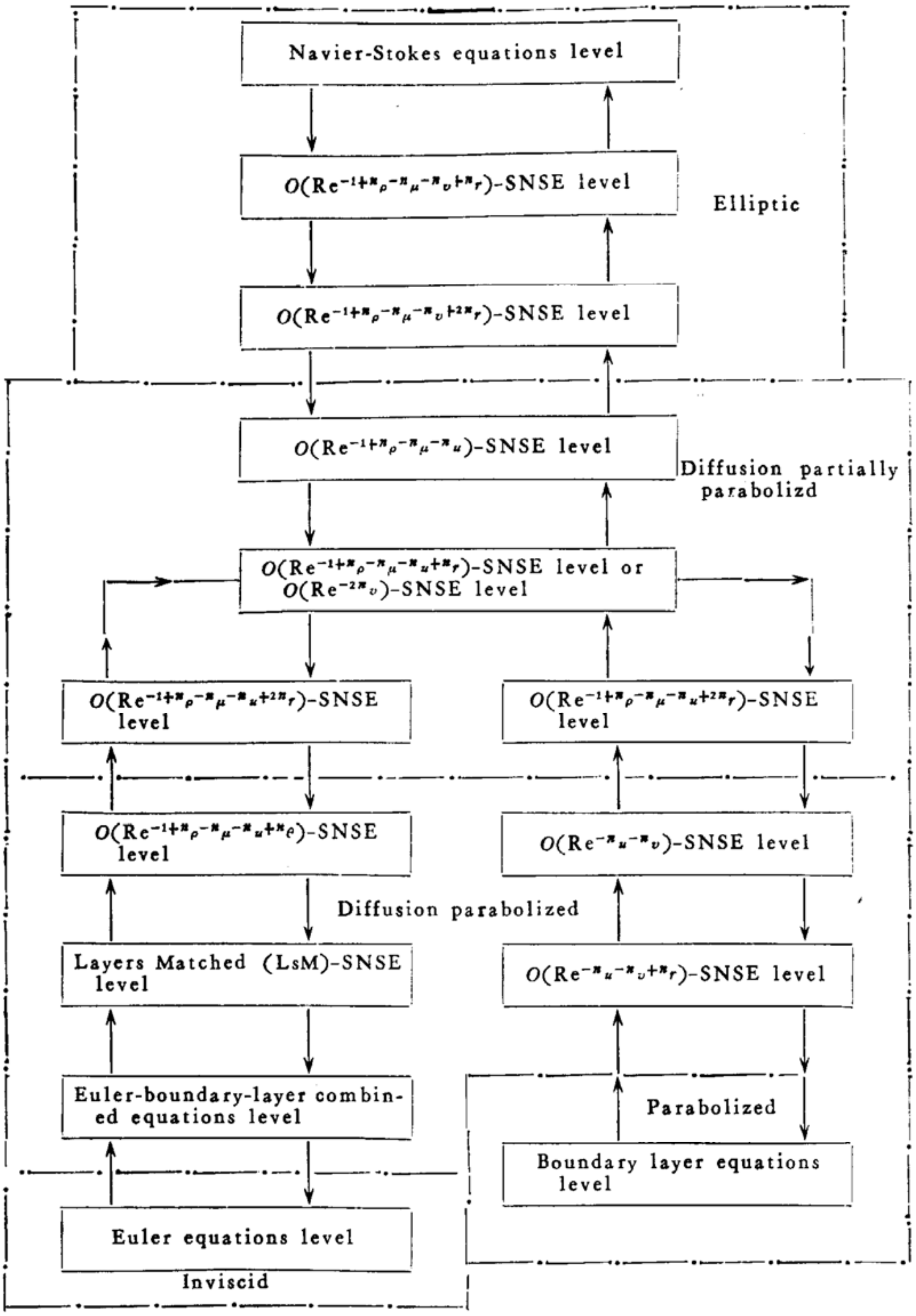
The order-of-magnitude exponents of all the inertial terms in the NS equations (2.2) and (2.3) satisfy the following relationships

$$\begin{aligned}&(-n_u + n_t),\ (-2n_u + n_r),\ (-n_u - n_v + n_\theta),\ (-n_p + n_\rho + n_r) > (-n_v + n_t),\\ &\quad (-n_u - n_v + n_r),\ (-2n_v + n_\theta) > (-n_u - n_v) > (-2n_v).\end{aligned} \tag{2.28}$$

