Turbulent passive scalar field of a small Prandtl number

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The statistical-mechanics theory of the passive scalar field convected by turbulence, developed in an earlier paper [Phys. Fluids **28**, 1299 (1985)], is extended to the case of a small molecular Prandtl number. The set of governing integral equations is solved by the equation-error method. The resultant scalar-variance spectrum for the inertial range is $F(k) \sim k^{-5/3} / [1 + 1.21x^{1.67} (1 + 0.353x^{2.32})]$, where x is the wavenumber scaled by Corrsin's dissipation wavenumber. This result reduces to the $-5/3$ law in the inertial-convective range. It also approximately reduces to the $-17/3$ law in the inertial-diffusive range, but the proportionality constant differs from Batchelor's by a factor of 3.6.

I. INTRODUCTION

While the molecular Prandtl numbers of most fluids are near unity, a number of important substances have Pr values much lower than this. For example liquid metals may have Prandtl numbers as low as $\frac{1}{250}$. When $Pr \ll 1$, the conduction cutoff (or the Corrsin wavenumber) $k_c = (\epsilon/\mu^3)^{1/4}$ will be much lower than the viscous cutoff $k_d = (\epsilon/\nu^3)^{1/4}$. Here ϵ is the dissipation rate of the energy of the velocity field, μ and ν are, respectively, the molecular diffusivity and the kinematic viscosity. In this case the inertial range consists of three sub-ranges: the inertial-convective range below k_c , the inertial-diffusive range above k_c , and a transition range around k_c . A statistical-mechanics theory was developed in Ref. 1 for the study of a passive scalar field convected by turbulence, and the resultant set of integral equations was solved to derive the scalar-variance spectrum for the inertial-convective range:

$$F(k) = 0.61\chi\epsilon^{-1/3}k^{-5/3}, \quad k \ll k_c. \quad (1)$$

Here χ is the dissipation rate of scalar variance. A comparison of this theoretical result with the experiments and the theoretical results of Obukhov,² Corrsin,³ Batchelor,⁴ Kraichnan,⁵ Gibson,⁶ and Lundgren⁷ is also given in Ref. 1.

There are some theoretical research works on the inertial-diffusive range. By neglecting the correlation between the passive scalar field and the velocity field while calculating the quadruple average in the variance equation, Batchelor, Howells, and Townsend obtained the following inertial-diffusive-range spectrum⁸:

$$F(k) = (Ko/3)\chi\epsilon^{-1/3}k^{-5/3}(k/k_c)^{-4}, \quad k \gg k_c. \quad (2)$$

Here Ko is the Kolmogorov constant for the inertial-range spectrum of the velocity field. Kraichnan⁵ also obtained a $k^{-17/3}$ inertial-diffusive-range spectrum by using the Lagrangian-history-direct-interaction approximation. Corrsin⁹ and Pao¹⁰ used an Onsager's conservative cascade model generalized to the nonconservative cases, and obtained an exponential scalar-variance spectrum for the inertial-diffusive range. An alternative theory for the inertial-diffusive range by Gibson¹¹ gives $F(k) \sim k^{-3}$. Rust and Sesonske¹² and Clay¹³ have made measurements of temperature fluctuations in mercury ($Pr = 0.018$) in order to observe an iner-

tial-diffusive range. As Hill^{14,15} pointed out, their experiments fail to observe the inertial-diffusive range, because the Prandtl number is not sufficiently low and the Reynolds number is not sufficiently high. At the present time it is not clear which form of the inertial-diffusive-range spectrum is supported by experiments.

As Corrsin⁹ pointed out, his exponential inertial-diffusive-range spectrum is the exact solution of a highly simplified "physical model," while Batchelor's $k^{-17/3}$ spectrum, Eq. (2), is obtained by a very rough estimation of a physically "exact" integral.⁴ In this paper we intend to obtain a good estimation of the physically exact integral. First of all the set of governing integral equations for a turbulent passive scalar field, developed in Ref. 1, is extended to the case of small Prandtl number. The resultant set of integral equations is very complicated; proper approximation is necessary. A reasonably approximate expression for the effective damping coefficient $\Omega(k)$ is used to replace the complicated Ω equation. The remaining variance equation is solved numerically by the equation-error method to obtain the scalar-variance spectrum for the whole inertial range (inertial-convective, transition, and inertial-diffusive ranges).

