Inverse energy cascade in two-dimensional turbulence

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The closure method developed in earlier papers [J. Qian, Phys. Fluids **26**, 2098 (1983); **27**, 2412 (1984)] is applied to the study of the inverse energy cascade in two-dimensional turbulence. The resultant inertial-range energy spectrum is $E(k) = 2.58g^{0.244}\epsilon^{2/3}k^{-5/3}$. Here g is a localization factor and ϵ is the rate of energy cascade. This result is compatible with the numerical experiments by Lilly [D. K. Lilly, Phys. Fluids Suppl. II **12**, 24 (1969)], Siggia and Aref [E. D. Siggia and H. Aref, Phys. Fluids **24**, 171 (1981)], and Frisch and Sulem [U. Frisch and P. L. Sulem, Phys. Fluids **27**, 1921 (1984)].

I. INTRODUCTION

A two-dimensional (2-D) inviscid flow has the property that the vorticity of each fluid element is unchanged. Hence in addition to kinetic energy, enstrophy is another inviscid constant of motion for 2-D turbulence. The presence of this additional constant of motion has profound effects on nonequilibrium as well as equilibrium statistics of 2-D turbulence. In contrast to the three-dimensional (3-D) case, in 2-D turbulence the energy dissipation rate approaches zero as the Reynolds number approaches infinity, so the inertial-range cascade transfer of energy to higher wavenumbers is excluded.[1] Instead of energy, the enstrophy is cascaded to higher wavenumbers through an inertial range of the form[1–4]

$$E(k) = C\chi^{2/3}k^{-3}. \tag{1}$$

Here $E(k)$ is the energy spectrum, χ is the dissipation rate of enstrophy, and C is a dimensionless constant and depends upon a localization factor.[4] As Kraichnan[2,5] pointed out, besides the k^{-3} enstrophy-cascade inertial range, there is another inverse energy-cascade inertial range through which energy is cascaded to lower wavenumbers and it has the energy spectrum

$$E(k) = A\epsilon^{2/3}k^{-5/3}. \tag{2}$$

Here ϵ is the rate of energy cascade, and A is Kolmogorov's constant. The existence of the inverse energy-cascade inertial range has been confirmed by Lilly,[6] Siggia and Aref,[7] and Frisch and Sulem[8] in their numerical experiments.

A new statistical-mechanics theory of 2-D turbulence has been developed in Ref. 4, and it was applied to the study of the enstrophy-cascade inertial-range spectrum Eq. (1) as well as the inviscid equilibrium spectrum.[4] In the present paper this theory is to be applied to the derivation of the inverse energy-cascade inertial-range spectrum Eq. (2); and it will be shown that the Kolmogorov constant A is not universal, but depends upon a localization factor, because of the nonlocalness of the cascade process in 2-D turbulence. This theoretical result could explain why the values of Kolmogorov's constant A determined by Frisch and Sulem[8] as well as Siggia and Aref[7] is much larger than the value obtained by Lilly.[6]

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II. ENERGY EQUATION AND η EQUATION

Let ω and v be, respectively, the vorticity and the norm of velocity vector of 2-D turbulence. We have[4]

$$\frac{\langle v^2\rangle}{2} = \int_0^\infty E(k)dk, \tag{3}$$

$$\frac{\langle \omega^2\rangle}{2} = \int_0^\infty S(k)dk. \tag{4}$$

Here $\langle\cdots\rangle$ means the statistical average, k is the wavenumber, and $E(k)$ and $S(k)$ are, respectively, the energy spectrum and the enstrophy spectrum. The relationship between $E(k)$ and $S(k)$ is

$$S(k) = k^2E(k). \tag{5}$$

Often it is more convenient to introduce the quantity

$$\tilde{q}(k) = E(k)/(2\pi k) \tag{6}$$

to replace $E(k)$. Hence

$$E(k) = 2\pi k\tilde{q}(k), \quad \tilde{q}(k) = q(k)/k^2, \tag{7}$$

where $q(k)$ is the average modal intensity introduced in Ref. 4.

