A closure theory of intermittency of turbulence

J. Qian[a)]
Institute of Mechanics, Chinese Academy of Sciences, Beijing, People's Republic of China and Department of Mechanical Engineering, The City College of New York, New York, New York 10031

(Received 17 April 1984; accepted 29 January 1986)

The variational approach to the closure problem of turbulence theory, proposed in an earlier article [Phys. Fluids **26**, 2098 (1983); **27**, 2229 (1984)], is extended to evaluate the flatness factor, which indicates the degree of intermittency of turbulence. Since the flatness factor is related to the fourth moment of a turbulent velocity field, the corresponding higher-order terms in the perturbation solution of the Liouville equation have to be considered. Most closure methods discard these higher-order terms and fail to explain the intermittency phenomenon. The computed flatness factor of the idealized model of infinite isotropic turbulence ranges from 9 to 15 and has the same order of magnitude as the experimental data of real turbulent flows. The intermittency phenomenon does not necessarily negate the Kolmogorov $k^{-5/3}$ inertial range spectrum. The Kolmogorov $k^{-5/3}$ law and the high degree of intermittency can coexist as two consistent consequences of the closure theory of turbulence. The Kolmogorov 1941 theory [J. Fluid Mech. **62**, 305 (1974)] cannot be disqualified merely because the energy dissipation rate fluctuates.

I. INTRODUCTION

The intermittency of turbulence was first inferred by Batchelor and Townsend.[1,2] Since then many theoretical and experimental works have been done on this subject. A quantitative measure of the intermittency is the amount by which the flatness factor of the derivative of the turbulent velocity field,

$$F = \langle (\partial_{x1} u_1)^4 \rangle / \langle (\partial_{x1} u_1)^2 \rangle^2, \tag{1}$$

exceeds the value 3.0, which corresponds to Gaussian probability distribution. Here u_1 is the velocity component along the x_1 direction, ∂_{x1} denotes the partial differentiation with respect to x_1, and $\langle \cdots \rangle$ means the statistical average. Experiments[3,4] show that the flatness factor F depends on the Reynolds number

$$R_\lambda = u_{\rm rms} \lambda / \nu. \tag{2}$$

Here $u_{\rm rms}$ is the rms velocity of one component of the turbulent velocity field, ν is the kinematic viscosity, and

$$\lambda = u_{\rm rms} / \langle (\partial_{x1} u_1)^2 \rangle^{1/2} \tag{3}$$

is Taylor's microscale.

Small Reynolds number grid turbulence experiments give $F \simeq 4$ at $R_\lambda \simeq 50$.[1-4] Kuo and Corrsin[5] present laboratory data that show F increasing to 10 at $R_\lambda \simeq 1000$, which is about the limit for laboratory flows. The atmospheric turbulence measurements give widely scattering values of F at much higher R_λ. Wyngaard and Tennekes[6] present data of F from 18 to 40 for R_λ from 2000 to 10 000, but Sheih–Tennekes–Lumley values[7] are much lower and range from 5 to 15 for R_λ from 2300 to 6000. As emphasized by Tennekes and Wyngaard,[8] there are some technical difficulties in making reliable measurements of F at high Reynolds number. Recently a series of further experiments have been performed by McConnell,[9] Park,[10] Williams and Paulson,[11] Gagne and Hopfinger,[12] and Antonia *et al.*[13]

Many efforts have been made to include the intermittency phenomenon in the turbulence theory. Kolmogorov[14] and Oboukhov[15] modified their original similarity hypothesis to take account of intermittency and further assumed that the logarithm of the average energy dissipation rate over a finite volume has a Gaussian distribution. Along this line many heuristic phenomenological models of the dissipation fluctuation have been proposed by Novikov and Steward,[16] Corrsin,[17] Gurvich and Yaglom,[18] Tennekes,[19] Saffman,[20] Kuo and Corrsin,[21] Kraichnan,[22] and Van Atta and Antonia.[23] Mandelbrot[24] applied the concept of "fractal dimension" to the study of intermittency. A "β model" of intermittency was developed by Frisch, Sulem, and Nelkin.[25] Nelkin[26-28] studied the similarity between the critical phenomenon and intermittency and proposed a phenomenological scaling theory to relate measurable scaling exponents to each other. All these phenomenological approaches do not meet the objective of the analytical turbulence theory, which evaluates the statistical properties of turbulence, e.g., the flatness factor, directly from the Navier–Stokes equation by means of the general method of statistical mechanics.

The central problem of the analytical turbulence theory is the closure problem. There are many methods for this problem.[3,4,29-32] A serious common drawback of many closure methods is the complete neglect of intermittency. As far as the author knows, there is no published paper in which a closure theory is applied to the evaluation of the flatness factor. The purpose of this article is to develop a closure theory that would enable us to evaluate the flatness factor directly from the Navier–Stokes equation by means of the general method of statistical mechanics. This will be done without appealing to any extra phenomenological model.