II. VARIANCE EQUATION

The variance equation derived in Ref. 1 is

$$(d_t + 2\mu k^2)F(k) = S(k), \quad (3)$$

where d_t means the differentiation with respect to the time t , and $S(k)$ is the scalar-variance transfer spectrum function. In the convective range,¹

$$S(k) = 16\pi^2 k^4 \int_0^\infty dp C(k,p) p^4 \times [g(p) - g(k)] / [\Omega(k) + \Omega(p)], \quad (4)$$

$$C(k,p) = C(p,k) = \int_0^\pi d\varphi \sin^3 \varphi q(|\bar{k} - \bar{p}|) / |\bar{k} - \bar{p}|^2. \quad (5)$$

Here $\Omega(k)$ is the effective damping coefficient, $q(k)$ is the energy spectrum of the turbulent velocity field, φ is the angle made by wave vectors $\bar{k}$ and $\bar{p}$. The relationship between the

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scalar-variance spectrums $g(k)$ and $F(k)$ is

$$F(k) = 4\pi k^2 g(k). \quad (6)$$

In the derivation of Eq. (4) the following formula is used:

$$\langle Y_i^2 \rangle = G_i, \quad (7)$$

where the Y_i are the real modal parameters for the passive scalar field convected by turbulence, and the G_i are the parameters in the perturbation solution of the Liouville equation of the passive scalar field. Equation (7) is valid only for the convective range where the diffusion (or conduction) effect can be neglected. In the diffusive range where the diffusion effect is important, Eq. (7) should be replaced by¹

$$\langle Y_i^2 \rangle = G_i [1 - \mu_i/\Omega_i], \quad \mu_i = \mu k^2. \quad (8)$$

For isotropic turbulence, $\langle Y_i^2 \rangle$, G_i , and Ω_i are functions of k only, so $G_i = G(k)$ and $\Omega_i = \Omega(k)$.

In order for the variance equation (3) with (4) to be extended to the diffusive range, it is necessary to replace Eq. (7) by (8) during its derivation. Following the procedure developed in Ref. 1, it can be proved that the variance equation extended to the diffusive range is still Eq. (3), but the scalar-variance transfer spectrum function $S(k)$ becomes

$$S(k) = 16\pi^2 k^4 \int_0^\infty dp C(k,p) p^4 \times [\tilde{g}(p) - \tilde{g}(k)] / [\Omega(k) + \Omega(p)], \quad (9)$$

where

$$\tilde{g}(k) = [\Omega(k)/\beta(k)]g(k), \quad \beta(k) = \Omega(k) - \mu k^2. \quad (10)$$

In the convective range, μk^2 can be neglected; $\beta(k) = \Omega(k)$ and Eq. (9) becomes (4). For high wavenumbers the spectrum dynamics may be considered stationary, i.e., $d_t F(k) = 0$ in Eq. (3). Hence, by using Eqs. (6) and (9), the extended variance equation becomes

$$\mu g(k) = 2\pi \int_0^\infty dp C(k,p) p^4 \times [\tilde{g}(p) - \tilde{g}(k)] / [\Omega(k) + \Omega(p)]. \quad (11)$$

In the inertial range,¹⁶ the energy spectrum of the velocity field is

$$q(k) = K_0 \epsilon^{2/3} k^{-11/3} / (4\pi), \quad K_0 = 1.19. \quad (12)$$

Substituting (12) into (5), we have

$$C(k,p) = K_0 \epsilon^{2/3} \tilde{C}(k,p) / (4\pi), \quad (13a)$$

with

$$\begin{aligned} \tilde{C}(k,p) = & 0.75(kp)^{-3} \left[\frac{1}{11} (k^2 - p^2)^2 \right. \\ & \times (|k+p|^{-11/3} - |k-p|^{-11/3}) \\ & - 0.4(k^2 + p^2) (|k+p|^{-5/3} - |k-p|^{-5/3}) \\ & \left. - (|k+p|^{1/3} - |k-p|^{1/3}) \right]. \end{aligned} \quad (13b)$$

By the way, please notice that there are misprints in Eq. (42b) of Ref. 1; the boldface $\mathbf{k}$ and $\mathbf{p}$ should be replaced by the plain ones. Equation (11) with (10) and (13) is the variance equation valid for the whole inertial range (the inertial-convective range, the inertial-diffusive range, and the transition range). In the following it will be numerically

solved by the equation-error method to get its approximate solution.

