Nonequilibrium statistical mechanics of one-dimensional turbulence

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The method of statistical mechanics is applied to the study of the one-dimensional model of turbulence proposed in an earlier paper. The closure problem is solved by the variational approach which has been developed for the three-dimensional case, yielding two integral equations for two unknown functions. By solving the two integral equations, the Kolmogorov $k^{-5/3}$ law is derived and the (one-dimensional) Kolmogorov constant Ko is evaluated, obtaining $\mathrm{Ko} = 0.55$, which is in good agreement with the result of numerical experiments on one-dimensional turbulence.

The argument, which led Kolmogorov to propose the $k^{-5/3}$ law for the inertial-range spectrum of turbulence,[1–3] is so general that it is logical to expect that the validity of the Kolmogorov law is independent of the dimensionality of turbulence. Unfortunately, the commonly used one-dimensional (1-D) model of turbulence, the Burgers equation, has a k^{-2} inertial-range spectrum.[4–9] In Ref. 10 it is shown that the Burgers equation is not a proper 1-D model of the Navier–Stokes turbulence and the so-called Burgers turbulence is not turbulence.

A new 1-D model of turbulence is proposed in Ref. 10 to simulate the Navier–Stokes turbulence. It is

$$\frac{d}{dt} U(k) = -ikW_m(k) + P(k) - \nu(k)U(k). \tag{1}$$

Here k is the normalized discrete wavenumber and $U(k)$ is a complex model parameter which satisfies the "realness" condition

$$U^{(1)}(-k) = U^{(1)}(k), \quad U^{(2)}(-k) = -U^{(2)}(k), \tag{2}$$

where $U^{(1)}(k)$ and $U^{(2)}(k)$ are the real and imaginary parts of $U(k)$, respectively. The term $-ikW_m(k)$ in (1) is a modified advection term, which transfers the energy from one mode to the others but conserves the total energy.[10] The term $P(k)$ is a pressure-type term to simulate the role of the pressure term in the Navier–Stokes turbulence as a high-frequency conservative random force. The term $\nu(k)U(k)$ is a generalized viscous term, corresponding to an energy sink at high wavenumbers. The explicit expressions of $W_m(k)$, $P(k)$, and $\nu(k)$ are given in Ref. 10. A series of numerical experiments on the 1-D model (1) have been made and are successful in obtaining the Kolmogorov inertial-range spectrum[10]:

$$E(k) = \langle U(k)U^*(k)\rangle = \mathrm{Ko}\,\epsilon^{2/3}\,k^{-5/3}. \tag{3}$$

Here ϵ is the energy dissipation rate. According to the numerical experiments,[10] Ko is between 0.5 and 0.65.

In this paper the method of statistical mechanics is applied to the study of the 1-D model (1), especially to the derivation of the Kolmogorov law (3) from the first principle of the statistical mechanics. The closure problem is solved by the variational approach which has been developed for the three-dimensional case.[11]

The real and imaginary parts of the model parameters $U(k)$ are not independent because of (2). In order to obtain a complete set of independent real parameters, the wavenumber k and the real-imaginary-part index $\alpha(=1 \text{ or } 2)$ are combined into one index i in the following way:

$$i = (\alpha, k), \quad -i = (\alpha, -k) \tag{4a}$$

and

$$i > 0 \quad \text{if} \quad k > 0. \tag{4b}$$

The real model parameters are defined as

$$X_i = S(i)U^{(\alpha)}(k). \tag{5}$$

Here $S(i) = -1$ if $i < 0$ and $\alpha = 2$, otherwise $S(i) = 1$. It is easy to prove that $X_{-i} = X_i$, hence $(X_i, i > 0)$ forms a complete set of independent real parameters for the 1-D model of turbulence.