II. FLATNESS FACTOR AND PROBABILITY DISTRIBUTION

For an isotropic turbulence[3]

$$\langle (\partial_{x1} u_1)^2 \rangle = \frac{2}{15} \int_0^\infty k^2 E(k) dk = \frac{\epsilon}{(15\nu)}, \tag{4}$$

[a)] Present address: Beijing Institute of Environmental Features, P. O. Box 142, Branch Box 207, Beijing, The People's Republic of China.

 0031-9171/86/072165-07$01.90

where $E(k)$ is the energy spectrum and ϵ is the energy dissipation rate. Substituting (4) into (1), we have

$$F=225(\nu/\epsilon)^2\langle(\partial_{x1}u_1)^4\rangle. \tag{5}$$

A complete set of independent real parameters $[X_i]$ has been worked out in Ref. 31 to describe the dynamic state of turbulence; they are the real or imaginary parts of turbulent velocity components in properly defined wave-vector-dependent coordinates. The statistical behavior of turbulence is described by the probability distribution $P=P([X_i])$ over an ensemble of numerous realizations of the turbulence, which satisfied the Liouville equation.[31] Assuming that the perturbation solution of the Liouville equation is

$$P=P^{(0)}+P^{(1)}+P^{(2)}+\cdots, \tag{6}$$

then the flatness factor can be expanded into a series

$$F=F^{(0)}+F^{(1)}+F^{(2)}+\cdots. \tag{7}$$

Here

$$F^{(n)}=225\left(\frac{\nu}{\epsilon}\right)^2\int(\partial_{x1}u_1)^4P^{(n)}\Pi\, dX_i, \quad (n=0,1,2,\ldots). \tag{8}$$

A homogeneous isotropic incompressible turbulence is assumed to be confined within a large cubic box with side L and cyclic boundary condition. The velocity component $u_1(\mathbf{x})$ can be expanded into a Fourier series,[31] where

$$u_1(\mathbf{x})=H\sum_{\mathbf{k}}u_1(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x}),\quad H=(2\pi/L)^3, \tag{9}$$

$$u_1(\mathbf{k})=(2\pi)^{-3}\int\int\int u_1(\mathbf{x})\exp(-i\mathbf{k}\cdot\mathbf{x})d\mathbf{x}, \tag{10}$$

are the Fourier components of the velocity field, and $\Sigma_{\mathbf{k}}$ means the summation over the discrete wave vector $\mathbf{k}$. By means of (9) we have

$$\langle(\partial_{x1}u_1)^4\rangle=H^4\sum_{\mathbf{kprs}}k_1p_1r_1s_1\langle u_1(\mathbf{k})u_1(\mathbf{p})u_1(\mathbf{r})u_1(\mathbf{s})\rangle \times\exp[i(\mathbf{k}+\mathbf{p}+\mathbf{r}+\mathbf{s})\cdot\mathbf{x}]. \tag{11}$$

Because of the homogeneity of turbulence, Eq. (11) is equivalent to

$$\langle(\partial_{x1}u_1)^4\rangle=H^4\sum_{\mathbf{kprs}}k_1p_1r_1s_1\langle u_1(\mathbf{k})u_1(\mathbf{p})u_1(\mathbf{r})u_1(\mathbf{s})\rangle \times\delta_{\mathbf{k}+\mathbf{p}+\mathbf{r}+\mathbf{s},0}, \tag{12}$$

where $\delta_{\mathbf{k}+\mathbf{p}+\mathbf{r}+\mathbf{s},0}$ is the Kronecker symbol; $\mathbf{k}$, $\mathbf{p}$, $\mathbf{r}$, and $\mathbf{s}$ are discrete wave vectors.

By means of the formulas in Appendix A of Ref. 31 we have

$$\langle u_1(\mathbf{k})u_1(\mathbf{p})u_1(\mathbf{r})u_1(\mathbf{s})\rangle =\sum_{abcd}\sum_{\alpha\beta\gamma\sigma}(-1)^{(\alpha+\beta+\gamma+\sigma)/2}n_{a1}(\mathbf{k})n_{b1}(\mathbf{p}) \times n_{c1}(\mathbf{r})n_{d1}(\mathbf{s})S(i)S(j)S(m)S(n)\langle X_iX_jX_mX_n\rangle, \tag{13}$$

with

$$i=(\alpha,a,\mathbf{k}),\quad j=(\beta,b,\mathbf{p}),\quad m=(\gamma,c,\mathbf{r}),\quad n=(\sigma,d,\mathbf{s}). \tag{14}$$

For the meanings of $n_{a1}(\mathbf{k})$ and $S(i)$ see Appendix A of Ref. 31. From (8), (12), and (13)

$$F^{(n)}=225\left(\frac{\nu}{\epsilon}\right)^2H^4\sum_{\mathbf{kprs}}k_1p_1r_1s_1 \times\langle u_1(\mathbf{k})u_1(\mathbf{p})u_1(\mathbf{r})u_1(\mathbf{s})\rangle^{(n)}\delta_{\mathbf{k}+\mathbf{p}+\mathbf{r}+\mathbf{s},0}$$
$$=225\left(\frac{\nu}{\epsilon}\right)^2H^4\sum_{ijmn}k_1p_1r_1s_1(-1)^{(\alpha+\beta+\gamma+\sigma)/2} \times n_{a1}(\mathbf{k})n_{b1}(\mathbf{p})n_{c1}(\mathbf{r})n_{d1}(\mathbf{s})S(i)S(j)S(m)S(n) \times\langle X_iX_jX_mX_n\rangle^{(n)}\delta_{\mathbf{k}+\mathbf{p}+\mathbf{r}+\mathbf{s},0}. \tag{15}$$

