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ELASTIC-PLASTIC FIELDS NEAR CRACK TIP GROWING STEP-BY-STEP IN A PERFECTLY PLASTIC SOLID

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Abstract

In this paper, based on the three-dimensional flow theory of plasticity, the fundametal equations for plane strain problem of elastic-perfectly plastic solids are presented. By using these equations the elastic-plastic fields near the crack tip growing step-by-step in an elastic incompressible-perfectly plastic solid are analysed.

The first order asymptotic solutions for the stress field and velocity fields near the crack tip are obtained. The solutions show the evolution process of elastic unloading domain and the development process of central fan domain and reveal the possibility of the presence of the secondary plastic domain. The second order asymptotic solution for stress field is also presented.

I. Introduction

Exact analysis of stress and strain fields near the crack tip is a leading subject all along. For linear elastic crack problem, the earlier work given by Irwin[1] william [2] revealed that the stress-strain field near the crack tip has a $r^{-1/2}$ singularity

$$\sigma_{ij}=\frac{K}{\sqrt{2\pi r}}\tilde{\sigma}_{ij}(\theta)$$

where K is called stress intensity factor, which chracterizes the intensity of stress singular field.

For ductile fracture, the fracture process zone is surrounded by plastic zone. The crack tip fields of stress and strain are dominated by the plastic singular field. Hence the analysis of the plastic singular field is an essential problem.

For a stationary crack, the HRR singularity field offers a clear description and the J integral is a useful critical parameter under some restrictions.

For a growing crack, the problem becomes complex and the progress in this field is very slow. up to now, only for a few subjects the correct solution has been worked out. The topic extracted from this kind of problem seems quite simple but the solution to the topic is quite difficult.

For a Mode I crack with steady growth in an incompressible perfectly plastic solid, the asymptotic solution near the crack tip was developed by Rice[3] in the case of plane strain. Cherepanov[4] gave the similar analysis for the case of non-steady crack growth. But they did not consider the existence of unloading zone behind the crack tip. The complete solution for this problem was worked out by Slepjan[5] Using the Tresca yield criterion and associated flow rule, he obtained the exact asymptotic solutions for the crack tip fields of pure Mode I and Pure Mode II. Six years later, Rice, Drugan and Sham[6] Gao[7] got the same result using the Mises yield criterion. It is not a surprise that their result is coincident with Slepjan's result, because the Tresca criterion is coincident with the Mises criterion in the case of plane strain and $\nu=0.5$.

If we take compressibility into account the problem becomes much more complex. Drugan, Rice and Sham[8]; Gao[8] Proposed their results respectively. But their results were different. There were some hot debates. Drugan, Rice and Sham[8] pointed out that the result given by Gao[9] contains some errors, On the other hand Gao[11] contends that the proof of non-singularity of strain field ahead of crack tip given by Drugan, Rice and Sham[8] is incorrect. However the work given by the auther[10] shows that their solutions[8,9] can not meet the requirement of the high order asymptotic equation in the central fan sector, therefore, their results can not be the suitable solution in near tip region.

In the case of plane stress, there is no correct solution, although many scientists have attempted to solve this problem but without success.

On the other hand, the ductile fracture processes are always related to the nucleation, growth and coalescence of micro voids.

Experimental results show that the macro crack grows usually step-by-step. Before crack growth the micro voids will be formed ahead of crack tip. The coalesence between the main crack and the nearst void will result in the crack growth by one step. Fig.1 shows a typical example of the presence of micor void ahead of crack tip in the aluminum thin plate.

For the plane strain problem, the micro voids can not become the thorough thickness cylindrical voids. This results in great difficulty for experimental observation. But the Zig-Zag path of crack growth also implies that the crack grows step-by-step. Hence it is worth while using the step-by-step growth model to investigate the stress and strain fields near the tip of growing crack.

II. Basic Equations for Plane Strain Problem

Let (x,y,z) denote the fixed cartesian coordinate system and (x_1, x_2, x_3) are tensors for (x,y,z). σ_{ij} and ε_{ij} are respectively stress and strain tensors. The corresponding physical components are σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} and ε_x, ε_y, ε_z, ε_{xy}, ε_{yz}, ε_{zx}

1. Yield condition

$$(\sigma_y-\sigma_x)^2/4+\tau_{xy}^2+3S_z^2/4=k^2 \qquad (2.1)$$

2. Constitutive relation

Three-dimensional Prandtl-Reuss flow rule can be expressed as

$$D_{ij}=\frac{(1+\nu)}{E}\dot{\sigma}_{ij}-\frac{\nu}{E}\delta_{ij}\dot{\sigma}_{kk}+\lambda S_{ij} \qquad (2.2)$$

where S_{ij} is the stress deviatoric tensor and D_{ij} is the strain rate tensor.

