## BRIEF NOTES

$$
\begin{equation*}
\omega(t)=\Omega_{2}-\Omega_{1} . \tag{3.2a}
\end{equation*}
$$

We now write $G(t)$ in the form

$$
\begin{equation*}
G(t)=N(t)\left[n g_{f}(t)+g_{T}(t)\right], n \cdot g_{T}(t)=0 \tag{3.5a}
\end{equation*}
$$

Here $N(t)$ is the normal force, $N n g_{f}$ is the frictional moment which is normal to the surface, and $N g_{T}$ is the nonfrictional moment which is tangential. We assume that $g_{f}$ is given by a law of angular friction of the following form, in which $\omega_{n}=\omega \cdot n$ :

$$
\begin{array}{r}
g_{f}=-\nu \operatorname{sgn} \omega_{n} \text { if } \omega_{n} \neq 0,  \tag{3.6a}\\
\left|g_{f}\right| \leq \nu \text { if } \omega_{n}=0 .
\end{array}
$$

The quantity $\nu$ is a coefficient of angular sliding friction with the dimensions of length. In principle it can be determined by analyzing the deformation of the surfaces during impact and using the usual law of friction. It ought to be proportional to $\mu$ and should depend upon $N(t)$.

When we use equations (3.5a) in (2.11a), and write $g_{f}$ and $g_{T}$ as functions of $\tau$, we get

$$
\begin{equation*}
\Gamma=\int_{0}^{(1+e) \tau_{0}}\left[n g_{f}(\tau)+g_{T}(\tau)\right] d \tau \tag{3.8a}
\end{equation*}
$$

Then equation (3.8a) is used in equation (3.10) with the extra term $(-1)^{j} J_{j}^{-1} \Gamma$ on the right side, and equation (3.10) becomes

$$
\begin{align*}
{\left[\Omega_{j}\right]=} & (-1)^{j} J_{j}^{-1}\left(R_{j} \times\left\{(1+e) \tau_{0} n-\mu \int_{0}^{(1+e) \tau_{0}} \hat{u}_{T}(\tau) d \tau\right\}\right. \\
& \left.+\int_{0}^{(1+e) \tau_{0}}\left[n g_{f}(\tau)+g_{T}(\tau)\right] d \tau\right) . \tag{3.10a}
\end{align*}
$$

Next we must add $\left(J_{j}^{-1} G\right) \times R_{j}$ to the summand in equation (4.1) and add $n g_{f}+g_{T}$ to $R_{j} \times(n+f)$ in equations (4.2), (4.3), (4.4), (4.6), and (4.9). We must also add $n \cdot\left[J_{j}^{-1}\right.$ $\left.\int_{0}^{\tau}\left(n g_{f}+g_{T}\right) d \tau^{\prime}\right] \times R_{j}$ to the summands in equations (4.7) and (4.8) with $\tau=\tau_{0}$ in (4.8).

To determine $\omega(\tau)$, which occurs in equation (3.6a), we differentiate equation (3.2a) and use equation (2.6) to get

$$
\begin{equation*}
\frac{d \omega}{d t}=\sum_{j=1}^{2} J_{j}^{-1}\left[G+R_{j} \times F\right] . \tag{4.1a}
\end{equation*}
$$

Now we use equation (3.5) for $F$ and equation (3.5a) for $G$ in equation (4.1a) and write $N^{-1} d / d t=d / d \tau$ to obtain

$$
\begin{equation*}
\frac{d \omega}{d \tau}=\sum_{j=1}^{2} J_{j}^{-1}\left[n g_{f}+g_{T}+R_{j} \times(n+f)\right] \tag{4.3a}
\end{equation*}
$$

From equation (3.2a) the initial value of $\omega$ is

$$
\begin{equation*}
\omega(0)=\Omega_{2}^{-}-\Omega_{1}^{-} . \tag{4.4a}
\end{equation*}
$$

Finally we take the inner product of $n$ with equations (4.3a) and (4.4a) to get

$$
\begin{gather*}
\frac{d \omega_{n}}{d \tau}=\sum_{j=1}^{2} n \cdot J_{j}^{-1}\left[n g_{f}+g_{T}+R_{j} \times(n+f)\right],  \tag{4.9a}\\
\omega_{n}(0)=n \cdot\left(\Omega_{2}^{-}-\Omega_{1}^{-}\right) . \tag{4.10a}
\end{gather*}
$$