III. APPROXIMATE FORM OF $\Omega(k)$

The variance equation (11) with (10) and (13) contains two unknowns $g(k)$ and $\Omega(k)$; another equation for $g(k)$ and $\Omega(k)$ is needed to solve the closure problem. One way to solve this closure problem is to apply the variation approach¹⁶ to derive the Ω equation extended to the inertial-diffusive range, following the same procedure as the derivation of the η equation for the universal equilibrium range described in Ref. 17. Since the resultant Ω equation obtained in this way is very complicated, some approximation method has to be used. In a sense any approximation method of this kind is essentially equivalent to the adoption of some approximate formula for the effective damping coefficient $\Omega(K)$. By Eq. (10) we have

$$\Omega(k) = \beta(k) + \mu k^2. \quad (14)$$

In the inertial-convective range the diffusion effect can be neglected $\Omega(k) = \beta(k)$, and¹

$$\beta(k) = 0.5\epsilon^{1/3} k^{2/3}. \quad (15)$$

The $\beta(k)$ (or β_i in Ref. 1) is introduced in the Langevin-Fokker-Planck model of Ref. 1 as the measure of the effect of turbulent velocity field on the passive scalar field. Physically its form should mainly depend on the structure of the turbulent velocity field, because the scalar field is passive. It is reasonable to assume that Eq. (15) is not only valid for the inertial-convective range, but also is approximately valid for the whole inertial range. Hence in the inertial range a simple approximate formula for the effective damping coefficient is

$$\Omega(k) = 0.5\epsilon^{1/3} k^{2/3} + \mu k^2, \quad (16)$$

which is the counterpart of Eq. (24) of Ref. 17. The variance equation (11) with (10), (13), (15), and (16) will be used as the basic equation to determine the scalar-variance spectrums $g(k)$ and $F(k)$.

IV. EQUATION-ERROR METHOD

The scalar-variance spectrum for the inertial-convective range is given by Eq. (1), which is the solution of the variance equation and the Ω equation derived in Ref. 1. By dimensional analysis, in the case of small Prandtl number, the scalar-variance spectrum for the whole inertial range is

$$F(k) = 0.61\chi\epsilon^{-1/3} k^{-5/3} / f(k/k_c), \quad (17)$$

where k_c is Corrsin's wavenumber and $f(x)$ is a universal dimensionless function. Of course, we can let $F(k) \sim k^{-5/3} \hat{f}(k/k_c)$ and $\hat{f}(x) = 1/f(x)$, but Eq. (17) is more convenient. In order for Eq. (17) to approach (1), we must have

$$f(0) = 1, \quad (18)$$

which is a constraint on the trial form of $f(x)$.

From Eqs. (10), (11), (13), (16), and (17), after some manipulation, we have

$$0.84z^{4/3} = W(z), \quad (19a)$$

$$W(z) = \int_0^\infty dx \tilde{C}(1,x) x^4 \times [x^{-7/3} b(x,z) f(z)/f(xz) - 1]/[1 + x^2 b(x,z)], \quad (19b)$$

and

$$b(x,z) = [1 + 0.5(xz)^{-4/3}]/[1 + 0.5z^{-4/3}]. \quad (19c)$$

Here $\tilde{C}(1,x)$ is given by Eq. (13b). Equation (19) is the integral equation for the determination of the universal dimensionless function $f(x)$.

By its definition, the dissipation rate of the scalar variance is

$$\chi = 2\mu \int_0^\infty k^2 F(k) dk. \quad (20)$$

It will be shown later that when the Reynolds number is very high and the Prandtl number is very low, the contribution of the variance-containing range and the viscous range to the dissipation rate χ can be neglected, and the scalar-variance spectrum $F(k)$ in Eq. (20) can be replaced by its expression for the inertial range, i.e., by Eq. (17). Substituting (17) into (20), we obtain

$$1.22 \int_0^\infty dx x^{1/3}/f(x) = 1, \quad (21)$$

which is another constraint on the trial form of $f(x)$.

