As an example let us consider a collision in which the last two terms in equation (4.9a) vanish. Then if \( \omega_n(\tau) \neq 0 \), (4.9a) reduces to

\[
\frac{d\omega_n}{d\tau} = -m(\tau_1^1 + \tau_2^1) nsgn \omega_n.
\]

The solution of equation (1) is

\[
\omega_n(\tau) = \omega_n(0) - m(\tau_1^1 + \tau_2^1) nsgn \omega_n(0).
\]

The right side of equation (2) vanishes at

\[
\tau^* = \omega_n(0) [m(\tau_1^1 + \tau_2^1) n^1 - 1].
\]

If \( \tau^* \geq (1 + e) \tau_0 \), equation (2) holds throughout the collision. However, if \( \tau^* < (1 + e) \tau_0 \) then \( \omega_n(\tau) = 0 \) for \( \tau \geq \tau^* \). This is so because equation (4.9a) with \( \omega_n(\tau) = 0 \) yields \( g_f = 0 \), and this satisfies the inequality in (3.6a). Thus \( \omega_n \) at the end of the collision is given by equation (2) with \( \tau = (1 + e) \tau_0 \). Finally equation (3.8a) yields

\[
\Gamma = -v \omega_n(0) \min \lbrack \tau^*, (1 + e) \tau_0 \rbrack.
\]

This example applies in particular to the normal collision of two spheres with \( N \neq 0 \), including the special case of a sphere and a plane. Then \( J_f = 2/5 \, mJ_i \), so equations (2)–(4) become

\[
\omega_n(\tau) = \omega_n(0) - \frac{5\tau}{2} \left( \frac{1}{m_1 J_1^0} + \frac{1}{m_2 J_2^0} \right) nsgn \omega_n(0),
\]

\[
\tau^* = \frac{2}{5} \left( \frac{1}{m_1 J_1^0} + \frac{1}{m_2 J_2^0} \right)^{-1} \omega_n(0) \Gamma,
\]

\[
\Gamma = -v(1 + e) \omega_n(0) \min \lbrack \tau^*, (1 + e) \tau_0 \rbrack.
\]

Finally since \( \Gamma = nr_1 \) and \( \Gamma = -nr_2 \), equation (3.10a) becomes

\[
\omega_n(\tau) = \omega_n(0) - \frac{5\tau}{2} \left( \frac{1}{m_1 J_1^0} + \frac{1}{m_2 J_2^0} \right) nsgn \omega_n(0),
\]

\[
\tau^* = \frac{2}{5} \left( \frac{1}{m_1 J_1^0} + \frac{1}{m_2 J_2^0} \right)^{-1} \omega_n(0) \Gamma,
\]

\[
\Gamma = -v(1 + e) \omega_n(0) \min \lbrack \tau^*, (1 + e) \tau_0 \rbrack.
\]

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Reference


Perturbation Solutions on J-Integral for Nonlinear Crack Problems

Ming-Yuan He

A crack of length \( 2a \) in an infinite two-dimensional body occupies \(-a \leq x_1 \leq a, x_2 = 0\) and the following load systems are
The incremental problem is again a linear problem. The two parallel to the crack, so that inplane shear stress \( Q \) perpendicular to the crack and an increment of remote stress this trivial state we superimpose an increment of remote tensile trivial state \( \sigma_{22} = T < 0, \sigma_{n} = 0, \) is used. Onto \( \sigma_{22} = T \) and \( \sigma_{n} = 0. \) We consider an incompressible solid under small strain deformation and characterized in simple tension by

\[
\epsilon / \epsilon_0 = \alpha (\sigma / \sigma_0)^n
\]

where \( \epsilon_0 \) and \( \sigma_0 \) are a reference strain and stress, \( \alpha \) and \( n \) are material constants. The tensile relation is generalized to multiaxial states by the \( J_2 \) deformation theory according to

\[
\epsilon_\sigma = \frac{3}{2} \frac{\sigma_\sigma}{\sigma_0} - \frac{1}{n-1} \frac{\sigma_y}{\sigma_0}
\]

where \( \sigma_y \) is the stress deviator and \( \sigma_\sigma \) is the effective stress defined by

\[
\sigma_\sigma = \left( \frac{3}{2} \sigma_\sigma^2 \right)^{1/2}
\]

Following procedures developed by He and Hutchinson (1981) and Abeyaratne (1985), we produce solutions for the relation between the \( J \)-integral and the remote stresses. The method is based on a perturbation about a specially selected uniform trivial state. With that state specified by \( \sigma \) and \( \epsilon \), the increments in deviatoric stress and strain satisfy

\[
\dot{\sigma}_y = 2\mu \left\{ \frac{1}{3} (\delta_{ik} \delta_{kl} + \delta_{il} \delta_{jk}) - \frac{1}{3} \delta_{ij} \delta_{kl} \right\} + \frac{2}{3} \left\{ \frac{1-n}{n} \epsilon_{ik} \epsilon_{jl} \epsilon_\sigma^2 \right\} \dot{\epsilon}_{kl}
\]

where

\[
\mu = \frac{\sigma_\sigma}{3 \epsilon_\epsilon}
\]

(5)

and

\[
\epsilon_\epsilon = (2\epsilon_\sigma \epsilon_\sigma / \sqrt{3})^{1/2}.
\]