We have now derived equations for the determination of $\omega_{n}(\tau)$ and $u_{T}(\tau)$. They are equations (4.9a), (4.10a), (4.9) with the extra term $\left(J_{j}^{-1} \cdot\left[n g_{f}+g_{T}\right]\right) \times R_{j}$ inside the braces, and equation (4.10). In these equations the nonfrictional contact moment $g_{T}(\tau)$ occurs. When $g_{T}=0$ and $\nu$ is constant these equations can be solved as in the case without any contact moment. However, it is necessary to determine if $\omega_{n}$ vanishes during the collision, and if so whether it remains zero. It will remain zero if $g_{f}$ satisfies the inequality in equation (3.6a). To find $g_{f}$ one must solve equation (3.5a) with $G$ determined from equation (4.1a) with $\omega_{n}=0$.

As an example let us consider a collision in which the last two terms in equation (4.9a) vanish. Then if $\omega_{n}(\tau) \neq 0,(4.9 a)$ reduces to

$$
\begin{equation*}
\frac{d \omega_{n}}{d \tau}=-\nu n \cdot\left(J_{1}^{-1}+J_{2}^{-1}\right) n \operatorname{sgn} \omega_{n} \tag{1}
\end{equation*}
$$

The solution of equation (1) is

$$
\begin{equation*}
\omega_{n}(\tau)=\omega_{n}(0)-\nu \tau n \cdot\left(J_{1}^{-1}+J_{2}^{-1}\right) n s g n \omega_{n}(0) \tag{2}
\end{equation*}
$$

The right side of equation (2) vanishes at

$$
\begin{equation*}
\tau^{* *}=\left|\omega_{n}(0)\right|\left[\nu n \cdot\left(J_{1}^{-1}+J_{2}^{-1}\right) n\right]^{-1} . \tag{3}
\end{equation*}
$$

If $\tau^{* *} \geq(1+e) \tau_{0}$, equation (2) holds throughout the collision. However, if $\tau^{* *}<(1+e) \tau_{0}$ then $\omega_{n}(\tau)=0$ for $\tau \geq$ $\tau^{* *}$. This is so because equation (4.9a) with $\omega_{n}(\tau)=0$ yields $g_{f}=0$, and this satisfies the inequality in (3.6a). Thus $\omega_{n}$ at the end of the collision is given by equation (2) with $\tau=(1+$ $e) \tau_{0}$, or by $\omega_{n}=0$ according as $\tau^{* *}$ is greater or smaller than $(1+e) \tau_{0}$. Finally equation ( $3.8 a$ ) yields

$$
\begin{equation*}
\Gamma=-\nu \operatorname{sgn} \omega_{n}(0) \cdot \min \left[\tau^{* *},(1+e) \tau_{0}\right] . \tag{4}
\end{equation*}
$$

This example applies in particular to the normal collision of two spheres with $g_{T}=0$, including the special case of a sphere and a plane. Then $J_{j}=2 / 5 m_{j} r_{j}^{2} I$, so equations (2)-(4) become

$$
\begin{gather*}
\omega_{n}(\tau)=\omega_{n}(0)-\frac{5 \nu \tau}{2}\left(\frac{1}{m_{1} r_{1}^{2}}+\frac{1}{m_{2} r_{2}^{2}}\right) \operatorname{sgn} \omega_{n}(0),  \tag{5}\\
\tau^{* *}=\frac{2}{5 \nu}\left(\frac{1}{m_{1} r_{1}^{2}}+\frac{1}{m_{2} r_{2}^{2}}\right)^{-1}\left|\omega_{n}(0)\right| \tag{6}
\end{gather*}
$$

$\Gamma \begin{cases}=-\nu(1+e) \tau_{0} \operatorname{sgn} \omega_{n}(0) & \text { if } \tau^{* *} \geq(1+e) \tau_{0} \\ =-\frac{2}{5}\left(\frac{1}{m_{1} r_{1}^{2}}+\frac{1}{m_{2} r_{2}^{2}}\right)^{-1} \omega_{n}(0) & \text { if } \tau^{* *} \leq(1+e) \tau_{0} .\end{cases}$
Finally since $R_{1}=n r_{1}$ and $R_{2}=-n r_{2}$, equation (3.10a) becomes
$\left[\Omega_{j}\right]=(-1)^{j} \frac{5}{2 m_{j} r_{j}^{2}}\left(\mu r_{j} n \times \int_{0}^{(1+e) \tau_{0}} \hat{u}_{T}(\tau) d \tau+\Gamma\right)$.

