Viscosity-induced mode splitting and potential energy criterion for mode stability

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This paper points out that viscosity can induce mode splitting in a uniform infinite cylinder of an incompressible fluid with self-gravitation, and that the potential energy criterion cannot be appropriate to all normal modes obtained, i.e., there will be stable modes with negative potential energy ($\Sigma < 0$). Therefore the condition $\Sigma > 0$ is not necessary, although sufficient, for the stability of a mode in an incompressible static fluid or magnetohydrodynamics (MHD) system, which is a correction of both Hare's [Philos. Mag. **8**, 1305 (1959)] and Chandrasekhar's [*Hydrodynamic and Hydromagnetic Stability* (Oxford U.P., Oxford, 1961), p. 604] stability criterion for a mode. These results can also be extended to compressible systems with a polytropic exponent.

I. INTRODUCTION

In a fluid or magnetohydrodynamics (MHD) system, the study on normal modes of small motion near the equilibrium state and mode changes, or mode splitting, under various conditions is of great significance for stability analysis, stability control, and bifurcations.[1–3] Therefore it is interesting to study the stability conditions not only for equilibrium states, but also for individual modes.

For rotating Newtonian stars, Friedman and Schutz[4] discussed the instability of normal modes, and showed that the sign of canonical energy E_c (not total energy of a mode) is irrelevant to the question of viscosity-induced instability. For static Newtonian stars, putting $\Omega = 0$ in Eq. (73) of Ref. 4, we obtain

$$\frac{dE_c}{dt} < 0, \tag{1}$$

where E_c is also the total energy and does not change sign with time. So, if

$$E_c < 0, \tag{2}$$

the mode is unstable; if

$$E_c > 0, \tag{3}$$

it is stable.

Is the sign of potential energy Σ of a mode relevant to the question of viscosity-induced instability? For the stability of a mode in a static, incompressible, viscous fluid, or MHD system with infinite electrical conductivity, a prevalent criterion, given by Hare[5] and Chandrasekhar,[6] is that a necessary and sufficient condition for stability should be $\Sigma > 0$. The deduction in Refs. 5 and 6 is as follows.

Suppose that all the quantities describing the perturbed state have the time dependence $e^{\sigma t}$, where σ takes the place of $i\sigma$ in Ref. 6; then an equation,

$$\sigma^2 I + \sigma\Phi + \Sigma = 0, \tag{4}$$

is obtained, where Σ, Φ, and I are real functionals of eigenfunction $\xi^{(\lambda)}$. (The dependence of Σ, Φ, and I on $\xi^{(\lambda)}$ of a mode also shows that the criterion of both Hare and Chandrasekhar refers only to a single mode, but not to the whole system.) Equation (4) gives

$$\sigma_\pm = (1/2I)[-\Phi \pm (\Phi^2 - 4I\Sigma)^{1/2}]. \tag{5}$$

Hare and Chandrasekhar arrived at their criterion from this equation immediately. In fact they had the implicit supposition, without any proof, that both σ_+ and σ_- are true eigenvalues of a given $\xi^{(\lambda)}$. Though it is obvious that there must be one true eigenvalue in the two roots at least, their supposition for two roots to be true eigenvalues cannot be proved in general. Schutz[7] pointed out that for some rotating systems, of the two roots of the equation similar to Eq. (4), only one is a true eigenvalue. However, even if we know that Eq. (5) gives one false eigenvalue without knowing which one is false, we can only say that neither the necessity nor the unnecessity of $\Sigma > 0$ for mode stability can be deduced from Eq. (5). In order to prove that the condition $\Sigma > 0$ is unnecessary for mode stability, we must prove that there are stable modes with $\Sigma < 0$ in the system discussed by Hare and Chandrasekhar. So far, such a proof has not yet appeared in the literature. In the present paper we show that viscosity can induce mode splitting in a uniform infinite cylinder of a static incompressible fluid with self-gravitation, and that among those modes obtained, there are an infinite number of stable modes with $\Sigma < 0$. The mode splitting induced by viscosity is also of great interest in itself, because it is similar to the Zeeman and Stark effects in atomic physics.