We have

$$D_{ij}=(\partial v_i/\partial x_j+\partial v_j/\partial x_i)/2 \qquad (2.3)$$

3. Governing equation of plane strain

For plane strain problem, the relevant physical quantities are only of the function of coordinates x_1, x_2, and independent of x_3. We have,

$$D_{33}=\frac{(1+\nu)}{E}\dot{\sigma}_{33}-\frac{\nu}{E}\dot{\sigma}_{kk}+\lambda S_{33}=0$$

From the above equation, it follows,

$$\dot{S}_{33}+\frac{2}{3}E\lambda S_{33}=-\frac{2}{3}\varepsilon\dot{\sigma}_{\alpha\alpha} \tag{2.4}$$

where $\varepsilon=-\nu+1/2$ and Greek indices represent only the numbers 1,2. The repeated index is summed up. Eq. (2.4) is the governing equation of the plane strain.

Let

$$P_{\alpha\beta}=\sigma_{\alpha\beta}-\delta_{\alpha\beta}(\sigma_{\rho\rho}/2)$$

By using Eq.(2.4) constitutive relation (2.2) becomes

$$D_{\alpha\beta}=\{\dot{\sigma}_{\alpha\beta}-\nu\delta_{\alpha\beta}\dot{\sigma}_{\rho\rho}\}/2\mu+\lambda P_{\alpha\beta}-\varepsilon\lambda S_{33}\delta_{\alpha\beta} \tag{2.5}$$

4. Stress function and stress components

Introducing stress function ϕ and putting the following expression for stress components

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2},\quad \sigma_y=\frac{\partial^2\phi}{\partial x^2},\quad \tau_{xy}=\frac{\partial^2\phi}{\partial x\partial y} \tag{2.6}$$

the equation of equilibrium will be automatically satisfied. Eqs. (2.1), (2.3)–(2.6) are the basic equations for the plane strain problem of elastic perfectly-plastic solid.

Eq. (2.5) can be rewritten as

$$D_{\alpha\beta}=\frac{1}{2\mu}\dot{P}_{\alpha\beta}+\lambda P_{\alpha\beta}+\frac{3\varepsilon}{2E}[\dot{\sigma}_{\rho\rho}+\dot{S}_{33}]\delta_{\alpha\beta} \tag{2.7}$$

For the elastic incompressible perfectly plastic solids, we have $\nu=1/2,\ \varepsilon=0$. By using Eq.(2.4), we find $S_{33}=0$. The basic equaitons become

$$(\sigma_y-\sigma_x)^2/4+\tau_{xy}^2=k^2 \tag{2.8}$$

$$D_{\alpha\beta}=\frac{1}{2\mu}\dot{P}_{\alpha\beta}+\lambda P_{\alpha\beta} \tag{2.9}$$

$$D_{\alpha\beta}=(\partial v_\alpha/\partial x_\beta+\partial v_\beta/\partial x_\alpha)/2 \tag{2.10}$$

$$\sigma_x=\phi_{,yy}\quad \sigma_y=\phi_{,xx}\quad \tau_{xy}=-\phi_{,xy} \tag{2.11}$$

III. Asymptotic Field for the Case of First Step Crack Growth

Consider plane strain problem of an incompressible perfectly plastic solid. The elastic-plastic field near the stationary crack tip is

$$\sigma_x=\pi k,\quad \sigma_y=(\pi+2)k,\quad \tau_{xy}=0 \quad \text{in domain } A$$

$$\sigma_r=\sigma_\theta=\left(1+\frac{3}{2}\pi-2\theta\right)k,\quad \tau_{r\theta}=k \quad \text{in domain } B$$

$$\sigma_x=2k,\quad \sigma_y=0,\quad \tau_{xy}=0, \quad \text{in domain } C$$

After crack first step growth, the crack tip will remove to point o. Set up the polar coordinate system $(r,\ \theta)$ with the center at point o.