The order-of-magnitude exponents of all the diffusion terms satisfy the following relationships

$$\begin{aligned}&(-n_u + 2n_\theta) > (-n_v + 2n_\theta) > (-n_u + n_\theta) > (-n_u + 2n_r) > (-n_u + n_r) >\\ &\quad (-n_u) > (-n_v + 2n_r) > (-n_v + n_r) > (-n_v),\end{aligned} \tag{2.29}$$

in which $(-1 + n_\rho - n_\mu)$ included in all the order-of-magnitude exponents has been omitted in order to save the space and we also have assumed that $n_\theta > 3n_r$. For the



$$\frac{v^2}{r} \geqslant \frac{1}{\rho r^2 \mathrm{Re} \sin\theta} \frac{\partial}{\partial\theta}\left(\mu \sin\theta \frac{\partial u}{\partial\theta}\right) \qquad u\frac{\partial u}{\partial r} \sim \frac{1}{\rho r^2 \mathrm{Re}\sin\theta}\frac{\partial}{\partial\theta}\left(\mu\sin\theta\frac{\partial u}{\partial\theta}\right)$$

Fig. 4. Hierarchial structure of the Simplified Navier-Stokes Equations (SNSE) for an axially symmetrical viscous jet of a compressible fluid (length scales in the r- and θ-directions: $r \sim \mathrm{Re}^{-n_r}$, $r\theta \sim \mathrm{Re}^{-n_\theta}$, $n_\theta > n_r \geqslant 0$).

case of $n_r < n_\theta < 3n_r$, an inequality similar to the relationship (2.29) can be easily found.

For the case where the maximum inertial term and the maximum viscous term have the same order of magnitude, i. e.

$$u\frac{\partial u}{\partial r} \sim \frac{1}{\rho \mathrm{Re}\, r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\mu \sin\theta \frac{\partial u}{\partial \theta}\right),$$

we neglect the small order of magnitude terms from Eqs. (2.1) to (2.5) in sequence of the inequalities (2.28) and (2.29) and then obtain eight kinds of SNSE for the jet. These eight kinds of SNSE belong to the inner hierarchy and are shown in Fig. 4. Their details are omitted for brevity.

For the case where the maximum order of magnitude of viscous term is comparable to the minimum order of magnitude of inertial term, i. e.

$$\frac{1}{\rho r^2 \mathrm{Re}\sin\theta}\frac{\partial}{\partial \theta}\left(\mu \sin\theta \frac{\partial u}{\partial \theta}\right) \leqslant \frac{v^2}{r},$$

we neglect the small order of magnitude terms from Eqs. (2.1) to (2.5) in sequence of the inequalities (2.28) and (2.29) and then obtain nine kinds of SNSE, each of which contains all the terms in the Euler equations. These nine kinds of SNSE in the outer hierarchy are shown in Fig. 4. Their details are also omitted in order to save the space. From Fig. 4 we see that the inner hierarchy and the outer hierarchy begin to intersect (or say, begin to branch) at the level of $O(\mathrm{Re}^{-2n_v})$-SNSE (refer to Fig. 4). Therefore, if one of the inertial terms is of a very small order-of-magnitude, for the present example, $\frac{v^2}{r}$ being of the small order of magnitude of Re^{-2n_v}, all the inertial terms can be reasonably treated only at a level with higher accuracy in the inner hierarchy.

In a word, for the viscous jet and the viscous flow over a sphere, both the inner hierarchy from BLE to NS equations and the outer hierarchy from the Euler equations to NS equations contain more than ten kinds of SNSE (refer to Fig. 2 and Fig. 4). If we think over the fact that most of the SNSEs appearing in literature are not consistent with the present hierarchial SNSE[1,8], the number of SNSE is obviously still larger. Therefore, it would be necessary to make a further study on various SNSEs and to sift out the best.