Here

$$\langle\cdots\rangle^{(n)}=\int(\cdots)P^{(n)}\Pi\, dX_i\quad (n=0,1,2,\cdots). \tag{16}$$

For later use we define

$$\langle\cdots\rangle^{(01)}=\int(\cdots)(P^{(0)}+P^{(1)})\Pi\, dX_i. \tag{17}$$

III. PROOF THAT $F^{(0)}+F^{(1)}=3.0$

By the variational approach to the closure problem we obtain[31,32]

$$P^{(0)}=\Pi(2\pi\phi_i)^{-1/2}\exp[-X_i^2/(2\phi_i)], \tag{18}$$

$$P^{(0)}+P^{(1)}=\left(1-\sum_i\frac{(\nu_i-\nu_i')(X_i^2-\phi_i)}{(2\eta_i\phi_i)} +\sum_{ijm}\frac{A_{ijm}X_iX_jX_m}{[\phi_i(\eta_i+\eta_j+\eta_m)]}\right)P^{(0)}. \tag{19}$$

The ϕ_i is related to the energy spectrum. The energy spectrum and the dynamic damping coefficient η satisfy two integral equations: the energy equation and the η equation. When the four wave vectors $\mathbf{k}$, $\mathbf{p}$, $\mathbf{r}$, and $\mathbf{s}$ are such that the modes i,j,m, and n in (13) are all different, from (13), (17), (18), and (19), we have

$$\langle X_iX_jX_mX_n\rangle^{(01)}=0, \tag{20}$$

$$\langle u_1(\mathbf{k})u_1(\mathbf{p})u_1(\mathbf{r})u_1(\mathbf{s})\rangle^{(01)}=0. \tag{21}$$

Because of the presence of $\delta_{\mathbf{k}+\mathbf{p}+\mathbf{r}+\mathbf{s},0}$ in (15), when $L\to\infty$ and $H\to 0$, the contribution of $\langle X_iX_jX_mX_n\rangle^{(01)}$ to $F^{(0)}+F^{(1)}$ is not zero only when the four wave vectors can be divided into two groups—one is $(\mathbf{k},-\mathbf{k})$, and the other is $(\mathbf{p},-\mathbf{p})$. After some manipulation we have

$$F^{(0)}+F^{(1)}=675(\nu/\epsilon)^2H^4 \times\sum_{\mathbf{kp}}k_1^2p_1^2\langle|u_1(\mathbf{k})|^2|u_1(\mathbf{p})|^2\rangle^{(01)}. \tag{22}$$

Here $|\cdots|$ means the absolute value. From (13), (18), and (19), if $\mathbf{k}$ is not equal to $\mathbf{p}$ or $-\mathbf{p}$, we have

$$\langle|u_1(\mathbf{k})|^2|u_1(\mathbf{p})|^2\rangle^{(01)}=\langle|u_1(\mathbf{k})|^2|\rangle^{(01)}\langle|u_1(\mathbf{p})|^2\rangle^{(01)}. \tag{23}$$

Hence (22) becomes

$$F^{(0)} + F^{(1)} = 675\left(\frac{\nu}{\epsilon}\right)^2 H^4\left(\sum_{\mathbf{kp}} k_1^2 p_1^2 \langle |u_1(\mathbf{k})|^2\rangle^{(01)}\right.$$
$$\times \langle |u_1(\mathbf{p})|^2\rangle^{(01)} + 2\sum_{\mathbf{k}} k_1^4 [\langle |u_1(\mathbf{k})|^4\rangle^{(01)}$$
$$\left. - (\langle |u_1(\mathbf{p})|^2\rangle^{(01)})^2]\right). \tag{24}$$

When $L \to \infty$ and $H \to 0$ the second summation in (24) approaches zero. After some manipulation we have

$$F^{(0)} + F^{(1)} = 675\left(\frac{\nu}{\epsilon}\right)^2\left(\frac{2}{15}\int_0^\infty k^2 E(k)dk\right)^2, \tag{25}$$

where

$$E(k) = 4\pi k^2 q(k) = 4\pi k^2 (2H)\langle X_i^2\rangle^{(01)} \tag{26}$$

is the 3-D energy spectrum defined in Refs. 31 and 32. From (4) and (26)

$$F^{(0)} + F^{(1)} = 3.0. \tag{27}$$

So long as we use (19) as an approximate expression for the probability distribution P, according to (20) we neglect the correlation between four different modes. In this case the flatness factor is 3.0 by (27), so the degree of intermittency is zero. In order to study the intermittency phenomenon, the higher-order term $P^{(2)}$ in the perturbation solution (6) has to be considered. Many closure methods neglect these higher-order terms and fail to explain the intermittency phenomenon.