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## Perturbation Solutions on J-Integral for

 Nonlinear Crack Problems
## Ming-Yuan He ${ }^{18}$

A crack of length $2 a$ in an infinite two-dimensional body occupies $-a \leq x_{1} \leq a, x_{2}=0$ and the following load systems are

[^0]Table 1 Comparison between the perturbation solutions (13) and numerical results for plane strain crack with $T=0$

|  |  |  | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=7$ | $n=10$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Mode$\frac{\stackrel{\text { II }}{S}}{Q}=0$ | $\frac{J}{a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}}$ | from (13) | 0.785 | 1.120 | 1.387 | 1.611 | 1.804 | 2.112 | 2.439 |
|  |  | from He (1983) | 0.785 | 1.111 | 1.360 | 1.571 | 1.756 | 2.078 | 2.483 |
|  |  | \% difference | * | 0.8 | 1.95 | 2.48 | 2.65 | 1.61 | 1.83 |
| Mode$\frac{\mathrm{S}-\mathrm{II}}{Q}=0.5$ | $\frac{J}{a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}}$ | from (13) | 0.929 | 1.327 | 1.650 | 1.932 | 2.181 | 2.603 | 3.068 |
|  |  | from He (1983) | 0.929 | 1.314 | 1.609 | 1.858 | 2.077 | 2.458 | 2.938 |
|  |  | \% difference | * | 0.98 | 2.48 | 3.83 | 4.77 | 5.57 | 4.24 |
| Mode I-II$\frac{S}{Q}=1.96$ | $\frac{J}{a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}}$ | from (13) | 1.941 | 2.746 | 3.341 | 3.807 | 4.139 | 4.744 | 5.248 |
|  |  | from He (1983) | 1.941 | 2.745 | 3.362 | 3.882 | 4.341 | 5.136 | 6.139 |
|  |  | \% difference | * | 0.004 | 0.06 | 1.97 | 4.88 | 8.26 | 17.0 |
| $\begin{aligned} & \begin{array}{c} \text { Mode } \\ \text { I } \\ \frac{S}{Q} \end{array}=0 \end{aligned}$ | $\frac{J}{a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}}$ | from (13) | 3.142 | 4.470 | 5.511 | 6.390 | 7.152 | 8.421 | 9.870 |
|  |  | from He (1983) | 3.142 | 4.443 | 5.441 | 6.283 | 7.025 | 8.312 | 9.935 |
|  |  | \% difference | * | 0.60 | 1.27 | 1.67 | 1.78 | 1.16 | 0.66 |

Table 2 Comparison between the perturbation solution (15) and numerical results for Mode III crack ( $\tau=0$ )

|  |  | $n=1.0$ | $n=1.5$ | $n=2.0$ | $n=3.0$ | $n=5.0$ | $n=10$ |
| :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $J$ | from Amazigo (1974) | 1.571 | 1.939 | 2.271 | 2.864 | 3.865 | 5.788 |
|  | from (15) | 1.571 | 1.924 | 2.222 | 2.721 | 3.513 | 4.968 |
| $e_{e}^{\infty} \sigma_{e}^{\infty}$ | $\%$ difference | 0 | 0.76 | 2.17 | 4.99 | 9.11 | 14.17 |

considered: (1) remote uniform shear stress $\sigma_{12}=Q$ (Mode II); (2) remote uniform tensile stress $\sigma_{11}=T, \sigma_{22}=S$ and shear stress $\sigma_{12}=Q$ (mixed Mode I and II); (3) remote antiplane shear stress $\sigma_{23}=R$ (Mode III). We consider an incompressible solid under small strain deformation and characterized in simple tension by