II. MODE SPLITTING INDUCED BY VISCOSITY

Consider a uniform infinite cylinder of a static, incompressible, self-gravitational, and viscous liquid with equilibrium radius R, density ρ, and kinematic viscosity ν. In a cylindrical coordinate system, we take the axis of symmetry as the z axis. In the unperturbed state, the pressure p_0 and gravitational potential B_0 are

$$p_0 = \pi G\rho^2(R^2 - r^2), \tag{6}$$

$$B_0 = \begin{cases} 2\pi G\rho R^2 \ln(r/R) + \pi G\rho R^2, & \text{if } r \geqslant R, \\ \pi G\rho r^2, & \text{if } r < R, \end{cases} \tag{7}$$

where G is the gravitational constant. Consider the axisymmetrical perturbation of surface radius ($m = 0$),

$$r_s = R + \epsilon_0 \exp(ikz + \sigma t), \tag{8}$$

where ϵ_0 is a small constant. The equations governing small departures from equilibrium are

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$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\Pi + \nu\nabla^2\mathbf{u}, \quad (9)$$

$$\nabla\cdot\mathbf{u} = 0, \quad \nabla^2\delta B = 0, \quad (10)$$

$$\Pi = \delta B + \delta p/\rho, \quad (11)$$

with boundary conditions $\mathbf{u}|_{r=0}$, $\delta B|_{r=0}$, and $\Pi|_{r=0}$ all finite;

$$\lim_{r\to\infty} \delta B = 0, \quad (12)$$

$$B|_{r=r_s+0} = B|_{r=r_s-0}, \quad (13)$$

$$\left.\frac{\partial B}{\partial r}\right|_{r=r_s+0} = \left.\frac{\partial B}{\partial r}\right|_{r=r_s-0}, \quad (14)$$

$$\frac{\partial r_s}{\partial t} = u_r|_{r=R}, \quad (15)$$

$$\left.\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\right|_{r=R} = 0, \quad (16)$$

$$\left.\left(p_0 + \delta p - 2\nu\rho\frac{\partial u_r}{\partial r}\right)\right|_{r=r_s} = 0, \quad (17)$$

where $\mathbf{u}(u_r,u_z)$ is the velocity perturbation of fluid, and δp is the pressure perturbation. Equations (6)–(17) give

$$u_r = \frac{-\nu\epsilon_0(k^2+\kappa^2)}{I_1(x)}\left(\frac{2x^2I_1(x)I_1(\kappa r)}{(x^2+y^2)I_1(y)} - I_1(kr)\right)\exp(ikz+\sigma t), \quad (18)$$

$$u_z = \frac{\nu\epsilon_0(k^2+\kappa^2)}{iI_1(x)}\left(\frac{2xyI_1(x)I_0(\kappa r)}{(x^2+y^2)I_1(y)} - I_0(kr)\right)\exp(ikz+\sigma t), \quad (19)$$

and the characteristic equation[8]

$$2x^2(x^2+y^2)\frac{I_1'(x)}{I_0(x)} \times\left(1 - \frac{2xyI_1(x)I_1'(y)}{(x^2+y^2)I_1(y)I_1'(x)}\right) - (x^4-y^4) = \frac{4\pi G\rho R^4 xI_1(x)}{\nu^2 I_0(x)}\left(K_0(x)I_0(x) - \frac{1}{2}\right), \quad (20)$$

where

$$\sigma = \nu(y^2-x^2)/R^2, \quad y=\kappa R, \quad x=kR; \quad (21)$$

$I_0(x)$, $I_1(x)$, and $K_0(x)$ are, respectively, the Bessel functions of order 0 and 1, and the Hankel function of order 0, for a purely imaginary argument.