III. Mathematical Property and Classification of the Hierarchial Simplified Navier-Stokes Equations (SNSE)

We take the viscous compressible flow over a sphere for example and study the characteristics and subcharacteristics of all the hierarchial SNSEs. The study of the characteristic and subcharacteristic surfaces determines the main structure of the flows, identifies the zones of influence and dependence. If the surface $\varphi(t, r, \theta) = 0$ is a characteristic surface in 3-dimensional space (t, r, θ) for Eqs. (2.1)—(2.5), the characteristic equation can be expressed as[9,10]

$$\left| \sum_{k_t+k_r+k_\theta=n_j} a_{ij}^{(k_t,k_r,k_\theta)} \left(\frac{\partial \varphi}{\partial t}\right)^{k_t} \left(\frac{\partial \varphi}{\partial r}\right)^{k_r} \left(\frac{\partial \varphi}{\partial \theta}\right)^{k_\theta} \right| = 0. \tag{3.1}$$

where $a_{ij}^{(k_t,k_r,k_\theta)}$ are the coefficients of the highest order partial derivatives of the dependent variables in Eqs. (2.1)—(2.5). $a_{ij}^{(k_t,k_r,k_\theta)}$ can be functions of the independent variables, the dependent variables and lower-order partial derivatives of the dependent variables. n_j is the order-number of the highest order partial derivative of the jth dependent variable. Note that $n_i \neq n_j (i \neq j)$ is permitted in the characteristic theory[9,10].

1) For NS equations, $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_\theta})$-SNSE and $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+2n_\theta})$-SNSE, we obtain a characteristic equation as follows:

$$S\left[\frac{4}{3}\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right]\left[\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{4}{3r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right] \cdot\left[\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right]=0, \tag{3.2}$$

where

$$S=S_t\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial r}+\frac{v}{r}\frac{\partial\varphi}{\partial\theta}. \tag{3.3}$$

S is the particle derivative. This indicates that the particle paths are characteristics of NS equations and these two kinds of SNSE. Other three factors of the characteristic equation (3.2) stem from the viscous terms and they have no real solutions, NS equations and these two SNSEs are therefore elliptic in mathematics (refer to the caption of Fig. 2).

2) For $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v})$-SNSE, $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_\theta})$-SNSE and $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+2n_\theta})$-SNSE, we obtain the following characteristic equation:

$$S\left(\frac{\partial\varphi}{\partial r}\right)^2\left[\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{4}{3r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right]\left[\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right]=0. \tag{3.4}$$

The particle paths are still characteristics of these three kinds of SNSE. The third and fourth factors of the characteristic equation (3.4), which stem from the momentum equation in the θ-direction and the equation of energy, have no real solutions. These three kinds of SNSE are therefore elliptic in mathematics. However, the second factor of the characteristic equation (3.4), which stems from the momentum equation in the r-direction, does not contain derivatives with respect to the variable θ. Therefore, θ is a partially parabolized variable. From here we see that the physical content of mathematical approximation of these three kinds of SNSE is different from that of $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+n_\theta})$-SNSE or $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_u+2n_\theta})$-SNSE. These three SNSEs are diffusions partially parabolized, as shown in Fig. 2.

3) For LsM-SNSE, $O(\mathrm{Re}^{-n_u-n_v})$-SNSE and $O(\mathrm{Re}^{-1+n_\rho-n_\mu-n_v+n_r})$-SNSE, we obtain a characteristic equation as follows:

$$S\left(\frac{\partial\varphi}{\partial r}\right)^6=0. \tag{3.5}$$

The particle paths are characteristics of these three kinds of SNSE. The second factor of the characteristic equation (3.5) does not contain derivatives with respect to the variable θ. These three SNSEs are therefore parabolic in mathematics and an integration can proceed by marching in the positive direction of the parabolic variable θ for steady flows.

4) For the Euler-boundary-layer combined equations, we obtain the following characteristic equation,

$$\left(\frac{\partial\varphi}{\partial r}\right)^4\left[S^2-\frac{p}{\rho}\left(\frac{\partial\varphi}{\partial r}\right)^2\right]=0, \tag{3.6}$$

in which the first factor does not contain derivatives with respect to the variable θ and the second factor has real solutions. Therefore, the Euler-boundary-layer combined equations have parabolic-hyperbolic double property in mathematics.