IV. HIGHER-ORDER TERMS IN PROBABILITY DISTRIBUTION

The Liouville operator corresponding to the Navier–Stokes equation is[31]

$$\mathscr{L} = -\sum_i\left((\nu_i - \nu_i')\partial_{xi}X_i - \sum_{jm} A_{ijm}X_jX_m\partial_{xi}\right). \tag{28}$$

According to the Langevin–Fokker–Planck model,[31] the Liouville operator (28) can be approximated by the Fokker–Planck operator

$$\mathscr{L}^{(f)} = -\sum_i \eta_i(\partial_{Xi}X_i + \phi_i\partial_{Xi}^2). \tag{29}$$

Hence

$$\delta\mathscr{L} = \mathscr{L} - \mathscr{L}^{(f)} \tag{30}$$

can be considered a small perturbation operator. The Liouville equation of a stationary turbulence is

$$(\mathscr{L}^{(f)} + \delta\mathscr{L})P = 0. \tag{31}$$

Substituting (6) into (31), we have

$$\mathscr{L}^{(f)}P^{(0)} = 0, \tag{32a}$$
$$\mathscr{L}^{(f)}P^{(1)} = -\delta\mathscr{L}P^{(0)}, \tag{32b}$$
$$\mathscr{L}^{(f)}P^{(2)} = -\delta\mathscr{L}P^{(1)}, \tag{32c}$$

and so on. We have solved (32a) and (32b) to get $P^{(0)}$ and $P^{(1)}$ in Ref. 31; they are given by (18) and (19). From (32c)

$$P^{(2)} = -(\mathscr{L}^{(f)})^{-1}\delta\mathscr{L}P^{(1)}. \tag{33}$$

By the eigenfunction expansion method of the Fokker–Planck operator, the right side of (33) is a complicated linear combination of the following terms:

$$(X_i^2 - \phi_i);\ X_iX_j;\ (X_i^2 - \phi_i)(X_j^2 - \phi_j);\ X_iX_jX_m;$$
$$(X_i^2 - \phi_i)X_jX_m;\ (X_i^2 - \phi_i)X_iX_jX_m;$$
$$(X_i^2 - \phi_i)(X_j^2 - \phi_j)(X_m^2 - \phi_m);\ \cdots;$$
$$X_iX_jX_mX_n;\ \cdots;\ X_iX_jX_mX_nX_uX_v. \tag{34}$$

It is not necessary to evaluate the coefficients of all the above terms. Only the term $X_iX_jX_mX_n$ contributes to the correlation of four different modes; all other terms have no contribution. Hence the manipulation is greatly simplified, and we have

$$P^{(2)} = \left(\cdots + \sum_{ijmn} D_{ijmn}X_iX_jX_mX_n \Big/ [\phi_i(\eta_i + \eta_j + \eta_m + \eta_n)] + \cdots\right)P^{(0)}, \tag{35}$$

with

$$D_{ijmn} = \sum_v 2\phi_v A_{ivj}(A_{vmn}/\phi_v + 2A_{mvn}/\phi_m)/(\eta_v + \eta_m + \eta_n). \tag{36}$$

We did not make D_{ijmn} symmetrical with its subscripts j, m, and n, although it may be symmetrized.

From (35), for four different modes i, j, m, and n, we have

$$\langle X_iX_jX_mX_n\rangle^{(2)} = \int X_iX_jX_mX_nP^{(2)}\Pi\, dX_i = [\phi_i\phi_j\phi_m\phi_n/(\eta_i + \eta_j + \eta_m + \eta_n)]$$
$$\times (P_{jmn}D_{ijmn}/\phi_i + P_{mni}D_{jmni}/\phi_j + P_{nij}D_{mnij}/\phi_m + P_{ijm}D_{nijm}/\phi_n). \tag{37}$$

Here P_{jmn} is a perturbation operator, and

$$P_{jmn}D_{ijmn} = D_{ijmn} + D_{ijnm} + D_{imnj} + D_{imjn} + D_{injm} + D_{inmj}. \tag{38}$$

Notice that $\langle X_iX_jX_mX_n\rangle^{(0)} = \langle X_iX_jX_mX_n\rangle^{(1)} = 0$. The higher-order term $P^{(2)}$ of the perturbation solution (6) must be considered in order to re-evaluate the degree of intermittency of turbulence, which depends upon the correlation of four different modes.