The nonlinear integral equation (19) is so complicated that so far we do not know how to obtain its exact solution. It is necessary to appeal to some approximate method. Some trial form, which contains several undetermined parameters, is used to approximate the exact solution. Then the equation-error method¹⁷ is used to determine these undetermined parameters. When the exact solution of Eq. (19) is approximated by some trial solution, the (square) equation error at the sampling wavenumber k_i is defined as

$$D_w(k_i) = [w(z_i)/(0.84z_i^{4/3}) - 1]^2, \quad z_i = k_i/k_c. \quad (22)$$

The mean (square) equation error over n different sampling wavenumbers k_i ($i = 1, 2, \dots, n$) is

$$D = (1/n) \sum_{i=1}^n D_w(k_i). \quad (23)$$

The number of sampling wavenumbers n is chosen to be equal or greater than the number of the undetermined parameters in the trial solution. The optimum estimation of the undetermined parameters of the trial solution is obtained by minimizing the mean equation error D under the constraints (18) and (21). The corresponding minimum of D can be used as a measure of the accuracy of the trial solution.

V. NUMERICAL PROCEDURE AND RESULTS

The $W(z)$ in Eq. (19) is a functional of the universal function $f(x)$. A good trial form of $f(x)$ should give $W(z) \sim z^{4/3}$ for a wide range of z , i.e., give a long segment of straight line of slope $\frac{4}{3}$ on a log-log plot. Many trial forms of $f(x)$ have been tested. For each of them the $W(z)$ is numerically calculated and their diagrams on a log-log plot are compared. This is a very boring procedure of "trial and er-

ror." Fortunately it is found that the following form is a good one:

$$f(x) = 1 + c_1 x^{m_1} (1 + c_2 x^{m_2}), \quad (24)$$

which satisfies the constraint (18) when m_1 and m_2 are positive and contains four undetermined parameters c_1 , c_2 , m_1 , and m_2 . The values of c_1 , c_2 , m_1 , and m_2 are to be determined by the equation-error method described in the last section. The number of the sampling wavenumbers $n = 6$ and they are

$$k_i = 10 \cdot 2^i k_c \quad (i = 1, 2, \dots, 6). \quad (25)$$

By using Eqs. (19), (22), (23), and (25) the mean equation-error D is numerically calculated for numerous different sets of parameters c_1 , c_2 , m_1 , and m_2 under the constraint (21). The integral from 0 to ∞ in Eq. (19) is transformed into an integral from 0 to 1, which is numerically calculated by the adaptive Simpson method. The optimum c_1 , c_2 , m_1 , and m_2 which minimize D are determined by the golden-section-search method¹⁸ together with the trial and error method. Finally we obtain

$$f(x) = 1 + 1.21x^{1.67}(1 + 0.353x^{2.32}). \quad (26)$$

It is easy to show that the $f(x)$ given by Eq. (26) satisfies the constraints (18) and (21). The mean equation error for the formula (26) is $D = 0.000\,02$. The diagrams of the function $W(z)$ for the formula (26) are given in Fig. 1 on a log-log plot, which almost coincides with the straight line corresponding to $0.84z^{4/3}$ for a wide range of z . Hence Eq. (26) is a very good approximate formula for the universal dimensionless function $f(x)$.

VI. DISCUSSION

From Eqs. (17) and (26), for the inertial-diffusive range ($k \gg k_c$) we have

$$F(k) = 1.2\text{Ko} \chi \epsilon^{-1/3} k^{-5/3} (k/k_c)^{-3.39}, \quad (27)$$

FIG. 1. The functions $W(z)$ and $0.84z^{4/3}$: — $0.84z^{4/3}$; --- represents $W(z)$ for the $f(x)$ in Eq. (26).

with $Ko = 1.19$ obtained in Ref. 16. Equation (27) is in agreement with the $k^{-17/3}$ spectrum proposed by Batchelor *et al.*⁸ and Kraichnan.⁵ Batchelor's result Eq. (2) and the present result Eq. (27) are almost the same except the proportionality coefficients are different by a factor 3.6. According to Corrsin,⁹ Batchelor's result Eq. (2) is a very rough estimation of a physically exact integral, because the correlation between the passive scalar field and the velocity field is neglected when calculating the scalar-variance transfer spectrum function $S(k)$ in the variance equation (3). In the present theory this correlation between the scalar and the velocity is considered and is deliberately calculated; the resultant spectrum would be a good estimation of the physically exact integral.