Because the crack growth step is quite small, circumstantial stress σ_θ alone line element O^*O is uniform (strictly speaking, the stress field near point o differ a little with the asymptotic field

Fig.1

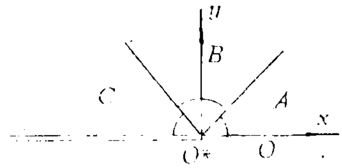

Fig.2

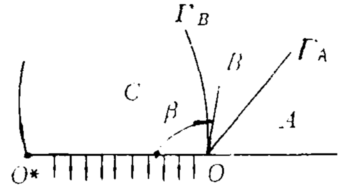

Fig.3

in domainA. But the asymptotic sense, the stress field near point o can be taken to be uniform). The new crack faces will be formed by cutting line element O^*O . The new crack faces are subjected to the traction of normal stress σ_θ . The crack growth process can be simulated by releasing traction σ_θ to zero.

The we have the following boundary condition,

$$\left.\begin{aligned} &\sigma_\theta(r,\ \pi,\ t)=(1-t)\sigma_\theta(r,\ \pi,\ 0)\\ &\tau_{r\theta}(r,\ \pi,\ t)=0 \end{aligned}\right\} \tag{3.1}$$

The generalized time $t=0$ corresponds the initial instance of release process. The $t=1$ corresponds the terminal instance of release process.

Now consider the stress-strain field near tip o of the crack during the release process. We need only consider the upper half plane due to symmetry.

Assume the near tip field is assembled by three sectors (as shown in Fig.3. Domain A is a uniform stress zone, B is a centered fan sector and C is an unloading zone.

Before traction relase, the asymptotic field near point o is a uniform stress field and there is no centered fan sector.

The corresponding stress function ϕ_0^* is,

$$\phi_0^*=\frac{k}{2}r^2[A_1(0)+\cos 2\theta]+O(r^3)$$

where

$$A_1(0)\doteq 1+\pi$$

During release process, the centered fan sector B extends. At time t stress function ϕ can be expressed as

$$\phi=\begin{cases} kr^2[A_1(t)+\cos 2\theta]/2 & (0\leqslant\theta\leqslant\pi/4)\\ \dfrac{kr^2}{2}\left[A_1(t)-2\left(\theta-\dfrac{\pi}{4}\right)\right] & \left(\dfrac{\pi}{4}\leqslant\theta\leqslant\pi-\beta\right)\\ kr^2[C_1(t)+C_2(t)\theta+C_3(t)\cos 2\theta+C_4(t)\sin 2\theta]/2+\phi_0^*(r,\theta) & (\pi-\beta\leqslant\theta\leqslant\pi) \end{cases} \tag{3.2}$$

It is obvious that all of the stress components are continuously across boundary Γ_A .

From the continuity of all stress components on Γ_B we obtain,

$$[\phi]_{\Gamma_B}=[\phi']_{\Gamma_B}=[\phi'']_{\Gamma_B}=0 \tag{3.3}$$

Substituting (3.2) into (3.1) and (3.2), we find,

$$\left.\begin{aligned}
&C_1+C_2(\pi-\beta)=A_1^*-2\left(\frac{3}{4}\pi-\beta\right)\\
&C_2+2C_3\sin2\beta+2C_4\cos2\beta=-2[1+\sin2\beta]\\
&C_3\cos2\beta-C_4\sin2\beta=-\cos2\beta\\
&C_1+C_2\pi+C_3=-(1+A_1(0))t\\
&C_2++2C_4=0, \qquad A_1^*=A_1(t)-A_1(0)
\end{aligned}\right\} \tag{3.4}$$

Solving the above equations, we obtain,

$$\left.\begin{aligned}
&C_1=-\{(1+A_1(0))t+C_2\pi+C_3\}\\
&C_2=\frac{2\cos2\beta}{(1-\cos2\beta)}, \qquad C_3=\frac{\cos2\beta-1-\sin2\beta}{(1-\cos2\beta)}\\
&C_4=\frac{-\cos2\beta}{(1-\cos2\beta)}, \qquad A_1^*=C_1+C_2(\pi-\beta)+2\left(\frac{3}{4}\pi-\beta\right)
\end{aligned}\right\} \tag{3.5}$$