V. EXPRESSION FOR $F^{(2)}$

After a long manipulation of Eqs. (15), (36), (37), and (38), and the formulas in Appendixes A and B of Ref. 31, we have

$$F^{(2)} = 43\,200H^6\left(\frac{\nu}{\epsilon}\right)^2 \sum_{\mathbf{kprst}} G(\mathbf{k,p,r,s,t})\phi(p)\phi(r)\phi(s)\delta_{\mathbf{t,k+p}}\delta_{\mathbf{t,r+s}}$$

$$\times\{[\eta(t)+\eta(r)+\eta(s)][\eta(k)+\eta(p)+\eta(r)+\eta(s)]\}^{-1}. \quad (39)$$

Here $G(\mathbf{k,p,r,s,t})$ is a complicated geometrical factor, and

$$G(\mathbf{k,p,r,s,t}) = k_1p_1r_1s_1\{2[\phi(t)/\phi(r)][a(\mathbf{k,p})d(\mathbf{k,t,r})a(\mathbf{r,s}) + b(\mathbf{k,p})c(\mathbf{k,t,r})a(\mathbf{r,s})$$
$$+ a(\mathbf{k,p})c(\mathbf{r,t,k})b(\mathbf{r,s}) + b(\mathbf{k,p})e(\mathbf{k,t,r})b(\mathbf{r,s})] - [a(\mathbf{k,p})d(\mathbf{k,t,r})a(\mathbf{t,s})$$
$$+ b(\mathbf{k,p})c(\mathbf{k,t,r})a(\mathbf{t,s}) + a(\mathbf{k,p})d(\mathbf{k,t,s})a(\mathbf{t,r}) + b(\mathbf{k,p})c(\mathbf{k,t,s})a(\mathbf{t,r})]\}, \quad (40a)$$

$$a(\mathbf{k,p}) = k_1 - p_1(\mathbf{k}\cdot\mathbf{p})/p^2, \quad (40b)$$

$$b(\mathbf{k,p}) = 1 - (k_1/k)^2 - (p_1/p)^2 + p_1k_1(\mathbf{k}\cdot\mathbf{p})/(kp)^2, \quad (40c)$$

$$c(\mathbf{k,p,r}) = k_1 - p_1(\mathbf{k}\cdot\mathbf{p})/p^2 - r_1(\mathbf{k}\cdot\mathbf{r})/r^2 + r_1(\mathbf{k}\cdot\mathbf{p})(\mathbf{r}\cdot\mathbf{p})/(rp)^2, \quad (40d)$$

$$d(\mathbf{k,p,r}) = 1 - (k_1/k)^2 - (p_1/p)^2 - (r_1/r)^2 + p_1k_1(\mathbf{k}\cdot\mathbf{p})/(kp)^2$$
$$+ r_1k_1(\mathbf{k}\cdot\mathbf{r})/(kr)^2 + p_1r_1(\mathbf{r}\cdot\mathbf{p})/(rp)^2 - k_1r_1(\mathbf{r}\cdot\mathbf{p})(\mathbf{k}\cdot\mathbf{p})/(kpr)^2, \quad (40e)$$

$$e(\mathbf{k,p,r}) = \mathbf{k}\cdot\mathbf{r} - (\mathbf{k}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{r})/p^2. \quad (40f)$$

For the evaluation of the average modal intensity, Eq. (19) is a good approximate expression for the probability distribution, and we have[31,32]

$$\langle X_i^2\rangle = e_i\phi_i, \quad e_i = 1 - \nu_i/\eta_i. \quad (41)$$

For isotropic turbulence ϕ_i, $\langle X_i^2\rangle$, and η_i are functions of k only. Let

$$q(k) = 2H\langle X_i^2\rangle, \quad \tilde{q}(k) = 2H\phi_i; \quad (42)$$

we have

$$q(k) = e(k)\tilde{q}(k), \quad e(k) = 1 - \nu k^2/\eta(k), \quad (43)$$

and

$$E(k) = 4\pi k^2 q(k) = 4\pi k^2 e(k)\tilde{q}(k). \quad (44)$$

Here $E(k)$ is the 3-D energy spectrum. Let $L\to\infty$ and $H\to 0$, and by means of the formulas in Appendix B of Ref. 31, Eq. (39) becomes

$$F(2) = 5400\left(\frac{\nu}{\epsilon}\right)^2 \int\int\int d\mathbf{p}\, d\mathbf{r}\, d\mathbf{s} G(\mathbf{k,p,r,s,t})\tilde{q}(p)\tilde{q}(r)\tilde{q}(s)$$
$$\times\{[\eta(t)+\eta(r)+\eta(s)]$$
$$\times[\eta(k)+\eta(p)+\eta(r)+\eta(s)]\}^{-1}, \quad (45)$$

with $\mathbf{t} = \mathbf{r}+\mathbf{s}$ and $\mathbf{k} = \mathbf{t}-\mathbf{p}$. The right side of (45) is a 9-D (nine-dimensional) integral.

VI. FORMULA FOR $E(k)$ AND $\eta(k)$

In order to calculate the integrand of the 9-D integral in (45), it is necessary to determine $\tilde{q}(k)$ and $\eta(k)$. Since $\tilde{q}(k)$ is related to $E(k)$ by (44), it is equivalent to determining $E(k)$ and $\eta(k)$. In Ref. 31 the variational approach to the closure problem is successful in determining $E(k)$ and $\eta(k)$ for the inertial range, and gives

$$E(k) = 1.19\epsilon^{2/3}k^{-5/3}, \quad \eta(k) = 0.268\epsilon^{1/3}k^{2/3}. \quad (46)$$

In Ref. 32 the variational approach is extended to determine $E(k)$ and $\eta(k)$ for the universal equilibrium range by means of the equation error method of the control theory, and gives that

$$E(k) = \epsilon^{2/3}k^{-5/3}F(k/k_d),$$
$$F(x) = 1.19[1+5.3x^{2/3}]\exp(-5.4x^{4/3}), \quad (47a)$$

$$\eta(k) = 0.268\epsilon^{1/3}k^{2/3}[1+3.73(k/k_d)^{4/3}]. \quad (47b)$$

Here $k_d = (\epsilon/\nu^3)^{1/4}$ is the Kolmogorov wavenumber.