5) For the Euler equations, we obtain a characteristic equation as follows:

$$S^2\left\{S^2-a^2\left[\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right]\right\}=0, \tag{3.7}$$

where $a^2=\gamma\frac{p}{\rho}$, a is the sound speed. The particle paths are characteristics of the Euler equations. The second factor of the characteristic equation (3.7) indicates that when $M<1$, Eq. (3.7) has no real solutions and the Euler equations are therefore elliptic for $M<1$ and that when $M>1$, Eq. (3.7) has real solutions and the Euler equations are therefore hyperbolic for $M>1$, where M is the Mach numbers, $M^2=(u^2+v^2)/a^2$. For the cases of incompressible flows, the characteristic equation (3.7) changes into

$$S^2\left[\left(\frac{\partial\varphi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2\right]=0. \tag{3.8}$$

The characteristic equation (3.8) shows that for incompressible flows the Euler equations are elliptic and the particle paths are characteristics. These two conclusions, of course, agree with those proved in the preceding paragraph.

6) For the boundary layer equations (BLE) and the modified boundary layer equations (MBLE), we obtain a characteristic equation as follows:

$$\left(\frac{\partial\varphi}{\partial r}\right)^6=0. \tag{3.9}$$

The characteristic equation (3.9) does not contain the partial derivatices with respect to the independent variables, t and θ. BLE and MBLE are therefore parabolic in mathematics and an integration can proceed by marching in the positive directions of the parabolic variables both t and θ.

The subcharacteristics (or the limiting characteristics as called in [10]) of a hierarchial SNSE are determined by the system of partial differential equations which are obtained by letting $\mathrm{Re}\to\infty$, i. e. by neglecting all the viscous and heat conduction terms from the original hierarchial SNSE[1,10]. It is obvious that the subcharacteristics of all the hierarchial SNSEs except the Euler equations and the classical and modified boundary layer equations are just the characteristics of the Euler equations, that is to say, their subcharacteristic equations are just Eq. (3.7). Therefore, the particle paths are subcharacteristics of these hierarchial SNSEs., Especially, in accordance with subcharacteristics, these hierarchial SNSEs are hyperbolic for $M > 1$, and yet they are elliptic for $M < 1$.

For BLE and MBLE, we obtain a subcharacteristic equation as follows:

$$S^2 \left(\frac{\partial \varphi}{\partial r}\right)^2 = 0. \tag{3.10}$$

It is interesting to note that BLE and MBLE are still parabolic in accordance with their subcharacteristics.

For the cases of the viscous compressible jet and other viscous flows, an analysis of characteristics and subcharacteristics will give the same results as we have listed above. As an example, the analytical results for the viscous compressible jet are demonstrated in Fig. 4. As a result, we can draw out some general conclusions as follows.

1) The mathematical properties of all the hierarchial SNSE and NS equations are represented by the higher-order diffusion terms as well as the pressure gradient terms. The mathematical properties represented by the former are independent of Mach numbers M. However, the mathematical properties represented by the latter are dependent on M: when $M < 1$, all the hierarchial SNSEs except the classica land modified boundary layer equations are always elliptic in mathematics. This means that the elliptic character represented by the pressure gradient terms is restricted within the subsonic flow region of $M < 1$, in other words, the pressure gradient terms allow an upstream propagation of disturbances in the flow field through the elliptic subsonic regions; when $M > 1$, all the hierarchial SNSEs are hyperbolic-parabolic.

2) Although there are many SNSEs, they can be divided into only three types according to their mathematical properties. These three types are: elliptic, diffusion parabolic (i. e. type of the Euler equations) and hyperbolic-parabolic. On the other hand, they may be also divided into five groups according to their physical content. This five groups are: elliptic approximation, diffusion partially parabolized approximation, diffusion parabolized approximation, inviscid approximation and the boundary layer approximation (refer to Table 1, Fig. 2 and Fig. 4)

Table 1

Classified Catalogue for Simplified Navier-Stokes Equations (SNSE)

Physical aspects	ellitic approximation and NS equations	diffusion partially parabolized approximation	boundary-layer approximation (BLE and MBLE)
Mathematical character	elliptic		parabolic-hyperbolic
Physical aspects	diffusion parabolized approximation	inviscid approximation (Euler equations)	
Mathematical character	parabolic-hyperbolic(or hyperbolic) for $M>1$; elliptic for $M<1$.		

Note: BLE and MBLE are abbreviations of the classical and modified boundary layer equations, respectively. M indicates Mach number.

3) The diffusion parabolized SNSEs are always elliptic in flow region in the vicinity of the solid wall, whether the flow far from the wall is either supersonic of $M > 1$ or subsonic of $M < 1$. This forms a most important difference between them and the boundary layer equations (BLE) or modified BLE, the former can represent the interaction between two flows near and far from the solid wall, while the latter cannot. The mathematical properties of the diffusion parabolized SNSE are consistent with that of the Euler equations (refer to Table 1). As a result, the diffusion parabolized SNSE may be regarded as the inviscid Euler equations with physical viscosity.