Consider equivalent stress τ_e in domain C. Stress function ϕ and the stress components in domain C are

$$\left.\begin{aligned}
&\phi=\frac{k}{2}r^2\left\{A_1(0)+C_1+\frac{2\cos2\beta}{(1-\cos2\beta)}\theta-\frac{\sin2(\theta+\beta)}{(1-\cos2\beta)}\right\}+O(r^3)\\
&S_r=\frac{1}{2}(\sigma_r-\sigma_\theta)=\frac{k\sin2(\theta+\beta)}{(1-\cos2\beta)}+O(r)\\
&\tau_{r\theta}=-k[\cos2\beta-\cos2(\theta+\beta)]/(1-\cos2\beta)+O(r)
\end{aligned}\right\} \tag{3.6}$$

Hence we have

$$\tau_e^2=\frac{k^2}{(1-\cos2\beta)^2}[1+\cos^22\beta-2\cos2\beta\cos2(\theta+\beta)]+O(r) \tag{3.7}$$

If $\cos2\beta\leqslant0$, it follows

$$1+\cos^22\beta-2\cos2\beta\cos2(\theta+\beta)=(1+\cos2\beta)^2-2\cos2\beta(1+\cos2(\theta+\beta))\geqslant0 \tag{3.8}$$

and

$$\begin{aligned}
&(1-\cos2\beta)^2-\{1+\cos^22\beta-2\cos2\beta\cos2(\theta+\beta)\}\\
&=-2\cos2\beta\{1-\cos2(\theta+\beta)\}\geqslant0
\end{aligned} \tag{3.9}$$

From Eqs. (3.8), (3.9), we find,

$$0\leqslant\frac{1+\cos^22\beta-2\cos2\beta\cos2(\theta+\beta)}{(1-\cos2\beta)^2}\leqslant1 \tag{3.10}$$

It means that the following inequality will be satisfied

$$\tau_e^2\leqslant k^2 \qquad \text{When} \qquad \pi/4\leqslant\beta\leqslant3\pi/4 \tag{3.11}$$

ie.e. the yield constraint could be met in the elastic unloading sector.

On the other hand, from Eq. (3.7), we obtain,

$$(\tau_e^2)^{\cdot}=2\tau_e\dot{\tau}_e=\frac{4k^2\cos 2\beta}{(1-\cos 2\beta)^2}\sin 2(\theta+\beta)\frac{d\beta}{dt}$$

Therefore the following unloading condition on Γ_B has been satisfied,

$$[\dot{\tau}_e]_{\Gamma_B}=0 \qquad \text{at} \qquad \theta=\pi-\beta$$

It can be seen that the asymptotic field defined by Eqs. (3.2), (3.5) satisfies the basic equations, the continuity of stresses between neighboring sectors, boundary condition and unloading condition. The continuity conditions of velocities will be discussed in the next section.

In brief, the asymptotic field defined by (3.2) (3.5) is really a solution of our problem. This solution contains a free parameter β which can be determined by taking the solution of the whole field.

When $\beta=\pi/4$ or $\beta=3\pi/4$, the yield condition will be met anywhere on sector C. $\beta=3\pi/4$ corresponds to the initial instance of release process, i.e. $t=0$. In this instance, there is no sector B and domain C is a uniform stress sector just like domain A. $\beta=\pi/4$ may be corresponding to time t_e, at which the centered fan sector B will become the biggest one $(\pi/4\leqslant\theta\leqslant 3\pi/4)$ and the whole sector C will be translated into a secondary plastic zone. We have in domain C,

$$\left.\begin{array}{ll} C_2=C_4=0, & C_3=-2 \\ C_1=2-(1+A_1(0))t_e & \\ A_1^*=(1+A_1(0))(1-t_e) & \end{array}\right\} \tag{3.12}$$

$$\phi=kr^2\{A_1^*(t)+1-\cos 2\theta\}/2 \tag{3.13}$$

After that time the further releasing traction will not change the deveatoric stresses. It will only change the average stress in the three sectors. The plastic strain will continuously increase until $t=1$, $A_1^*=0$ at which time the whole stress field will coincide with that of stationary crack but the crack tip will remove to point o. Whether or not the centered fan sector will be extended to $\theta=3\pi/4$ is a problem that needs further investigation.