Let k_0 be the characteristic wavenumber of the energy-containing eddies; Eqs. (46) and (47) are valid for $k \gg k_0$ only. Although the variational approach to the closure problem has not yet been extended to determine the accurate behavior of $E(k)$ and $\eta(k)$ for $k \approx k_0$, the general characteristics of $E(k)$ and $\eta(k)$ for small k are known. The energy spectrum $E(k)$ has a maximum around k_0. When k approaches zero,[3] firstly $E(k)\approx k$, then $E(k)\approx k$.[4] As long as the Reynolds number R_λ is very high and $k_d \gg k_0$, there is a wide inertial range; the detailed structure of $E(k)$ and $\eta(k)$ near k_0 is not important for the evaluation of the 9-D integral in (45). For the numerical computation of the flatness factor of high-Reynolds-number turbulence, the following formula is used for $E(k)$:

$$E(k) = \epsilon^{2/3}k^{-5/3}F(k/k_d)/[1+(k_0/k)^{8/3}],$$
$$\text{if } 0.01k_0 \leqslant k \leqslant 2k_d, \quad (48a)$$

$$E(k) = 0, \quad \text{if } k < 0.01k_0 \text{ or } k > 2k_d. \quad (48b)$$

The formula (47b) is used for $\eta(k)$ for numerical computation. The cutoff at $0.01k_0$ and $2k_d$ is for convenience in numerical computation. Since $E(k)$ is nearly zero when $k < 0.01\,k_0$ or $k > 2k_d$, this cutoff is acceptable. When $k \gg k_0$, (48) becomes (47a). When $0.01\,k_0 < k < 0.5k_0$, by (48), $E(k)\approx k$ for small k, which is the expected behavior of $E(k)$ for small k. It has to be emphasized that (48) is a good approximation only when R_λ is very high. A much simpler formula for $E(k)$ may be

$$E(k) = \epsilon^{2/3}k^{-5/3}F(k/k_d), \quad \text{if } k_0 \leqslant k \leqslant 2k_d, \quad (49a)$$

$$E(k) = 0, \quad \text{if } k < k_0 \text{ or } k > 2k_d. \quad (49b)$$

Tentative numerical computation (by the Monte Carlo method, see the next section) shows that when R_λ is very high, (48) and (49) give nearly the same results.

VII. NUMERICAL COMPUTATION BY MONTE CARLO METHOD

The Monte Carlo method is used to evaluate the 9-D integral in (45). Because of the cutoff at $0.01k_0$ and $2k_d$, the integration domain is finite instead of being infinite. By (45) the integrand of the 9-D integral is a function of wave vectors **p**, **r**, *and* **s** only. The random sampling points in the 9-D space (**p**,**r**,**s**) are generated by the standard method of producing uniformly distributed pseudorandom numbers, based on the recurrent use of the following residue formula,[33,34]

$$\beta_{n+1} = c\beta_n \,(\mathrm{Mod}\, M).$$

Here c and M are properly selected positive numbers.

The ratio $\phi(t)/\phi(r)$ in (40a) is to be calculated by the following formula:

$$\phi(t)/\phi(r) = \tilde{q}(t)/\tilde{q}(r) = r^2 e(r)E(t)/[t^2 e(t)E(r)]. \quad (50)$$

When $\mathbf{r} \simeq -\mathbf{s}$ and $t \simeq 0$ the ratio $\phi(t)/\phi(r)$ makes the geometrical factor $D(\mathbf{k},\mathbf{p},\mathbf{r},\mathbf{s},\mathbf{t})$ oscillate between large positive and negative values. This will lead to some anomalous fluctation, although the probability is very small. Actually when $t = 0, \mathbf{r} = -\mathbf{s}$ and $\mathbf{k} = -\mathbf{p}$, the four modes i, j, m, and n are not different; their correlation has been considered in Sec. III. Hence in the numerical computation we exclude any sampling point for which $t/(r+s) < 0.05$.

The relationship between u_{rms} and $E(k)$ is

$$\frac{3}{2} u_{\mathrm{rms}}^2 = \int_0^\infty E(k)dk. \quad (51)$$

By (2), (3), (4), (48), and (51) the corresponding R_λ was calculated for many different k_0/k_d. Then by interpolation k_0/k_d can be obtained for a given R_λ. The value of k_0/k_d in Table I is obtained in this way. In the numerical computation all wavenumbers are made dimensionless by using k_d as unit.