In Ref. 4 a complete set of independent real modal parameters and its dynamic equation are worked out to describe the vorticity dynamics of 2-D turbulence. The closure problem was solved by the variation-perturbation method developed originally for the 3-D case.[9] Thereby two integral equations, the enstrophy equation and the η equation, were obtained for two unknown functions: the enstrophy spectrum and the dynamic damping coefficient. For the study of the inverse energy cascade, the enstrophy equation has to be replaced by a proper energy equation, and the η equation is still valid. By using Eqs. (5)–(7) and the enstrophy equation of Ref. 4, after a boring manipulation we obtain the following energy equation:

$$(d_t + 2\nu k^2)E(k) = U(k), \tag{8a}$$

where ν is the kinematic viscosity, d_t means differentiation with respect to time t, and

$$U(k) = \frac{1}{2}\iint^{\Delta} V(k\,|p,r)dp\,dr \tag{8b}$$

is the energy transfer spectrum function, and

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$$V(k|p,r) = (8\pi/k)(p^2 - r^2)d(k,p,r) \times q(k,p,r)/\eta(k,p,r). \quad (8c)$$

Here

$$d(k,p,r) = 2[s(s-k)(s-p)(s-r)]^{1/2}/(pr), \quad s = (k+p+r)/2, \quad (8d)$$

$$q(k,p,r) = (k^2 - p^2)\tilde{q}(k)\tilde{q}(p) + (p^2 - r^2)\tilde{q}(p) \times \tilde{q}(r) + (r^2 - k^2)\tilde{q}(r)\tilde{q}(k), \quad (8e)$$

and

$$\eta(k,p,r) = \eta(k) + \eta(p) + \eta(r). \quad (8f)$$

Here $\eta(k)$ is the dynamic damping coefficient.[4] The Δ in Eq. (8b) indicates that the integration is restricted to the following infinite slot in the first quarter of the p-r plane:

$$\Delta: (r \geqslant 0, |k-r| \leqslant p \leqslant k+r)$$

or

$$(p \geqslant 0, |k-p| \leqslant r \leqslant k+p). \quad (9)$$

The energy equation (8) contains two independent unknown functions, $\tilde{q}(k)$ and $\eta(k)$; $E(k)$ is related to $\tilde{q}(k)$ by Eq. (7). Another equation of $\tilde{q}(k)$ and $\eta(k)$ is needed to solve the closure problem; it is the η equation derived in Ref. 4. By using $\tilde{q}(k)$ to replace $q(k)$, it becomes

$$k^2\tilde{q}(k)\eta(k) = \iint^{\Delta} dp\, dr\, p^2 r^2\, d(k,p,r) q(k,p,r) \times \{(k^{-2} - r^{-2})[2\eta(p) + \eta(r)] - (k^{-2} - p^{-2})[\eta(p) + 2\eta(r)]\}/[\eta(k,p,r)]^2. \quad (10)$$

Here $d(k,p,r)$, $q(k,p,r)$, and $\eta(k,p,r)$ are defined in Eq. (8).

III. ENERGY TRANSFER FUNCTION IN ENERGY-CASCADE INERTIAL RANGE

The energy equation (8) and the η equation (10) constitute a closed set of equations for $\tilde{q}(k)$ [or $E(K)$] and $\eta(k)$. From them it will be proved that in the energy-cascade range the direction of energy transfer is backward, from higher to lower wavenumbers. Moreover they allow us to derive the inertial-range spectrum Eq. (2) and calculate the Kolmogorov constant A. First of all we introduce the energy transfer function

$$\Pi(k) = \int_k^\infty U(k')dk', \quad (11)$$

which represents the rate of energy flow across the spectrum. Here $U(k)$ is the energy transfer spectrum function defined in the energy equation (8). As in the 3-D case,[9] from Eqs. (8b) and (11), after some manipulation we obtain

$$\Pi(k) = \int_k^\infty dk' \int_0^k dp \int_b^c dr\, V(k'|p,r), \quad b = \max[p, k'-p], \quad c = k'+p. \quad (12)$$

In the energy-cascade inertial range the energy transfer function $\Pi(k)$ is a constant, independent of k. Hence Eq. (12) becomes

$$\int_k^\infty dk' \int_0^k dp \int_b^c dr\, V(k'|p,r) = e\epsilon, \quad b = \max[p, k'-p], \quad c = k'+p, \quad (13)$$

which can be considered the special case of the energy equation (8) while it is applied to the energy-cascade inertial range. In Eq. (13) ϵ is the absolute value of the rate of energy cascade, and e is its sign. When $e = +1$ the energy transfer is forward, from lower to higher wavenumbers; when $e = -1$, the energy transfer is backward, from higher to lower wavenumbers. It will be shown that $e = -1$, so the energy transfer is backward.