IV. Asymptotic Analysis of Velocity Field

Consider the velocity field in the elastic unloading sector. The stress rate function $\dot{\phi}$ is

$$\dot{\phi}=kr^2[\dot{C}_1(t)+\dot{C}(t)\theta+\dot{C}_3(t)\cos 2\theta+\dot{C}_4(t)\sin 2\theta]/2+O(r^3) \tag{4.1}$$

From Eqs. (2.11), (2.12), we find

$$\left.\begin{array}{l} \dfrac{\partial v_r}{\partial r}=\dfrac{1}{2\mu}\dot{P}_r=-\dfrac{k}{2\mu}[\dot{C}_3\cos 2\theta+\dot{C}_4\sin 2\theta]+O(r) \\ \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial\theta}+\dfrac{v_r}{r}=\dfrac{k}{2\mu}[\dot{C}_3\cos 2\theta+\dot{C}_4\sin 2\theta]+O(r) \\ \dfrac{1}{r}\dfrac{\partial v_r}{\partial\theta}+\dfrac{\partial v_\theta}{\partial r}-\dfrac{v_\theta}{r}=\dfrac{-k}{2\mu}\dot{C}_2+O(r) \end{array}\right\} \tag{4.2}$$

Integrating the first two formulas of Eq. (4.2) leads to

$$v_r=-\frac{k}{2\mu}r[\dot{C}_3\cos 2\theta+\dot{C}_4\sin 2\theta]-f'(\theta,t)+O(r^2)$$

$$v_\theta=\frac{k}{2\mu}r[\dot{C}_3\sin 2\theta-\dot{C}_4\cos 2\theta]+f(\theta,t)+g(r,t)+O(r^2) \tag{4.3}$$

Substituting EQ. (4.3) into the third formula of Eq. (4.2), we obtain

$$f''(\theta,t)+f(\theta,t)=0 \tag{4.4}$$

$$\frac{\partial g}{\partial r}(r,t)-\frac{1}{r}g(r,t)=-\frac{k}{2\mu}\dot{C}_2(t) \tag{4.5}$$

Hence we have

$$g(r,t)=-\frac{k}{2\mu}\dot{C}_2(t)r\ln r+\dot{C}_7(t)r \tag{4.6}$$

$$f(\theta,t)=\dot{C}_5(t)\cos\theta+\dot{C}_6(t)\sin\theta \tag{4.7}$$

The opening displacement rate $\dot{\delta}$ will be,

$$\dot{\delta}=2v_\theta(r,\pi,t)=-\frac{k}{\mu}r[\dot{C}_4+\dot{C}_2(t)\ln r]+2\dot{C}_7(t)r-2\dot{C}_5(t)+O(r^2)$$

Noting that δ is equal to zero at the initial instance of release process therefore we have

$$\delta=-\frac{k}{\mu}r[C_4+C_2\ln r]+2C_7(t)r-2C_5(t)+O(r^2) \tag{4.8}$$

Hence we take the covention in the form

$$C_5(0)=C_7(0)=0$$

Coefficient C_5 must be equal to zero, because the opening displacement at the crack tip should be equal to zero. In the centered fan sector B, we have

$$\dot{\phi}=\frac{k}{2}\dot{A}_1(t)r^2+O(r^3) \tag{4.9}$$

Eqs. (2.11), (2.12) give rise to the following equations

$$\left.\begin{aligned}&\frac{\partial v_r}{\partial r}=\frac{1}{2\mu}\dot{P}_r+\lambda P_r,\qquad \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}=\frac{1}{2\mu}\dot{P}_\theta+\lambda P_\theta\\&\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}=\frac{1}{\mu}\dot{P}_{r\theta}+\lambda P_{r\theta}\end{aligned}\right\} \tag{4.10}$$

Taking into account the continuity of velocities on Γ_B, we can assume that

$$\lambda=\frac{\lambda_{-1}(\theta,t)}{r}+\lambda_0(\theta,t)+r\lambda_1(\theta,t)+\cdots \tag{4.11}$$

$$\phi=\sum_{n=0}^{\infty}r^{n+2}F_n(\theta,t) \tag{4.12}$$

It is easily shown that

$$P_r=S_r=(\sigma_r-\sigma_\theta)/2=r\tilde{S}_{r1}(\theta,t)+O(r^3) \tag{4.13}$$

Substituting Eqs. (4.11), (4.12) into (4.10), we arrive at

$$\left.\begin{aligned} v_r &= f'(\theta,t)+O(r) \\ v_\theta &= f(\theta,t)+g(r,t)+O(r) \end{aligned}\right\} \tag{4.14}$$