4) As for the solution of various SNSEs, if their mathematical types are the same, there is little difference between them, though they embrace numbers of terms. Therefore, according to the viewpoint of numerical computation, both the elliptic SNSE and the diffusion partially parabolized SNSE are nearly identical with NS equations; the diffusion parabolized SNSE may be identical with the Euler equations. Fortunately, solving the diffusion parabolized SNSE will give a solution for the whole flow-field including both inviscid and viscous regions.

5) Some common SNSEs appeared in literature, such as the SNSE in which the streamwise viscous derivatives are ignored[12], Davis's viscous layer equations[13], parabolized NS equations (PNS)[14], modified PNS (MPNS)[15], thin-layer approximation NS equations[16] and slender thin-layer approximation NS equations[17] and so on. All SNSEs, whether they coincide with the present hierarchial SNSE or not, can be proved to be diffusion parabolized, which reveals that there are a lot of formulations of diffusion parabolized SNSE, therefore, it would be necessary to systematize them and to choose the best.

IV. Mechanical Connotation of the Hierarchial Simplified Navier-Stokes Equations (SNSE) and the Diffusion Parabolized Approximation

1) For moderate and high Reynold's number flows with main stream direction, the inner and outer hierarchies of SNSE elucidate the equational evolution from the most simplest equations, i. e. the classical boundary layer equations and the Euler equations, to NS equations. On the other hand, a moderate or high Re shear flow with a main stream direction can be divided into several layers (see next paragraph) in the normal direction of the main stream. We are first concerned with the simplest SNSE being suitable to each one of the layers as well as the solution of the whole flow-field obtained by solving simultaneously all the simplest SNSE above-mentioned with physical and mathematical conditions at the boundaries and junctures of all the neighbouring two layers. Secondly, we are also concerned with the simplest SNSE being simultaneously suitable to all the layers as well as the solution of the whole flow-field obtained by solving this SNSE with physical boundary conditions. It is obvious that the diffusion parabolized SNSE within all of the SNSEs being suitable to the whole flow-field are of most importance.

2) Mechanical connotation of the diffusion parabolized approximation. We illuminate the present proposition with an example of a viscous compressible flow over a thin body. In this case, the main stream direction is parallel to the surface of the thin body. The distance of viscous diffusion is of the order of magnitude of $L\mathrm{Re}^{-1/2}$

during the characteristic transit time L/U of a fluid particle crossing the flow region near the surface of the body, where $\mathrm{Re} = \rho LU/\mu$, L and U are the characteristic length and velocity, respectively.

This means that during the transit time L/U the direct influence range of viscous diffusion is of the order of magnitude of $L\mathrm{Re}^{-1/2}$. On the other hand, as was stated above the pressure gradient allows upstream propogation of disturbances in the flow-field through subsonic flow region. We call this phenomenon the pressure propogation of disturbances for convenience. The range of the pressure propogation of disturbances is of the order of magnitude of aL/U during the transit time L/U, where a is the sound speed. The propagation velocity of disturbances is approximately unaffected by viscosity. When $\mathrm{Re} \gg 1$, aL/U is much larger than $L\mathrm{Re}^{-1/2}$ for subsonic flows as well as flows within the boundary layer. Therefore, for a moderate or high Re flow with a main stream direction the influence of downstream disturbances on upstream region is mainly decided by the pressure propogation of disturbances, the direct influence of viscous diffusion is rather small and can be neglected. In other words, the downstream boundary conditions of the closed boundary conditions demanded by the highest-order viscous derivatives can be ignored. The above-mentioned state is mechanical connotation of diffusion parabolized approximation. This demonstrates that the standard judging merit and demerit of all the diffusion parabolized SNSEs ought to see whether it is able to predict correctly the pressure distribution of the flow-field. In the cases of 2- and 3-dimensional viscous incompressible flows for which the Navier-Stokes equations are in Cartesian coordinates, we can prove that within all the hierarchial SNSEs only L_sM-SNSE can take viscous effects into account as well as lead to an exact pressure-velocity equation, which is

$$\mathrm{div}\left(\frac{d\boldsymbol{v}}{dt}\right) = -\frac{1}{\rho}\,\Delta p,$$

where Δ is the Laplace operator.