Ten different sets of 10^6 random sampling points are used to calculate the 9-D integral in (45) by the Monte Carlo method[35] so that ten different values of $F^{(2)}$ are obtained. The $F^{(2)}$ obtained in this way is a random variable because the sampling points are random. Then the standard statistical method is applied to the evaluation of the expected value and the standard deviation of $F^{(2)}$. The final results are given in Table I for R_λ from 250 to 10^5. The comparison with the experimental data is given in Fig. 1 for R_λ from 250 to 2000.

VIII. DISCUSSION

We have succeeded in developing a closure theory to evaluate the degree of intermittency of turbulence directly

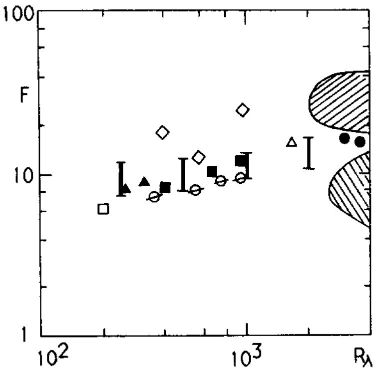

FIG. 1. Variation of flatness factor with Reynolds number: I Present theoretical result; ⊘ Wyngaard and Tennekes (1970); ◇ Gibson, Stegen, and Williams (1970); ⊖ Sheih, Tennekes, and Lumley (1971); ∅ Kuo and Corrsin (1971); △ McConnell (1976); ● Park (1976); □ Williams and Paulson (1977); ▲ Gagne and Hopfinger (1979); ■ Antonia, Satyaprakash, and Hussian (1982).

from the Navier–Stokes equation by the general method of statistical mechanics, without appealing to any extra phenomenological model. The expected values of the degree of intermittency given in Table I are a sample average and, therefore, it is a random variable, so there is some fluctuation in its change with R_λ. The standard deviations given in Table I are quite large. In order to reduce the standard deviation, it is necessary to increase the number of sampling points. Since the convergence of the Monte Carlo method is extremely slow, it would increase the computation time greatly to achieve a little more accuracy. The computed values of the flatness factor range from 9 to 15 for Reynolds numbers from 250 to 10^5 and have the same order of magnitude as the experimental data. The change in F is small, and asymptotically F seems to approach a constant as R_λ becomes very high. This result is not in agreement with the conclusions of most phenomenological theories,[14–28] which show that

$$F \sim R_\lambda^h, \quad (52)$$

where h is a constant between 0.2 and 1.5. This would indicate a significant increase in F with R_λ. The available experimental data of F for high Reynolds number vary widely.[5–13] Most experimental data support (52), while the Sheih–Tennekes–Lumley values seem to be in favor of the present theoretical results. This disagreement between the present theo-

TABLE I. Degree of intermittency of isotropic turbulence.

Reynolds number R_λ	(k_0/k_d)	Expected value of $F^{(2)}$	Standard deviation of $F^{(2)}$	Flatness factor $F = F^{(0)} + F^{(1)} + F^{(2)}$
250	4.10×10^{-3}	6.7	2.3	9.7
500	1.26×10^{-3}	7.4	2.2	10.4
1000	4.16×10^{-4}	8.6	1.9	11.6
2000	1.40×10^{-4}	10.5	2.6	13.5
5000	3.38×10^{-5}	10.1	2.4	13.1
10^4	1.18×10^{-5}	11.7	2.3	14.7
2×10^4	4.11×10^{-6}	11.5	2.0	14.5
5×10^4	1.04×10^{-6}	12.0	2.2	15.0
10^5	3.65×10^{-7}	11.8	2.5	14.8

retical results and Eq. (52) is very interesting and reflects the conflict of the Kolmogorov 1941 and 1962 theories.

The 1941 theory is intended to describe an asymptotic universal statistical state attained by the small-scale motion after a long cascade process.[22] The 1941 theory does not deny the weak dependence of the small-scale statistics on the Reynolds number or other mean-flow parameters; it only asserts that this dependence becomes negligible when the Reynolds number is very high. On the contrary the 1962 theory denies the existence of such a universal statistical state, and asserts that the dependence of the small-scale statistics on the macrostructure persists even when the Reynolds number approaches infinity. In this sense we say that the 1962 theory disqualifies the 1941 theory. This point can be illustrated by the energy spectrum $E(k)$ in the small-scale range. According to the 1941 theory we may assume that

$$E(k) = \epsilon^{2/3} k^{-5/3} [A_0 + A_1 (kL)^{-1} + A_2 (kL)^{-2} + \cdots], \quad kL \gg 1. \quad (53)$$

Here L is the characteristic length of the large-scale motion. According to the 1962 theory we have,

$$E(k) = \epsilon^{2/3} k^{-5/3} (kL)^{-\mu/9} [B_0 + B_1 (kL)^{-1} + B_2 (kL)^{-2} + \cdots], \quad kL \gg 1. \quad (54)$$