Substituting Eq. (4.14) into the third formula of Eq. (4.10), we obtain,

$$f''+f=-k\lambda_{-1}(\theta,t) \tag{4.15}$$

$$\frac{\partial g(r,t)}{\partial r}-\frac{1}{r}g(r,t)=k\lambda_0(\theta,t) \tag{4.16}$$

Eq.(4.16) gives

$$\left.\begin{aligned} g(r,t) &= k\dot{B}_2(t)r\ln r+\dot{B}_6(t)r \\ \lambda_0(\theta,t) &= \dot{B}_2(t) \end{aligned}\right\} \tag{4.17}$$

The continuity conditions of velocities v_r, v_θ on Γ_B become

$$[f]_{\Gamma_B}=[f'(\theta,t)]_{\Gamma_B}=0 \tag{4.18}$$

V. Second Order Asymptotic Field of Stresses

Assume that stress function ϕ can be expanded into the following series near crack tip,

$$\phi=\sum_{n=0,1}^{\infty} r^{2+n}F_n(\theta,t)=\sum_{n=0,1}^{\infty}\phi_n(r,\theta,t) \tag{5.1}$$

The yield condition can be expressed as

$$S_r^2+\tau_{r\theta}^2=k^2 \tag{5.2}$$

Substituting (5.1) into (5.2) and comparing the m-th power of r, we get

$$S_{r0}^2+\tau_{r\theta 0}^2=k^2 \qquad (m=0) \tag{5.3a}$$

$$\sum_{\substack{i=0,1\\ j=m-i}}^{m}\{S_{ri}S_{rj}+\tau_{r\theta i}\tau_{r\theta j}\}=0 \qquad (m=1,2,\cdots) \tag{5.3b}$$

where

$$S_{rm}=(\sigma_{rm}-\sigma_{\theta m})/2$$

and σ_{rm}, $\sigma_{\theta m}$, $\tau_{r\theta m}$ are respectively the stress components given by stress function ϕ_m .

For the case of $m=0$, we have

$$S_{r0}^2+\tau_{r\theta 0}^2=k^2 \tag{5.4}$$

The first order asymptotic solution is based on Eq. (5.4).

For the case of $m=1$, we obtain,

$$S_{r0}\,S_{r1}+\tau_{r\theta 0}\tau_{r\theta 1}=0 \tag{5.5}$$

Eq.(5.5) can be represented in the cartisian coordinate system,

$$S_{x0}\,S_{x1}+\tau_{xy0}\,\tau_{xy1}=0 \tag{5.6}$$

Using the first order asymptotic stress field and the stress boundary, Eqs. (5.5), (5.6) give

$$\phi_1=\begin{cases} A_{11}r^3[(\cos\theta+\sin\theta)^3+(\cos\theta-\sin\theta)^3] & \text{(in domain A)} \\ B_{11}r^3 & \text{(in domain B)} \\ r^3\{C_{11}\cos3\theta+C_{12}\sin3\theta+C_{13}\cos\theta+C_{14}\sin\theta\}+\phi_1^*(r,\theta) & \text{(in domain C)} \end{cases} \quad (5.7)$$

Coefficients C_{1i} satisfy the initial condition

$$C_{1i}(0)=0 \qquad (i=1,\ \cdots,\ 4) \quad (5.8)$$

From the continuity of stress components σ_θ, $\tau_{r\theta}$, it follows

$$B_{11}=8A_{11}$$

Considering the stress fields near crack tip O before traction releasing, we find,

$$\begin{aligned}\phi_1{}^*(r,\theta)&=A_{11}(0)r^3[(\cos\theta+\sin\theta)^3+(\cos\theta-\sin\theta)^3]\\&=A_{11}(0)r^3[2-\cos2\theta]2\cos\theta\end{aligned} \quad (5.9)$$

The continuity of all stress components on Γ_B result in,

$$[\phi]_{\Gamma_B}=[\phi']_{\Gamma_B}=[\phi'']_{\Gamma_B}=0$$

Substituting Eq.(5.1) into the above formulas, we obtain,

$$\left.\begin{aligned} &r^2\{[F_0(\theta,t)]+r[F_1(\theta,t)]+r^2[F_2(\theta,t)]+\cdots\}=0\\ &r^2\{[F_0'(\theta,t)]+r[F_1'(\theta,t)]+r^2[F_2'(\theta,t)]+\cdots\}=0\\ &r^2\{[F_0''(\theta,t)+r[F_1''(\theta,t)]+r^2[F_2''(\theta,t)]+\cdots\}=0\end{aligned}\right\} \quad (5.10)$$