In terms of X_i and the explicit expressions for $W_m(k)$ and $P(k)$ in Ref. 10, after some manipulation, (1) becomes

$$\frac{d}{dt} X_i = -\nu_i X_i + \sum_{j,m} A^{(1)}_{ijm} X_j X_m + F^{(p)}_i, \tag{6a}$$

$$A^{(1)}_{ijm} = \begin{cases} 0, & \text{if any two of } i, j, m \text{ are equal,} \\ (k/2)S(i)C^{\alpha\beta\gamma}[S(j)S(m)\delta_{k,p+r} + S(j) \\ \quad \times S(-m)\delta_{k,p-r} + S(-j) \\ \quad \times S(m)\delta_{k,-p+r} + S(-j) \\ \quad \times S(-m)\delta_{k,-p-r}], & \text{otherwise,} \end{cases} \tag{6b}$$

$$F^{(p)}_i = \begin{cases} -S(i)A(k)|U(1)|^2 U^{(2)}(k)/|U(k)|, & \text{if } \alpha = 1, \\ S(i)A(k)|U(1)|^2 U^{(1)}(k)/|U(k)|, & \text{if } \alpha = 2. \end{cases} \tag{6c}$$

Here $\nu_i = \nu(k)$, $\delta_{k,p+r}$ is the Kronecker symbol, and $C^{\alpha\beta\gamma}$ is defined by Table I of Ref. 11. By (4) $i = (\alpha, k)$, $j = (\beta, p)$, and $m = (\gamma, r)$. $A(k)$ is a random variable independent of X_i, see Ref. 10. When there is some external force $\nu_i' X_i$ acting at low wavenumbers to maintain stationarity of 1-D turbulence, (6a) beomes

$$\frac{d}{dt} X_i = -(\nu_i - \nu_i')X_i + \sum_{j,m} A^{(1)}_{ijm} X_j X_m + F^{(p)}_i. \tag{7}$$

The form of (7) is different from its 3-D counterpart in Ref. 11 because of the presence of the extra term $F^{(p)}_i$ and the replacement of A_{ijm} by $A^{(1)}_{ijm}$. In (6) and (7) i, j, and m are positive.

All possible dynamic states of 1-D turbulence, or all possible sets $(X_i, i > 0)$, constitute a phase space. The probability distribution in the phase space, denoted by

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$P = P[(X_i, i>0)]$, satisfies the Liouville equation corresponding to the dynamic equation (7)

$$\frac{\partial}{\partial t} P + (\bar{L} + \bar{L}^{(p)})P = 0, \tag{8a}$$

$$\bar{L} = -\sum_i \left((\nu_i - \nu_i') \frac{\partial}{\partial X_i} X_i - \sum_{j,m} A_{ijm}^{(1)} X_j X_m \frac{\partial}{\partial X_i} \right), \tag{8b}$$

$$\bar{L}^{(p)} = \sum_i \frac{\partial}{\partial X_i} F_i^{(p)}. \tag{8c}$$

Similar to the Langevin–Fokker–Planck model described in Ref. 11, the following simplified model is assumed for 1-D turbulence:

$$\sum_{j,m} A_{ijm}^{(1)} X_j X_m + F_i^{(p)} \simeq -\zeta_i X_i + f_i. \tag{9}$$

Here f_i is a random force of the type of white noise. The approximation (9) means that

$$\bar{L} + \bar{L}^{(p)} \simeq \bar{L}^{(f)} \equiv -\sum_i \eta_i \left(\frac{\partial}{\partial X_i} X_i + \phi_i \frac{\partial^2}{\partial X_i^2} \right). \tag{10}$$