It is interesting to discuss Onsager's cascade model which is used by Corrsin⁹ and Pao¹⁰ to derive an exponential inertial-diffusive-range spectrum. The scalar-variance transfer function is defined as¹

$$U(k) = \int_k^\infty S(r) dr. \quad (28)$$

According to the Onsager–Corrsin cascade model,⁹

$$d_k U(k) = -2\mu k^2 F(k), \quad (29a)$$

$$U(k) = kF(k)/\tau(k). \quad (29b)$$

Here $\tau(k)$ is a characteristic time, or (29b) can be considered the definition of $\tau(k)$. In the inertial-diffusive range it is generally assumed that

$$1/\tau(k) = Ck^m, \quad (30)$$

where C and m are some parameters independent of k . By Eq. (30), Eq. (29) becomes

$$d_k [Ck^{m+1}F(k)] = -2\mu k^2 F(k). \quad (31)$$

We distinguish two cases of solutions of Eq. (31): (a) In the first case, $m \neq 2$; from Eq. (31), $F(k) \sim k^{-m-1} \times \exp(-C'k^{2-m})$ with C' being a constant. For example, in Corrsin's theory⁹ $m = \frac{2}{3}$ and $F(k) \sim k^{-5/3} \times \exp(-C'k^{4/3})$. (b) In the second case, $m = 2$; from Eq. (31), $F(k) \sim k^{-3-2\mu/C}$. For example, when $m = 2$ and $C = 3\mu/4$, $F(k) \sim k^{-17/3}$. Therefore in the frame of Onsager's cascade model, Corrsin's exponential inertial-diffusive-range spectrum corresponds to the first case, while the power-law spectrum (the $k^{-17/3}$ spectrum or Gibson's k^{-3} spectrum) corresponds to the second case. Of course Onsager's cascade model is a highly simplified physical model for the scalar-variance transfer function $U(k)$. In the present theory a much more accurate and complicated model is used to calculate $S(k)$ or $U(k)$; and it is shown that if Eq. (29b) is to be considered the definition of $\tau(k)$, then approximately $\tau(k) \sim k^{-2}$ in the inertial-diffusive range.

The integrand $H(k) = 2\mu k^2 F(k)$ in Eq. (20) is called the dissipation spectrum of the passive scalar field. In the case of small Prandtl number and high Reynolds number, by Eqs. (17) and (26), for the inertial range we have

$$H(k) = (\chi/k_c) h(k/k_c),$$

$$h(x) = 1.22x^{1/3} / [1 + 1.21x^{1.67}(1 + 0.353x^{2.32})]. \quad (32)$$

From Eq. (32) it is easy to show that $H(k)$ has a maximum at $k = 0.368k_c$ and approaches zero when $k \ll k_c$ or $k \gg k_c$. By (32) it is easy to prove that so long as the inertial-convective range and the inertial-diffusive range are sufficiently wide, the contribution of the inertial range to the dissipation rate χ is dominant. That justifies the replacement of the scalar-variance spectrum $F(k)$ in Eq. (20) by its expression for the inertial range and confirms the validity of the constraint Eq. (21).

The $-\frac{1}{3}$ inertial-diffusive range can be estimated as follows:

$$\alpha k_c < k < \gamma k_d; \quad (33)$$

here αk_c and γk_d are, respectively, the lower and upper limits of the inertial-diffusive range. According to Eqs. (17) and (26) $\alpha \approx 3$, and Hill^{14,15} assumes $\gamma = 0.05$. In order for Eq. (33) to be valid, the Prandtl number must be

$$Pr \equiv \nu/\mu = (k_c/k_d)^{4/3} < (\gamma/\alpha)^{4/3} \approx 0.004. \quad (34)$$

Hence, as one of the referees pointed out, it is difficult to observe the $-\frac{1}{3}$ inertial-diffusive range, because no scalar-fluid pair known today has such a small Prandtl number. Of course it is still a controversial issue how to evaluate α and γ .

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