In an inviscid fluid, we know that each mode can be represented by a set of numbers (k,m) or (x,m),[9] and that the modes $(x,0)$ in the range

$$0 < x < x_0(=1.0668) \quad (22)$$

are unstable, and those in the range $x > x_0$ are stable, x_0 being the root of equation

$$K_0(x)I_0(x) - \tfrac{1}{2} = 0. \quad (23)$$

Consider the case of condition (22). It can be proved that for a given x, Eq. (20) has a pair of real roots $\pm y_0$, which satisfy

$$y_0^2 > x^2.$$

Then, Eq. (21) gives $\sigma_0 > 0$, so that Eqs. (18) and (19) give out an exponentially divergent mode $\mathbf{u}_0$. Furthermore, Eq. (20) also has denumerably infinite pairs of conjugate, purely imaginary roots. In order to prove this, let $y = i\beta$; then Eq. (20) becomes

$$\left[2x^2(x^2-\beta^2)\frac{I_1'(x)}{I_0(x)}\left(1-\frac{2x\beta I_1(x)J_1'(\beta)}{(x^2-\beta^2)J_1(\beta)I_1'(x)}\right) - x^4+\beta^4\right]\frac{I_0(x)}{xI_1(x)[K_0(x)I_0(x)-\frac{1}{2}]} = 4\pi G\rho R^4/\nu^2, \quad (24)$$

where $J_1(\beta)$ is the Bessel function of order 1. Here $J_1(\beta)$ has denumerably infinite zero points on the real axis. They are denoted by $\pm a_l$ $(1 = 0,1,2,...)$, where a_0 is equal to zero, and the rest of the a's are all positive. For $1\neq 0$, when $\beta\to a_l \pm 0$, the left-hand side of Eq. (24) tends to $\mp\infty$, so Eq. (24) has at least one real root between a_l and a_{l+1} $(l\neq 0)$. Thus Eq. (20) has denumerably infinite pairs of purely imaginary roots,

$$\pm y_n = \pm i\beta_n \quad (n=1,2,...). \quad (25)$$

Substituting Eq. (25) into Eq. (21), we obtain σ_n $(n\neq 0)$ <0. Let $k = k_n = y_n/R$ $(n\neq 0)$; then Eqs. (18) and (19) give denumerably infinite exponentially damping velocity functions. Since different velocity functions $\mathbf{u}_1,\mathbf{u}_2,...,$ represent different distributions of velocity, thus they really represent different modes of motion, which are stable, because $\sigma_n < 0$ $(n\neq 0)$. It is obvious that for mode splitting, the condition (22) is not necessary; so, in viscous fluid, each mode is represented by a set of numbers (x,m,n). We believe that viscosity can also induce mode splitting in other systems of fluid or MHD. The mode splitting induced by viscosity is similar to the Zeeman and Stark effects in atomic physics, induced by external fields.

III. DISCUSSION ON POTENTIAL ENERGY CRITERION FOR MODE STABILITY

Let the potential energy of the equilibrium state of the cylinder be zero. The potential energy is determined by surface shape, but is independent of viscosity. Therefore the formula of potential energy per unit length under inviscid conditions,[10]

$$\Sigma = -2\pi^2 G\rho^2 R^2[K_0(x)I_0(x) - \tfrac{1}{2}]\epsilon_0^2\exp(2\sigma_n t), \quad (26)$$

can be adopted, where Σ must be negative in the case of Eq. (22). But all modes $(x,0,n)$, except $n=0$ modes, are stable, so that $\Sigma > 0$ is not a necessary condition for mode stability. Since the zero magnetic field is a special instance of MHD, our liquid cylinder can also take the role of an example in MHD opposite to the stability criterion given by Hare and Chandrasekhar. This conclusion can also be extended to compressible systems with a polytropic exponent.

In addition, for the stable modes given above, each canonical energy E_c should be positive in conjunction with Eq. (3).

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