IV. LOCALIZATION FACTOR

In order to derive the energy-cascade inertial-range spectrum [Eq. (2)], suppose that $\tilde{q}(k)$ and $\eta(k)$ are of the type of power function

$$\tilde{q}(k) = C_1 k^m \quad \text{and} \quad \eta(k) = C_2 k^n. \quad (14)$$

Substitute Eq. (14) into Eqs. (10) and (13). We have

$$m - 2n + 4 = 0 \quad \text{and} \quad 2m - n + 6 = 0. \quad (15)$$

Therefore $m = -\frac{8}{3}$ and $n = \frac{2}{3}$. Let $C_1 = [A/(2\pi)]\epsilon^{2/3}$ and $C_2 = B\epsilon^{1/3}$, Eq. (14) becomes

$$\tilde{q}(k) = [A/(2\pi)]\epsilon^{2/3}k^{-8/3}, \quad (16a)$$

$$\eta(k) = B\epsilon^{1/3}k^{2/3}. \quad (16b)$$

From Eqs. (7) and (16a), we obtain the energy-cascade inertial-range spectrum [Eq. (2)].

When substituting the simple power-function solution [Eq. (16)] into the η equation (10), the integral on its right-hand side is divergent. This related to the nonlocalness of the energy transfer in 2-D turbulence. As a consequence, the Kolmogorov constant is not a universal constant. As discussed in Ref. 4, in the enstrophy-cascade process the nonlocalness of the enstrophy transfer leads to the divergence of the enstrophy equation while the η equation is convergent, we call it a strong form of the nonlocalness of the cascade process in 2-D turbulence. Now in the energy-cascade process the nonlocalness of the energy transfer leads to the divergence of the η equation while the energy equation is convergent. We call it a weak form of the nonlocalness of the cascade process in 2-D turbulence, since the energy (or enstrophy) equation is more essential than the η equation. If the solutions of Eqs. (10) and (13) are not restricted to the simple power-function solution Eq. (16), there might be no divergence. But it is a difficult task to solve Eqs. (10) and (13) to get solutions that are expressed by more complicated functions, because Eqs. (10) and (13) are nonlinear integral equations. As proposed in Ref. 4, in order to overcome this difficulty, a localization procedure is introduced to transform the energy equation (13) and the η equation (10) into corresponding localized forms. This localization procedure is to omit the contribution of any triad interaction of (k,p,r) for which the ratio of the maximum wavenumber of (k,p,r) to the minimum wavenumber is greater than g, which is called the localization factor.[4] The actual spectrum of 2-D turbulence in numerical experiments is cut off or bounded at both low and high wavenumbers.[6-8] This cutoff or bound is roughly corresponding to the localization procedure, although not exactly.

V. NUMERICAL SOLUTION OF LOCALIZED ENERGY AND η EQUATIONS

It can be proved that the simple power-function solution Eq. (16) does satisfy the localized energy and η equations when the localization factor g is not too small and there is no divergence. Substitute Eq. (16) into the localized form of energy equation (13) and let $k'=k/u, p=vk'$, and $r=wk'$. After some manipulation we obtain

$$\frac{eB}{A^2}=\frac{2}{\pi}\int_{g^{-1}}^{1} dv \ln\left(\frac{1}{v}\right)\int_b^c dw\, F(v,w),$$
$$b=\max[v,1-v],\quad c=1+v, \tag{17a}$$
$$F(v,w)=d(1,v,w)(v^2-w^2)[(1-w^{-2})w^{-2/3}\times(1-v^{-2/3})-(1-v^{-2})v^{-2/3}\times(1-w^{-2/3})]/[1+v^{2/3}+w^{2/3}]. \tag{17b}$$

Here $d(1,v,w)$ is defined in Eq. (8d). Substitute Eq. (16) into the localized form of the η equation (10) and let $p=vk$ and $r=wk$. After some manipulation we obtain

$$\frac{B^2}{A}=\frac{1}{\pi}\int_{g^{-1}}^{g} dv\int_b^c dw\, G(v,w),$$
$$b=\max[v,1-v],\quad c=1+v, \tag{18a}$$
$$G(v,w)=F(v,w)[v^2w^2/(v^2-w^2)]\times[(v^{2/3}-w^{2/3})+(v^{-4/3}-w^{-4/3})+2(w^{2/3}v^{-2}-v^{2/3}w^{-2})]/[1+v^{2/3}+w^{2/3}]. \tag{18b}$$