V. Multilayer Structure Model For Viscous Shear Flow, Transition Layer and Practical Example

1) Multilayer structure model. As was stated above, simplifying NS equations as well as subdividing the flow-field should be based on the resolving of the spatial scales of the flow-field and the comparing of the orders the of magnitude between the inertial and viscous terms. The subdividing of a viscous shear flow with a main stream direction leads to a multilayer-structure model. The main points of the model are as follows: a moderate or high Reynold's number shear flow with a main stream direction can be divided into several layers in the normal direction of the main stream according to the resolving of the spatial scales of the shear flow and the comparison of the orders of magnitude between the inertial and viscius terms. Each of the layers is expressed by a set of length scales in the coordinate directions. The streamwise length scale of any one of the layers in which viscous effects cannot be neglected is always larger than that in the normal direction, as a result, the orders of magnitude of various terms in the NS equations are different. For example, a jet has five (or three) layers in the normal direction of the jet axis; a viscous flow over a thin body

Table 2

Essential Elements of Transition and Viscous Layers

Transition Layer and LsM-SNSE	Viscous Layer and BLE
1. Moderate and high Reynold's numbers(Re)	1. Moderate and high Reynold's numbers (Re)
2. Length scale L_n in the normal direction is much smaller than length scale L_s in the main stream direction($L_n \ll L_s$).	2. Thickness δ of the layer is much smaller than length scale L_s in the main stream direction($\delta \ll L_s$).
3. The minimum inertial term and the maximum viscous term are of the same order of magnitude ($U_s U_n / L_s$ or $U_n^2 / L_s \sim U_s / \mathrm{Re} L_n^2$).	3. The maximum inertial term and the maximum viscous term are of the same order of magnitude ($U_s^2 / L_s \sim U_s / \mathrm{Re} L_n^2$).
4. Influence of solid wall on transition layer is weak, diffusion proceeds in the various coordinate directions.	4. Influence of solid wall on viscous layer is strong, diffusion proceeds mainly in the normal direction.
5. When NS equations are simplified, all the inertial terms are retained and the viscous terms are retained up to the order of magnitude of $U_n / \mathrm{Re} L_n^2$, then we obtain LsM-SNSE.	5. When NS equations are simplified, the inertial and viscous terms are retained up to the order of magnitude of $U_s / \mathrm{Re} L_n^2$, then we obtain BLE.

Note: LsM-SNSE and BLE are abbreviations of the layers matched Simplified Navier-Stokes Equations and the boundary layer equation, respectively. U_n and U_s are the characteristic velocities of the normal and tangential directions of the layer, respectively.

or over a flat plate has three layers. The three layers are respectively viscous layer, inviscid layer and transition layer being situated between the former two layers, see next section. Now we discuss briefly two problems: how to determine the length scales of the above-mentioned three layers and which of all the hierarchial SNSEs suitable to each of the three layers is the simplest? All the length scales of the viscous layer are determined by four supplementary relationships which are respectively that between the maximum viscous term and the maximum inertial term being of the same order of magnitude, two relationships between Re and the shear stress and thermal flux at the wall, and that between the maximum inertial term and the pressure gradient term being of the same order of magnitude. Details are in the following paragraph. The boundary layer equation within all the SNSEs being suitable to the viscous layer is the simplest and the most important. All the length scales of the inviscid layer are determined by four supplementary conditions which are respectively the maximum viscous term being much smaller than the minimum inertial term, the length scale of the main stream direction as well as the orders of magnitude of both the pressure and temperature being matched with those of the transition layer. The simplest and the most important SNSE being suitable to the inviscid layer is the Euler equations. All the length scales of the transition layer are determined by four supplementary relationships which are respectively the maximum viscous term and the minimum inertial term being of the same order of magnitude, the length scale of the main stream direction as well as the order of magnitude of both the pressure and temperature being matched with those of the viscous layer. The simplest SNSE being suitable to the transition layer is the

Euler-boundary-layer combined equations, the next simplest SNSE is the LsM-SNSE. A further supplement for the transition layer as well as LsM-SNSE is as follows.