Mode II Crack

The trivial state is taken as \( \sigma_{11} = T < 0, \sigma_{22} = 0. \) We consider the plane strain problem, so that \( \sigma_{33} = T/2 \) and \( \sigma_{22} = (\sqrt{3}/2) T. \) In this uniform trivial state the crack remains closed, \( J \) is zero and the nonzero strain components from equation (2) are

\[
\epsilon_{22} = -\epsilon_{22} = -\frac{\sqrt{3}}{2} \epsilon_\epsilon
\]

(6)

Onto this trivial state we superimpose an increment of remote inplane shear stress \( Q \) so that

\[
\sigma_{22}^0 = T, \quad \sigma_{22}^0 = 0, \quad \sigma_{33}^0 = T / 2, \quad \dot{\sigma}_{22} = Q
\]

(7)

Following procedures which closely parallel those mentioned above, we have solved the linear incremental problem for a crack in a body with uniform incremental moduli specified by equation (4) where the strains in the trivial state are given by equation (6). The result is a relation between the \( J \)-integral and \( Q \) which is exact to lowest order in \( Q \). The result so obtained is

\[
J = \frac{3 \sqrt{n}}{4} \left( \frac{Q}{\sigma_\sigma^0} \right)^2
\]

(8)

where we have taken

\[
\sigma_\sigma^0 = \left\{ \frac{3}{4} (T^2 + 4Q^2) \right\}^{1/2}
\]

(9)

and

\[
\epsilon_\epsilon^0 / \epsilon_0 = \alpha (\sigma_\sigma^0 / \sigma_0)^n
\]

(10)

This formula is exact for \( n = 1 \), but at this point we can only argue for its accuracy for \( n > 1 \) when \( Q \) is small compared to \( T \). The accuracy at arbitrary \( Q \) will be assessed below.

Mode I-II Crack

For the crack subjected to a general plane loading, the same trivial state \( \sigma_{11} = T < 0, \sigma_{22} = 0, \) and \( \sigma_{33} = T / 2 \) is used. Onto this trivial state we superimpose an increment of remote tensile stress \( S \) perpendicular to the crack and an increment of remote inplane shear stress \( Q \) parallel to the crack, so that

\[
\sigma_{11}^0 = T, \quad \dot{\sigma}_{22} = S, \quad \dot{\sigma}_{22} = Q
\]

(11)

The incremental problem is again a linear problem. The two
incremental problems \((\delta_{12}^n = S, \delta_{12}^n = 0, \delta_{12}^n = Q)\) therefore can be solved respectively, and the final solution is simply given by superimposing the two incremental solutions.

From He and Hutchinson (1981), for \(\delta_{12}^n = S, \delta_{12}^n = 0\), the incremental solution is

\[
J = \frac{3\pi\sqrt{n}}{4} \left( \frac{S}{\sigma_e^n} \right)^2
\]

The incremental solution for \(\delta_{12}^n = 0, \delta_{12}^n = Q\) is given by equation (8). Because the two incremental solutions are purely symmetric and antisymmetric, respectively, with respect to the crack, the resulting expression for \(J\) for the combined increments of \(S\) and \(Q\) is obtained by the sum of equations (12) and (8) as

\[
J = \frac{3\pi\sqrt{n}}{4} \left( \frac{S^2 + Q^2}{\sigma_e^n} \right)
\]

where now

\[
\sigma_e^n = \left[ \frac{3}{4} \left( (S - T)^2 + 4Q^2 \right) \right]^{1/2}
\]

Formula (13) includes equation (8) as a special case. It is exact for \(n = 1\) and it is consistent with the requirement that \(J\) be homogeneous of degree \(n + 1\) in \(\sigma_e^n\). For \(Q = 0\) it reduces to the result of He and Hutchinson (1981), which has been shown to be highly accurate for arbitrary combinations of \(S\) and \(T\) except those for which \(S = T\). Table 1 compares the prediction from formula (13) with numerical results for \(J/(ae_0\tau_e^n)\) based on a lower bound modified energy principle which was presented by He and Hutchinson (1983) and He (1983). It is seen that the predictions from formula (13) are in close agreement with the numerical results for pure Mode II \((S = T = 0)\) and in pure Mode I. The greatest discrepancy is for the mixed-mode case with \(S/Q = 1.96\) where for \(n = 10\) the result from formula (13) is 17 percent above the numerical lower bound. It is very possible that part of this discrepancy may stem from the difference between the bound and the actual result.

Mode III Crack

The same procedures can be applied to the Mode III problem. In this case the trivial reference state is \(\sigma_{13} = \tau\) and an increment of remote stress \(\delta_{13}^n = R\) is applied. The resulting relation between \(J\) and \(R\) which is exact to the lowest order in \(R\) is

\[
J = \frac{3\pi\sqrt{n}}{4} \left( \frac{R}{\sigma_e^n} \right)^2
\]

where

\[
\sigma_e^n = \left[ \frac{3}{4} \left( (\tau^2 + R^2) \right) \right]^{1/2}
\]

This result agrees with that of Abeyaratne (1983) with \(\tau = 0\). As noted by that author, the result with \(\tau = 0\) is in excellent agreement with the exact results computed by Amazigo (1974), as can be seen in Table 2.

**References**