Dividing Eq.(5.10) by r^2 and let r approach to zero, we find,

$$[F_0(\pi-\beta,t)]=[F_0'(\pi-\beta,t)]=[F_0''(\pi-\beta,t)]=0 \quad (5.11)$$

Condition (5.11) is equal to Eq. (3.3). As shown in Fig.3, the curve equation of Γ_B is

$$r=\sum_{n=1}^{\infty}\rho_n(\theta-\theta_2)^n \qquad (\theta_2=\pi-\beta) \quad (5.12)$$

Dividing Eq.(5.10) by r^2 and taking the drivative of (5.10) with respect to θ and using (5.10), we arrive at,

$$\frac{dr}{d\theta}\{[F_1\ (\theta,t)]+2r[F_2\ (\theta,t)]+\cdots\}=0 \quad (5.13)$$

$$\frac{dr}{d\theta}\{[F_1'(\theta,t)]+2r[F_2'(\theta,t)]+\cdots\}=0 \quad (5.14)$$

$$\frac{dr}{d\theta}\{[F_1''(\theta,t)]+2r[F_2''(\theta,t)]+\cdots\}+\{F_0'''(\theta,t)]+r[F_1'''(\theta,t)]+\cdots\}=0 \quad (5.15)$$

Let r approach to zero, we obtain,

$$\left.\begin{aligned}&[F_1(\pi-\beta,t)]=0, \qquad [F_1'(\pi-\beta,t)]=0\\ &[F_1''(\pi-\beta,t)]+\rho_1[F_0'''(\pi-\beta,t)]=0\end{aligned}\right\} \quad (5.16)$$

Substituting Eq. (5.7) into Eq. (5.16), we get,

$$-C_{11}\cos 3\beta + C_{12}\sin 3\beta - C_{13}\cos\beta + C_{14}\sin\beta + A_{11}(0)[(\sin\beta-\cos\beta)^3 - (\sin\beta+\cos\beta)^3] = B_{11} \tag{5.17}$$

$$3(C_{11}\sin 3\beta + C_{12}\cos 3\beta) + C_{13}\sin\beta + C_{14}\cos\beta = 6A_{11}(0)\cos 2\beta\sin\beta \tag{5.18}$$

$$9(-C_{11}\cos 3\beta + C_{12}\sin 3\beta) - C_{13}\cos\beta + C_{14}\sin\beta + 6A_{11}(0)(\cos\beta\cos 2\beta + \cos 3\beta - \cos\beta) = \rho_1[F_0'''(\pi-\beta,t)] \tag{5.19}$$

The boundary conditions on crack faces lead to

$$3C_{12} + C_{14} = 0 \tag{5.20}$$

$$C_{11} + C_{13} = -2A_{11}(0)t \tag{5.21}$$

Eqs. (5.17) – (5.12) consist of five equations. But there are six unknown coefficients (besides coefficient β). ρ_1 chosen as a free coefficient, the remaining coefficients can be easily obtained. Due to limitation of space, the detailed results are not listed here. It is easily shown from Eq. (5.7),

$$S_{r1} = (\sigma_{r1} - \sigma_{\theta 1})/2 = B_{11}(t)3r/2 \tag{5.22}$$

VI. Asymptotic Field Near Crack Tip Growing Step-by-Step

We have considered the asymptotic field for the case of the first step crack growth in the previous sections. Now we consider the asymptotic field for the next step crack growth. It is worth while noting that after first step crack growth the stress field ahead of the new crack tip is also a uniform stress field.

If the centered fan sector B could be extended to $\theta = 3\pi/4$, then coefficient $A_i(0)$ will be equal to that for stationary crack. Therefore the analysis of asymptotic field for the next step crack growth will be the same procedure as in the previous sections.

It the centered fan sector B can not extend to $\theta = 3\pi/4$, coefficient $A_i(0)$ will differ with that for the stationary crack. It does not matter for the analysis of asymptotic field for the next step crack growth.

In brief, the asymptotic field for the next step crack growth can be obtained with the same procedure as the first step crack growth.

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