2) Transition layer. The transition layer is such a flow layer where inertial force is the main force, viscous force begins to play an indispensable role in equilibrating forces and the maximum viscous term is comparable to the minimum inertial term. A comparison between essential elements of the transition layer and the viscous layer is given in Table 2. An equation governing the transition layer flow should be the LsM-SNSE, because LsM-SNSE is superior to the Euler-boundary-layer combined equations and LsM-SNSE has the possibility of predicting correctly the pressure field and many solutions of LsM-SNSE are completely consistent with the exact solutions of NS equations[8] and LsM-SNSE is not only suitable to all the layers but also satisfy mathematical conditions of smooth transition at the junctures between all the neighbouring two layers, that is why we call this SNSE as layers-matched (LsM) SNSE. The meaning of introducing a concept of the transition layer shall be further discussed in the following paragraph. In addition, LsM-SNSE is exactly an extension of the inner-outer-layers-matched SNSE[11] to multilayer case, furthermore, the formulation of the former is always consistent with that of the latter. The inner-outer-layers-matched SNSE was developed on the basis of both an assumption of two-layers model for viscous shear flow and an estimation of the order of magnitude of the classical boundary layer theory[6,7].

3) Practical examples. The multilayer structure model is applied to viscous incompressible flow over a flat plate (including the separated flow region and the attached flow region, i. e. the flow region before the separation point). The flow-field is divided into three layers in the normal direction of the wall. The length scales in the individual coordinate directions and the order-of-magnitude of the individual terms related to the problem are given below.

i) The viscous layer. For a laminar flowover a flat plate the relationships between Re and the pressure and the shearing stressat the wall are as follows [6,18]:

$$(p-p_\infty)\sim \mathrm{Re}^{-m}\quad \left(m<\frac{1}{4}\ \text{for the attached-flow},\right.\qquad \left. m=\frac{1}{4}\ \text{for the separated-flow}\right), \tag{5.1}$$

$$C_f=2\tau_w/\rho_\infty U_\infty^2\sim \mathrm{Re}^{-1/2}\sim\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}, \tag{5.2}$$

where τ_w is the shearing stress at the wall. Again using the relationships among $u\dfrac{\partial u}{\partial x}\sim\dfrac{1}{\mathrm{Re}}\dfrac{\partial^2 u}{\partial y^2}$, $u\dfrac{\partial u}{\partial x}\sim\dfrac{\partial p}{\partial x}$ and $\dfrac{\partial u}{\partial x}\sim\dfrac{\partial v}{\partial y}$ we deduce the following results:

$$n_x=\frac{3}{2}m,\quad n_u=\frac{1}{2}m,\quad n_y=\frac{1+m}{2},\quad n_v=\frac{1-m}{2}. \tag{5.3}$$

For the attached flow with $m=0$, we have $n_x=n_u=0$, $n_y=n_v=\dfrac{1}{2}$, $u\dfrac{\partial u}{\partial x}\sim\dfrac{1}{\mathrm{Re}}\dfrac{\partial^2 u}{\partial y^2}\sim \mathrm{Re}^0$, $\dfrac{1}{\mathrm{Re}}\dfrac{\partial^2 v}{\partial y^2}\sim \mathrm{Re}^{-1/2}$, these are exactly the results of the classical boundary layer

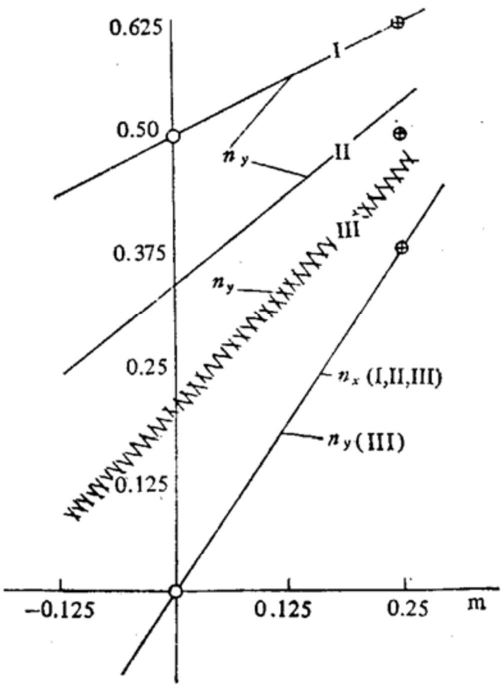


Fig. 5. Variation of the length scale exponents n_x and n_y with m for viscous incompressible flow over a flat plate ($x \sim \mathrm{Re}^{-n_x}$, $y \sim \mathrm{Re}^{-n_y}$, $(p-p_\infty) \sim \mathrm{Re}^{-m}$; o indicates the classical boundary layer theory[6]; ● indicates the triple deck theory[18] for separated flow).

