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On the Behaviour of Gravity Waves on Porous Slopes

Suppose that a sinusoidal wave in water of constant depth progresses in positive x-direction and is incident on a slope which intersects the x-axis at the angle α. A two-dimensional co-ordinate system is used with the y-axis pointing upwards, the x-axis located on the undisturbed free surface and the origin 0 set just above the intersection point of the bottom line and the sloping beach. It is assumed that the fluid is inviscid, incompressible and without surface tension and that the flow is irrotational. It is also assumed that the slope angle α is so small that the parameter $\frac{1}{k_0}\frac{dh/dx}{h_0}$ can be used as a small parameter where h is the vertical distance from the slope to x-axis, $k_0 = \frac{2\pi}{\lambda_0}$ is the wave number of the incident wave and h_0 is the constant depth of the channel. It is apparent that $h = h_0 - ax$ where $a = \tan\alpha$. Let us introduce the small positive parameter

$$\varepsilon = O\left(\frac{1}{k_0}\frac{|dh/dx|}{h_0}\right) = O\left(\frac{\lambda_0 a}{2\pi h_0}\right) \ll 1\,. \tag{1}$$

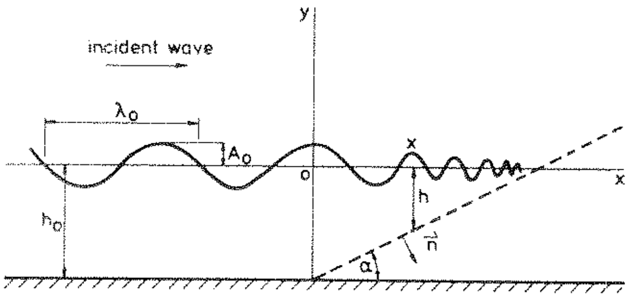

Fig. 1. Schematic diagram of a thin porous slope fixed in a channel with constant depth

Suppose the velocity potential of the small-amplitude surface wave over the slope is $\Phi(x, y, t)$ which satisfies the following equations:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0\,, \qquad x \geqq 0\,, \qquad -h < y < 0; \qquad \frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial y} = 0\,, \qquad x \geqq 0\,,\ y = 0\,, \tag{2; 3}$$

$$\frac{a\dfrac{\partial\Phi}{\partial x} - \dfrac{\partial\Phi}{\partial y}}{\sqrt{1 + a^2}} = v_n\,, \qquad x > 0\,, \qquad y = -h(x); \tag{4}$$

$$\Phi(x, y, t) = -\frac{igA_0}{\omega}\,\frac{\cosh\left[k_0(y + h_0)\right]}{\cosh(k_0 h_0)}\,e^{i(k_0 x - \omega t)}\,, \qquad x \leqq 0\,, \tag{5}$$

where v_n is the outward normal velocity on the beach, g is the gravitational accelleration, k_0 is the wave number of the incident wave, A_0 is the amplitude of the incident wave, $\omega^2 = gk_0 \tanh(k_0 h_0)$ and ω is the angular frequency.

Let us introduce the shrunk co-ordinates

$$\bar{x} = \varepsilon x\,, \qquad \bar{t} = \varepsilon t\,. \tag{6}$$

Then Φ becomes a function of $\bar{x}$, y and $\bar{t}$. Let Φ still be denoted by $\Phi = \Phi(\bar{x}, y, \bar{t})$. In order to solve the problem (2) to (5), according to KELLER [1], we assume the following asymptotic expansion for Φ:

$$\Phi(\bar{x}, y, \bar{t}) \sim \left[\varphi_0(\bar{x}, y, \bar{t}) + (-i\varepsilon)\,\varphi_1(\bar{x}, y, \bar{t}) + (-i\varepsilon)^2\,\varphi_2(\bar{x}, y, \bar{t}) + \dots\right] \exp\left\{\frac{i}{\varepsilon}\left[S(\bar{x}) - \omega\bar{t}\right]\right\}. \tag{7}$$

Substituting (7) into (2) to (4), from the coefficients of ε^0 we obtain

$$-k^2\varphi_0 + \frac{\partial^2\varphi_0}{\partial y^2} = 0\,, \qquad \bar{x} \geqq 0\,, \qquad -h < y < 0; \qquad -\omega^2\varphi_0 + g\frac{\partial\varphi_0}{\partial y} = 0\,, \qquad \bar{x} \geqq 0\,, \qquad y = 0\,, \tag{8; 9}$$

$$\frac{\partial\varphi_0}{\partial y} = -v_n \exp\left\{-\frac{i}{\varepsilon}\left[S(\bar{x}) - \omega\bar{t}\right]\right\}, \qquad \bar{x} \geqq 0\,, \qquad y = -h\,, \tag{10}$$

where

$$ds/d\bar{x} = S_{\bar{x}} = k\,. \tag{11}$$

It is easy to prove that v_n must be $O(\varepsilon)$ otherwise the porous slope will not affect the wave number of the incident wave and thus will not function like a wave absorber. Suppose $v_n = O(\varepsilon)$. Then (10) becomes

$$\frac{\partial \varphi_0}{\partial y} = 0\,, \qquad \bar{x} \geqq 0\,, \qquad y = -h\,. \tag{12}$$

φ_0 satisfying (8), (9) and (12) can be solved,

$$\varphi_0 = -\frac{igA_0}{\omega}\,\frac{\cosh\left[k_0(y + h_0)\right]}{\cosh\left(k_0 h_0\right)}\,\mathrm{e}^{i(k_0 x - \omega t)}\,, \tag{13}$$

where k satisfies $\omega^2 = gk \tanh(kh)$ and $A = A(\bar{x}) \geqq 0$.