Here $\bar{L}^{(f)}$ is a Fokker–Planck operator.[11] For stationary 1-D turbulence, (8a) becomes

$$[\bar{L}^{(f)} + (\bar{L} + \bar{L}^{(p)} - \bar{L}^{(f)})]P = 0, \tag{11}$$

$(\bar{L} + \bar{L}^{(p)} - \bar{L}^{(f)})$ is a small perturbation operator. The equations (8)–(11) are different from their 3-D counterparts in Ref. 11 because of the presence of $F_i^{(p)}$ or $\bar{L}^{(p)}$. By (6c) and (8c), it can be proved that

$$\sum_{\alpha=1}^{2} \bar{L}^{(p)} P^{(0)} = 0, \tag{12}$$

where

$$P^{(0)} = \prod_i (2\pi\phi_i)^{-1/2} \exp\left(\frac{-X_i^2}{(2\phi_i)} \right) \tag{13}$$

is the Gaussian density function. By using (12) we can prove that the perturbation solution of (11) has the same form as that obtained in Ref. 11 except for the replacement of A_{ijm} by $A_{ijm}^{(1)}$, i.e.,

$$P = \left(1 - \sum_i (\nu_i - \nu_i') \frac{X_i^2 - \phi_i}{2\eta_i \phi_i} + \sum_{ijm} \frac{A_{ijm}^{(1)} X_i X_j X_m}{\phi_i(\eta_i + \eta_j + \eta_m)} \right) P^{(0)}. \tag{14}$$

Here $\eta_i = \zeta_i + \nu_i - \nu_i'$. In the inertial range $\nu_i = \nu_i' = 0$, so $\eta_i = \zeta_i$.

Using the probability distribution (14), after long manipulation we have

$$\left(\frac{d}{dt} + 2\nu(k) \right) E(k) = T(k), \tag{15a}$$

where $T(k)$ is the 1-D energy transfer spectrum function and

$$T(k) = \int_{-\infty}^{+\infty} \frac{k^2 E(p)E(r) - kpE(k)E(r) - krE(k)E(p)}{\eta(k) + \eta(p) + \eta(r)} dp, \tag{15b}$$

with $k = p + r$. The 1-D energy transfer function $\Pi(k)$ is the energy flux across the spectrum and

$$\prod(k) = \int_k^\infty T(k')dk' = 2\int_{-k}^{k} dp \int_{p^*}^{\infty} dr \times \bar{k} \frac{\bar{k}E(p)E(r) - pE(\bar{k})E(r) - rE(\bar{k})E(p)}{\eta(\bar{k}) + \eta(p) + \eta(r)}, \tag{16a}$$

with $\bar{k} = p + r$ and $p^* = \max(p, k - p)$. In the inertial range the energy transfer function is independent of k and is equal to the energy dissipation rate, i.e.,

$$\prod(k) = \epsilon. \tag{16b}$$

The energy equation (15) or (16) contains two unknown functions $E(k)$ and $\eta(k)$. Another equation of $E(k)$ and $\eta(k)$ is needed to solve the closure problem of 1-D turbulence. In order for the approximation (9) or (10) to be as good as possible, we choose η_i or ζ_i in such a way that

$$\sum_i \left\langle \left(\sum_{j,m} A_{ijm}^{(1)} X_j X_m + F_i^{(p)} - (-\zeta_i X_i) \right)^2 \right\rangle$$

is minimum. Following the variational approach developed in Ref. 11, after long manipulation we obtain

$$E(k)\eta(k) = \int_{-\infty}^{+\infty} dp \frac{\eta(p)}{[\eta(k) + \eta(p) + \eta(r)]^2} \times \{kpE(r)[E(k) - E(p)] + k^2 E(p) \times [E(k) - E(r)] + p^2 E(k)[E(r) - E(p)]\}, \tag{17}$$

with $k = p + r$.

Equations (15) [or (16)] and (17) form a closed set of integral equations for two unknown functions $E(k)$ and $\eta(k)$. As an application of this closed set of integral equations, in the following we will derive the Kolmogorov law (3) and evaluate the 1-D Kolmogorov constant.