theory[6]. For the separated flow with $m = \frac{1}{4}$, we have $n_x = \frac{3}{8}$, $n_u = \frac{1}{8}$, $n_y = \frac{5}{8}$, $n_v = \frac{3}{8}$; $u\frac{\partial u}{\partial x} \sim \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} \sim \mathrm{Re}^{1/8}$, $\frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial y^2} \sim \mathrm{Re}^{-1/8}$, which are consistent with the lower deck of the triple deck theory[18] for separated flow (refer to Fig. 5).

ii) The transition layer. Based on the fact that the maximum viscous term and the minimum inertial term are of the same order of magnitude and that the length scale in the main stream direction and the order-of-magnitude of the tangential velocity (or the pressure) must be matched with those of the viscous layer, we deduce the following results:

$$n_x = \frac{3}{2}m,\ n_u = \frac{1}{2}m,\ n_y = \frac{2+5m}{6},\quad n_v = \frac{2+m}{6}. \tag{5.4}$$

For the attached flow with $m = 0$, we have $n_x = n_u = 0$, $n_y = n_v = \frac{1}{3}$, $u\frac{\partial u}{\partial x} \sim \mathrm{Re}^0$ $\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} \sim \mathrm{Re}^{-1/3}$, $\frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial y^2} \sim \mathrm{Re}^{-2/3}$. For the separated flow of $m = \frac{1}{4}$, we have $n_x = \frac{3}{8}$, $n_u = \frac{1}{8}$, $n_y = \frac{13}{24}$, $n_v = \frac{7}{24}$; $u\frac{\partial u}{\partial x} \sim \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} \sim \mathrm{Re}^{-1/24}$, $\frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial y^2} \sim \mathrm{Re}^{-5/24}$.

iii) The inviscid layer. Based on the fact that the maximum viscous term is much smaller than the minimum inertial term and that the length scale in the main stream direction and the order-of-magnitude of the tangential velocity (or the prsesure) must be matched with those of the transition layer, we deduce the following results:

$$n_y = \frac{3}{2} m, \; n_u = \frac{1}{2} m, \; 0 \leqslant n_y < \frac{2+5m}{6}, \; -n_v + n_y = m. \tag{5.5}$$

For the attached flow with $m=0$, we have $n_x = n_u = 0$, $0 \leqslant n_y < \frac{1}{3}$, if the length scale in the y-direction is equal to that in the x-direction we further have $n_y = 0$ and $n_v = 0$. For the separated flow of $m = \frac{1}{4}$, we have $n_x = \frac{3}{8}$, $n_u = \frac{1}{8}$, $0 \leqslant n_y < \frac{13}{24}$, if the length scale in the y-dirction is equal to that in the x-direction, we further have $n_y = \frac{3}{8}$, $n_v = \frac{1}{8}$(refer to Fig. 5).

From the above-mentioned results we can draw out that for the viscous incompressible flow over a flat plate, the present multilayer structure model construct not only new three layers models for both the attached and separated flows but also reveal the physical character of transition from the attached three-layers to the separated three-layers. The normal length scale of the transition layer decreases from $L\mathrm{Re}^{-1/3}$ to $L\mathrm{Re}^{-13/24}$ with increase of positive pressure gradient along the wall; the transition layer seems to jostle gradually into the classical boundary layer with a rise in the interaction between inviscid and viscous flows. In this case, the concept and equations

the transition and inviscid layers are corresponding to the main and upper decks, respectively. In the case of attached flow with $m = 0$, the classical boundary layer theory proposes a two layers model (refer to Fig. 5). It is obvious that viscous action region ought to contain both the viscous layer and transition layer. Therefore, the normal length scales of the viscous action region are $L\mathrm{Re}^{-1/3}$ for the attached flow with $m=0$ and $L\mathrm{Re}^{-13/24}$ for the separated flow of $m=\frac{1}{4}$, respectively. These length scales are distinctly different from those with the classical boundary layer theory and the triple-deck theory[18] for separated flow.

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