Here μ is a constant between 0.2 and 0.5.[28,36–40] Equation (53) means that the dependence of $E(k)$ on the large-scale motion in the small-scale range is weak, and when kL is very large we have asymptotically

$$E(k) = A_0 \epsilon^{2/3} k^{-5/3}, \quad (55)$$

with A_0 being a function of k/k_d only. This is the Kolmogorov $k^{-5/3}$ law. On the contrary Eq. (54) means that the dependence of $E(k)$ on the large-scale motion persists even when $kL \to \infty$, because μ is positive and B_0 might depend on the macrostructure.[4] The situation is the same for the relationship between the flatness factor F and the Reynolds number R_λ. The 1941 theory does not deny the weak dependence of F on R_λ, it asserts that F asymptotically approaches a constant as R_λ approaches infinity. On the contrary the 1962 theory asserts that F increases infinitely with R_λ according to (52). The exponent h in (52) is proportional to μ, for example, $h = \mu$ or $h = 1.5\,\mu$.[23] In the framework of the 1962 theory, the constant μ or some expression, for example, $\mu_{n/3} = (\mu n/18)(n-3)$ or $(\mu/3)(n-3)$ used in Ref. 39, can be considered as a measure of the degree of dependence of small scales on large scales.

In the analytical turbulence theory it is generally assumed that a fully developed turbulence is isotropic and confined in a cubic box with side L and cyclic boundary condition; then let L approach infinity.[29–31,41,42] This idealized model or infinite isotropic turbulence, which is also used in this article, has no macrostructure, so the role of the macrostructure is automatically discarded. Hence it comes in conflict with the 1962 theory. The model of infinite isotropic turbulence is in accordance with the 1941 theory, and corresponds to the universal statistical state attained by small scales after a long cascade process, as assumed in the 1941 theory. The present theoretical results in Table I represent an estimation of the flatness factor of this idealized model of infinite isotropic turbulence with a truncated energy spectrum. As explained in the last two sections, different values of the ratio k_d/k_0 of the truncated energy spectrum are used to correspond to different Reynolds numbers. In the present theoretical framework the flatness factor would approach a constant that is the F value of the infinite isotropic turbulence without truncating the energy spectrum, which is in disagreement with (52). This disagreement reflects the conflict of the Kolmogorov 1941 and 1962 theories.

The weight of experimental evidence is more in favor of some form of the 1962 theory than in favor of the 1941 theory, but this is very much an open problem. The 1941 theory is attractive because of its simplicity and self-consistency. As pointed out by Kraichnan,[22] once the 1941 theory is abandoned, a Pandora's box of possibilities is open. The 1962 theory means that the small-scale statistics can not be universal, but depend upon the Reynolds number and other mean-flow parameters,[4] while the small-scale motion of turbulence is random, the large-scale motion is organized or coherent and depends upon the boundary conditions. When the small-scale structure is dependent on the macrostructure of turbulence, the statistics of the random small-scale motion would have to include structures pertaining to the large-scale motion.[41] It would be a formidable task for the analytical turbulence theory. According to the 1962 theory,[4,14] the variance of the logarithm of the average dissipation rate ϵ_r over a sphere of radius r is[4]

$$\sigma^2 = A + \mu \ln(L/r), \quad (r \ll L), \quad (56)$$

and as a consequence,

$$F = BR_\lambda^h. \quad (57)$$

Here A depends upon the macrostructure of turbulence and B is a function of A, so B depends upon the macrostructure. Hence by the 1962 theory, the relationship between F and R_λ might not be universal, but may depend upon the macrostructure,[4] i.e., depend upon the conditions under which the turbulent flow develops.

The closure method itself does not necessarily mean that the flatness factor has to asymptotically approach a constant. The theoretical results in Table I are the consequence of the application of the closure theory to the idealized model of infinite isotropic turbulence. If the 1962 theory is correct, it must be in accordance with the Navier–Stokes equation, and its conclusion Eq. (52) will be a consequence of the application of the closure method to a proper model of turbulence that reflects the important role of the macrostructure in the small-scale statistics. At the present time there is no such application of the closure method because of the formidable mathematical difficulty. For the analytical turbulence theory, the infinite isotropic turbulence having no macrostructure is the unique workable theoretical model for the study of common features of small-scale statistics of various real turbulent flows at very high Reynolds numbers. The intermittency phenomenon indicated by a high flatness factor is a common feature of the kind. The work reported in this article shows that a closure theory can explain the intermittency phenomenon if the higher-order terms in the perturbation solution of the Liouville equation are considered. Most closure theories discard these higher-order terms and fail to explain the intermittency phenomenon.

Kolomogorov traces the origins of the 1962 theory to a remark by Landau that questioned the validity of the Kolmogorov $k^{-5/3}$ law because the local energy dissipation rate fluctuates.[4,22] In Ref. 31 the variational approach to the closure problem of turbulence theory is applied to derive the Kolmogorov $k^{-5/3}$ law and evaluate the Kolmogorov constant. In this article the same closure theory is extended to the calcuation of the flatness factor that indicates the degree of intermittency; the resultant F values are very high and have the same order of magnitude as the experimental data. As pointed out by Kraichnan,[22] the 1941 theory can not be disqualified merely because the dissipation rate fluctuates. The Kolmogorov $k^{-5/3}$ law and the high degree of intermittency can coexist as two consistent consequences of the closure theory.

ACKNOWLEDGMENT

The author would like to express his appreciation to Professor C. M. Tchen of The City College of New York for his guidance and encouragement.

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