The integrals in Eqs. (17) and (18) are numerically evaluated for many different values of the localization factor g; then Eqs. (17) and (18) are solved to get e, A, and B. The values obtained in this way depend upon g. For all values of g the integral in Eq. (17) is negative, so $e=-1$, i.e., the energy transfer is backward, from higher to lower wavenumbers. The resultant Kolmogorov constant A is given in Fig. 1. The numerical results can be approximately fitted by a power function

$$A=2.58g^{0.244}. \tag{19}$$

From Eqs. (2) and (19), the inverse energy-cascade inertial-range spectrum is

$$E(k)=2.58g^{0.244}\epsilon^{2/3}k^{-5/3}. \tag{20}$$

VI. DISCUSSION

It is interesting to compare the present theoretical result with numerical experiments. The Kolmogorov constant A determined by numerical experiments is given in Table I. In Lilly's numerical experiment, the wavenumber range is narrow and the corresponding localization factor g is small; hence according to the present theory A is also small. As for the numerical experiment by Frisch and Sulem, the wavenumber range is wider, so the corresponding g and A are

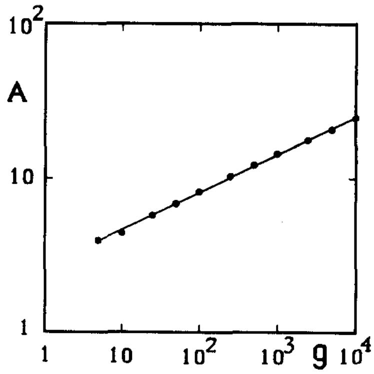

FIG. 1. Kolmogorov constant A versus localization factor g: ••• numerical solution;—Eq. (19).

larger. The large value of A obtained by Siggia and Aref may be as a result of their numerical experiment made "on a considerably larger scale than Lilly was able to run" (see Ref. 7), i.e., corresponding to a large g, and partly because of using a different model. Here we only make a qualitative comparison. A quantitative comparison needs evaluation of the equivalent localization factor g for each numerical experiment in Table I. That is equivalent to solving the nonlinear integral equations (10) and (13) for the complicated forms of real spectrum used in the numerical experiments. It is a very difficult problem.

The nonlocalness of the cascade process is a characteristic feature of 2-D turbulence. It assumes a strong form for the enstrophy transfer and leads to divergence of the enstrophy equation while the η equation is convergent. It assumes a weak form for the energy transfer and leads to the divergence of the η equation while the energy equation is convergent. As a consequence, the dimensionless constants C and A in Eqs. (1) and (2) are not universal. Physically the nonlocalness of the cascade process in 2-D turbulence means that the long-range correlation of velocity field or the organized motion is much more important in 2-D turbulence than in real 3-D turbulence. As Fornberg[10] pointed out, in 2-D turbulence some well-organized structure develops with time. Strictly speaking, the energy or enstrophy transfer in 2-D turbulence is not a cascade process.

It is interesting to compare the divergence of the response equation in Kraichnan's DIA theory with the divergence studied here. In 3-D turbulence the cascade process can be approximately considered to be local, i.e., the contribution of the triad interaction of (k,p,r) can be neglected when the ratio $\max\{k,p,r\}/\min\{k,p,r\}$ is quite large. In

TABLE I. Results of numerical experiments.

Authors	A	Model
D. K. Lilly[a] (1969)	4	2-D Navier–Stokes equation
U. Frisch and P. L. Sulem[b] (1984)	9	2-D Navier–Stokes equation
E. D. Siggia and H. Aref[c] (1981)	14	Point-vortex model

[a] D. K. Lilly, Phys. Fluids Suppl. II **12**, 240 (1969).
[b] U. Frisch and P. L. Sulem, Phys. Fluids **27**, 1921 (1984).
[c] E. D. Siggia and H. Aref, Phys. Fluids **24**, 171 (1981).

Kraichnan's DIA theory the divergence of the closure equation (the response equation) is a result of improper treatment of the effect of large scales on small scales,[11] and not due to the real nonlocalness of the cascade process in 3-D turbulence. For this case the divergence of the closure equation is said to be spurious, and is a deficiency of the closure method. In 2-D turbulence the divergence of the closure equation (the enstrophy equation or the η equation) is "genuine," because it is a consequence of the real nonlocalness of cascade processes in 2-D turbulence. Of course, at the present time there is no direct experimental verification of the nonlocalness of cascade processes in 2-D turbulence. If it should turn out that the cascade processes in 2-D turbulence could be considered local as in the 3-D case, then the divergence of the closure equations for 2-D turbulence is also spurious, and should be regarded as a deficiency of the present theory. In the author's opinion, as mentioned before, in 2-D turbulence the cascade processes cannot be considered to be local and the divergence of the closure equations is genuine.

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