Following the same procedure of Sec. IX of Ref. 11, it is easy to prove that Eqs. (16) and (17) have the following power function solution:

$$E(k) = \mathrm{Ko}\, \epsilon^{2/3} k^{-5/3}, \tag{18}$$

$$\eta(k) = D\epsilon^{1/3} k^{2/3}. \tag{19}$$

Substitute (18) and (19) into (16) and (17), then calculating the two resulting integrals by numerical method, we have

$$D/\mathrm{Ko}^2 = 2.05, \quad D^2/\mathrm{Ko} = 0.68. \tag{20}$$

From (20) the 1-D Kolmogorov constant is

$$\mathrm{Ko} = 0.55, \tag{21}$$

which is in good agreement with the result of the numerical experiments on 1-D turbulence reported in Ref. 10.

In Ref. 10 an 1-D model is proposed to simulate Navier–Stokes turbulence, a series of numerical experiments on it are made and are successful in obtaining the Kolmogorov $k^{-5/3}$ law. This can be used to test various approaches to the closure problem of turbulence theory. In this paper it is used to test the variational approach proposed in Ref. 11. The good agreement between the theory and the numerical experiments on the 1-D model is a heartening success of the variational approach.

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The commonly used 1-D model of turbulence is the Burgers equation which has a k^{-2} inertial-range spectrum instead of $k^{-5/3}$. An interesting problem arises. If the variational approach is applied to the Burgers equation, can we get the k^{-2} inertial-range spectrum? The answer is: the method of statistical mechanics and the variational approach cannot be applied to the Burgers equation. Contrary to the Navier–Stokes equation, solutions of the Burgers equation are not ergodic stochastic processes.[10] The basic assumption of the statistical mechanics of the many-body system is that the evolution of the dynamic state of the many-body system is an ergodic stochastic process.[12] This assumption is not valid for the dynamic system described by the Burgers equation.

A brief comment is made here on the issue, whether solutions of the Burgers equation are ergodic stochastic processes or not, although the discussion of it is not the objective of this paper. Both the Burgers equation and the Korteweg–de-Vries equation have only one nonlinear term, the advection term. By properly choosing the scales for time, length, and velocity, the Burgers equation (in the inviscid limit) can be obtained as an extreme case of the KdeV equation. The study of the self-generated (or intrinsic) chaos in nonlinear systems shows[13] that the KdeV equation is a limit case of the Toda chain and is integrable, so its solution cannot be ergodic. Hence solutions of the Burgers equation would not be ergodic, either. The nonlinear advection term reduces the chaos of the flow field and builds up correlation between modes, finally leads to a steady state of laminar flow.[10] In Navier–Stokes turbulence the pressure term plays the role of a high-frequency random force to limit the buildup of correlation between modes, finally leads to a turbulent flow.

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Thermal stability of radiating fluids: The scattering problem

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The problem of convective instability of a radiating fluid layer with scattering is treated with an extension of the Eddington approximation that allows the inclusion of anisotropic scattering into the solution of the radiative transfer equation. Introduction of scattering by keeping the optical depth of absorption constant reduces the critical Rayleigh number as well as the wavenumber, and thus, reduces the stabilizing influence of thermal radiation. It is shown that in cases of a narrow radiative boundary layer with a large temperature gradient, higher-order expansion terms are sometimes necessary to approximate the solution properly. In certain cases a two layer convection mode with a large critical wavenumber up to 50 sets in first with cells developing in and near the two radiative boundary layers.

Considerable attention has been given to the Rayleigh–Bénard problem of radiating fluids over the last 25 years. Goody[1] showed that in the thin as well as in the thick gas limit, the incorporation of emission and absorption of the fluid has a stabilizing effect on a layer heated from below, i.e., critical Rayleigh numbers are larger than in the radiation-free case. A review of papers on that subject can be found in Arpaci and Gözüm.[2] In their work they employed the radiative transfer equation (RTE) in the Eddington approximation to get results for any optical depth. Furthermore, they used the concept of Planck and Rosseland means of the absorption coefficient to retain some of the nongray

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