From the coefficients of ε^1 we obtain

$$\varphi_{1yy} - k^2\varphi_1 = k\varphi_{0\bar{x}} + (\varphi_0 k)_{\bar{x}}\,, \qquad \bar{x} \geqq 0\,, \qquad -h < y < 0\,, \tag{14}$$

$$\varphi_{1y} - (\omega^2/g)\,\varphi_1 = -\frac{1}{g}\left[\omega\varphi_{0\bar{t}} + (\omega\varphi_0)_{\bar{t}}\right], \qquad \bar{x} \geqq 0\,, \qquad y = 0\,, \tag{15}$$

$$\varphi_{1y} = -a_0 k_0 \varphi_0 - \frac{i}{\varepsilon}\,v_n \exp\left\{-\frac{i}{\varepsilon}\left[S(\bar{x}) - \omega\bar{t}\right]\right\}, \qquad \bar{x} \geqq 0\,, \qquad y = -h\,, \tag{16}$$

where the subscripts $\bar{x}$, y, $\bar{t}$ mean differentiating with respect to the corresponding variables and $a_0 = a/\varepsilon$. From (16) we see that v_n should include the same phase factor as Φ. Let

$$v_n = \varepsilon v_0(\bar{x}) \exp\left\{\frac{i}{\varepsilon}\left[S(\bar{x}) - \omega\bar{t}\right]\right\}, \tag{17}$$

where $v_0(\bar{x})$ is an arbitrary function. v_n should have the same phase as Φ_{0n} on $y = -h$. So $v_0(\bar{x}) \geqq 0$. Substituting (17) into (16), we obtain

$$\varphi_{1y} = -a_0 k_0 \varphi_0 - iv_0(\bar{x})\,, \qquad \bar{x} \geqq 0\,, \qquad y = -h\,. \tag{18}$$

Multiplying both sides of (14) by φ_0, integrating from $-h$ to 0 and taking into account (8) to (10), (15) and (18), we obtain

$$\frac{\partial}{\partial \bar{x}} \int\limits_{-h}^{0} k\varphi_0^2 \,\mathrm{d}y - iv_0(\bar{x})\left[\varphi_0\right]_{y=-h} = 0\,. \tag{19}$$

Substituting (13) for φ_0 in (19), we have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\left(\frac{A^2 C_g}{\omega}\right) + \frac{Av_0(\bar{x})}{\omega \cos(kh)} = 0\,, \tag{20}$$

where $C_g = \dfrac{\omega}{2k}\left[1 + \dfrac{2kh}{\sinh(2kh)}\right]$ is the local group velocity. Equation (20) can be simplified to obtain

$$\frac{\mathrm{d}A}{\mathrm{d}\bar{x}} + \frac{1}{2C_g}\,\frac{\mathrm{d}C_g}{\mathrm{d}\bar{x}}\cdot A + \frac{v_0(\bar{x})}{2C_g \cosh(kh)} = 0\,.$$

This equation can easily be solved to give

$$A(\bar{x}) = \frac{A_0\sqrt{C_{g0}}}{\sqrt{C_g}}\left[1 - \frac{1}{A_0\sqrt{C_{g0}}}\int\limits_0^{\bar{x}} \frac{v_0(\bar{x})}{2\sqrt{C_g}\cosh(kh)}\,\mathrm{d}\bar{x}\right], \tag{21}$$

where C_{g0} is the group velocity of the incident wave, i.e., $C_{g0} = \dfrac{\omega}{2k_0}\left[1 + \dfrac{2k_0h_0}{\sinh(2k_0h_0)}\right]$. When $v_0(\bar{x}) = 0$, (21) becomes exactly the solution of the amplitude for slowly varying smooth beach [2]. Since $v_0(\bar{x}) \geqq 0$, from (21) we see that the amplitude of the wave over the porous slope decays with the increase of x. Because v_n is proportional to the permeability of the slope and so does $v_0(\bar{x})$, from (21) the amplitude of the wave over the porous slope decreases with the increase of the beach permeability. Strictly speaking, equation (21) cannot be used for very shallow water. We will discuss the shallow water wave on a porous beach in another paper.

References

1 Keller, J. B., Surface Waves on Water on Non-uniform Depth, J. Fluid Mech. 4 (1958), 607–614.

2 Mei, C. C., The Applied Dynamics of Ocean Surface Waves, Chapter 3, John Wiley & Sons, Inc., New York, 1983.

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