$$
\begin{equation*}
\epsilon / \epsilon_{o}=\alpha\left(\sigma / \sigma_{o}\right)^{n} \tag{1}
\end{equation*}
$$

where $\epsilon_{o}$ and $\sigma_{o}$ are a reference strain and stress, $\alpha$ and $n$ are material constants. The tensile relation is generalized to multiaxial states by the $J_{2}$ deformation theory according to

$$
\begin{equation*}
\frac{\epsilon_{i j}}{\epsilon_{o}}=\frac{3}{2} \alpha\left(\frac{\sigma_{e}}{\sigma_{o}}\right)^{n-1} \frac{s_{i j}}{\sigma_{o}} \tag{2}
\end{equation*}
$$

where $s_{i j}$ is the stress deviator and $\sigma_{e}$ is the effective stress defined by

$$
\begin{equation*}
\sigma_{e}=\left(\frac{3}{2} s_{i j} s_{i j}\right)^{1 / 2} \tag{3}
\end{equation*}
$$

Following procedures developed by He and Hutchinson (1981) and Abeyaratne (1983), we produce solutions for the relation between the $J$-integral and the remote stresses. The method is based on a perturbation about a specially selected uniform trivial state. With that state specified by $\sigma$ and $\epsilon$, the increments in deviatoric stress and strain satisfy

$$
\begin{array}{r}
\dot{s}_{i j}=2 \mu\left\{\frac{1}{2}\left(\delta_{i k} \delta_{j \ell}+\delta_{i \ell} \delta_{j k}\right)-\frac{1}{3} \delta_{i j} \delta_{k \ell}\right. \\
\left.+\frac{2}{3} \frac{1-n}{n} \epsilon_{i j} \epsilon_{k \ell} \epsilon_{e}^{-2}\right\} \dot{\epsilon}_{k \ell} \tag{4}
\end{array}
$$

where

$$
\begin{equation*}
\mu=\frac{\sigma_{e}}{3 \epsilon_{e}} \tag{5}
\end{equation*}
$$

and

$$
\epsilon_{e}=\left(2 \epsilon_{i j} \epsilon_{i j} / 3\right)^{1 / 2}
$$

## Mode II Crack

The trivial state is taken as $\sigma_{11}=T<0, \sigma_{22}=0$. We consider
the plane strain problem, so that $\sigma_{33}=T / 2$ and $\sigma_{e}=(\sqrt{3} / 2)|T|$. In this uniform trivial state the crack remains closed, $J$ is zero and the nonzero strain components from equation (2) are

$$
\begin{equation*}
\epsilon_{22}=-\epsilon_{11}=\frac{\sqrt{3}}{2} \epsilon_{e} \tag{6}
\end{equation*}
$$

Onto this trivial state we superimpose an increment of remote inplane shear stress $Q$ so that

$$
\begin{equation*}
\sigma_{11}^{\infty}=T, \quad \sigma_{22}^{\infty}=0, \quad \sigma_{33}^{\infty}=\frac{T}{2}, \quad \dot{\sigma}_{12}^{\infty}=Q \tag{7}
\end{equation*}
$$

Following procedures which closely parallel those mentioned above, we have solved the linear incremental problem for a crack in a body with uniform incremental moduli specified by equation (4) where the strains in the trival state are given by equation (6). The result is a relation between the $J$-integral and $Q$ which is exact to lowest order in $Q$. The result so obtained is

$$
\begin{equation*}
\frac{J}{a \sigma_{e}^{\infty} \epsilon_{e}^{\infty}}=\frac{3 \pi \sqrt{n}}{4}\left(\frac{Q}{\sigma_{e}^{\infty}}\right)^{2} \tag{8}
\end{equation*}
$$

where we have taken

$$
\begin{equation*}
\sigma_{e}^{\infty}=\left\{\frac{3}{4}\left(T^{2}+4 Q^{2}\right)\right\}^{1 / 2} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\epsilon_{e}^{\infty} / \epsilon_{o}=\alpha\left(\sigma_{e}^{\infty} / \sigma_{o}\right)^{n} \tag{10}
\end{equation*}
$$

This formula is exact for $n=1$, but at this point we can only argue for its accuracy for $n>1$ when $|Q|$ is small compared to $|T|$. The accuracy at arbitrary $Q$ will be assessed below.

## Mode I-II Crack

For the crack subjected to a general plane loading, the same trivial state $\sigma_{11}=T<0, \sigma_{22}=0$, and $\sigma_{33}=T / 2$ is used. Onto this trivial state we superimpose an increment of remote tensile stress $S$ perpendicular to the crack and an increment of remote inplane shear stress $Q$ parallel to the crack, so that

$$
\begin{equation*}
\sigma_{11}^{\infty}=T, \quad \dot{\sigma}_{22}^{\infty}=S, \quad \dot{\sigma}_{12}^{\infty}=Q \tag{11}
\end{equation*}
$$

The incremental problem is again a linear problem. The two

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incremental problems ( $\dot{\sigma}_{22}^{\infty}=S, \dot{\sigma}_{12}^{\infty}=0$ and $\dot{\sigma}_{22}^{\infty}=0, \dot{\sigma}_{12}^{\infty}=Q$ ) therefore can be solved respectively, and the final solution is simply given by superimposing the two incremental solutions.

From He and Hutchinson (1981), for $\dot{\sigma}_{22}^{\infty}=S, \dot{\sigma}_{12}^{\infty}=0$, the incremental solution is

$$
\begin{equation*}
\frac{J}{a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}}=\frac{3 \pi \sqrt{n}}{4}\left(\frac{S}{\sigma_{e}^{\infty}}\right)^{2} \tag{12}
\end{equation*}
$$

The incremental solution for $\dot{\sigma}_{22}^{\infty}=0, \dot{\sigma}_{12}^{\infty}=Q$ is given by equation (8). Because the two incremental solutions are purely symmetric and antisymmetric, respectively, with respect to the crack, the resulting expression for $J$ for the combined increments of $S$ and $Q$ is obtained by the sum of equations (12) and (8) as

$$
\begin{equation*}
\frac{J}{a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}}=\frac{3 \pi \sqrt{n}\left(S^{2}+Q^{2}\right)}{4\left(\sigma_{e}^{\infty}\right)^{2}} \tag{13}
\end{equation*}
$$

where now

$$
\begin{equation*}
\sigma_{e}^{\infty}=\left[\frac{3}{4}\left((S-T)^{2}+4 Q^{2}\right)\right]^{1 / 2} \tag{14}
\end{equation*}
$$

Formula (13) includes equation (8) as a special case. It is exact for $n=1$ and it is consistent with the requirement that $J$ be homogeneous of degree $n+1$ in $\boldsymbol{\sigma}^{\infty}$. For $Q=0$ it reduces to the result of He and Hutchinson (1981), which has been shown to be highly accurate for arbitrary combinations of $S$ and $T$ except those for which $S \approx T$. Table 1 compares the prediction from formula (13) with numerical results for $J /\left(a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}\right)$ based on a lower bound modified energy principle which was presented by He and Hutchinson (1983) and He (1983). It is seen that the predictions from formula (13) are in close agreement with the numerical results for pure Mode II ( $S=T=0$ ) and in pure Mode I. The greatest discrepancy is for the mixedmode case with $S / Q=1.96$ where for $n=10$ the result from
formula (13) is 17 percent above the numerical lower bound. It is very possible that part of this discrepancy may stem from the difference between the bound and the actual result.

## Mode III Crack

The same procedures can be applied to the Mode III problem. In this case the trivial reference state is $\sigma_{13}=\tau$ and an increment of remote stress $\dot{\sigma}_{23}^{\infty}=R$ is applied. The resulting relation between $J$ and $R$ which is exact to the lowest order in $R$ is

$$
\begin{equation*}
\frac{J}{a \epsilon_{e}^{\infty} \sigma_{e}^{\infty}}=\frac{3 \pi \sqrt{n}}{4}\left(\frac{R}{\sigma_{e}^{\infty}}\right)^{2} \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{e}^{\infty}=\left[3\left(\tau^{2}+R^{2}\right)\right]^{1 / 2} \tag{16}
\end{equation*}
$$

This result agrees with that of Abeyaratne (1983) with $\tau=0$. As noted by that author, the result with $\tau=0$ is in excellent agreement with the exact results computed by Amazigo (1974), as can be seen in Table 2.

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