Nonequilibrium statistical mechanics of two-dimensional turbulence

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A complete set of independent real parameters and its dynamic equation are worked out to describe the vorticity dynamics of two-dimensional turbulence. The corresponding Liouville equation is solved by a perturbation method upon the basis of a Langevin–Fokker–Plank Model. The dynamic damping coefficient η of the LFP model is treated as an optimum control parameter to minimize the error of the perturbation solution. Thereby two integral equations, the enstrophy equation and the η equation, are obtained for two unknown functions: the spectrum and the η. The equilibrium spectrum for the inviscid case is obtained as a stationary solution of the enstrophy equation. The nonlocalness of the enstrophy transfer makes the enstrophy equation divergent for a simple power-law spectrum. In order to avoid the divergence problem, a localization factor g is introduced to characterize the actual spectrum. Finally, the localized forms of the two integral equations are numerically solved, leading to the inertial-range spectrum, $E(k) = 1.82(\ln g - 1.23)^{-2/3}\chi^{2/3}k^{-3}$ for $g \geqslant 10$, χ is the dissipation rate of the enstrophy.

I. INTRODUCTION

Much interest and effort have been directed to the study of two-dimensional (2-D) turbulence. The 2-D turbulence can be used as a simplified model of atmospheric turbulence.[1] The fundamental dynamic equation of 2-D turbulence is the same as that of guiding center plasmas.[2] It is easier to do numerical experiments on 2-D turbulence than on three-dimensional turbulence.[3,4]

In addition to kinetic energy, enstrophy is another inviscid constant of motion for 2-D turbulence. This has profound effects on its nonequilibrium, as well as equilibrium, statistical properties. In contrast to the three-dimensional case, for a 2-D turbulence the energy dissipation rate approaches zero as the Reynolds number approaches infinity, so the inertial-range cascade transfer of energy to higher wavenumbers is excluded.[5] Batchelor,[5] Kraichnan,[6] and Leith[7] proposed an enstrophy-cascade model in which the enstrophy is transferred from lower to higher wavenumbers with a constant dissipation rate χ, and the corresponding inertial-range spectrum is

$$E(k) = C\chi^{2/3}k^{-3}. \tag{1}$$

Kraichnan pointed out that the cascade transfer of enstrophy in a 2-D turbulence is not local in the wave vector space and the C in (1) is not a universal constant. He proposed to modify the spectrum (1) by a logarithmic correction term,

$$E(k) = C'[\ln(k/k_1)]^{-1/3}\chi^{2/3}k^{-3} \quad (k \gg k_1), \tag{2}$$

where k_1 is the lowest wavenumber of the enstrophy-cascade range. Kraichnan[8] estimates that $C' \simeq 2.6$.

Lilly[3] has made a series of numerical experiments on 2-D turbulence and, using a different argument, suggested another form of logarithmic correction term,

$$E(k) = C'[\ln(k/k_1)]^{-2}\chi^{2/3}k^{-3} \quad (k \gg k_1). \tag{3}$$

The available resolution of Lilly's numerical experiments is inadequate to test the validity of either (2) or (3).

A new approach to the closure problem of turbulence theory has been proposed in an earlier paper.[9] In this paper we apply it to study 2-D turbulence. First of all, a complete set of independent real modal parameters and its dynamic equation are worked out to describe the vorticity dynamics of 2-D turbulence. Following classical statistical mechanics, the phase space and the Liouville equation for the probability distribution are introduced. An approximate solution of the Liouville equation is obtained by a perturbation method, based on a Langevin–Fokker–Planck (LFP) model. The dynamic damping coefficient η of the LFP model is treated as an optimum control parameter to minimize the error of the perturbation solution. Two integral equations, the enstrophy equation and the η equation, are obtained for two unknown functions: the enstrophy spectrum $q(k)$ and the dynamic damping coefficient $\eta(k)$. The equilibrium spectrum of an inviscid 2-D turbulence is obtained as a stationary solution of the enstrophy equation for the inviscid case. Although the k^{-3} spectrum (1) is formally a solution of the two integral equations in the inertial range, the nonlocalness of the enstrophy transfer makes the enstrophy equation divergent logarithmically. That is the reason why (1) has to be modified by a logarithmic correction term. Unfortunately neither (2) nor (3) satisfies the two integral equations. A localization approximation of triad interactions among modes is introduced to correspond to the actual spectrum dynamics. The localized forms of the two integral equations are numerically solved to get the inertial-range spectrum.

II. VORTICITY EQUATION AND ITS FOURIER TRANSFORM

For the 2-D flow of an incompressible fluid, the Navier–Stokes equation becomes the vorticity equation,

$$\frac{\partial}{\partial t} w + \left(u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2}\right) w = \nu \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) w, \tag{4}$$

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where u_1 and u_2 are velocity components, ν is the viscosity, and

$$w=\frac{\partial}{\partial x_1}u_2-\frac{\partial}{\partial x_2}u_1\,, \tag{5}$$

is the vorticity. The continuity equation is

$$\frac{\partial}{\partial x_1}u_1+\frac{\partial}{\partial x_2}u_2=0\,. \tag{6}$$

For simplicity of mathematics, a homogeneous incompressible 2-D turbulence is supposed to be confined within a large square with sides L and periodic boundary conditions. The L will be let to approach infinity later. We can expand u_1, u_2, and w into Fourier series

$$u_n(\mathbf{x})=H\sum_{\mathbf{k}}U_n(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x}),\quad n=1\text{ or }2, \tag{7a}$$

$$w(\mathbf{x})=H\sum_{\mathbf{k}}W(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})\,. \tag{7b}$$

Here $H=(2\pi/L)^2$, $\mathbf{k}=[n_1(2\pi/L),n_2(2\pi/L)]$ with n_1 and n_2 being integers. From (5), (6), and (7) we have

$$U_1(\mathbf{k})=i(k_2/k^2)W(\mathbf{k}), \tag{8a}$$

$$U_2(\mathbf{k})=-i(k_1/k^2)W(\mathbf{k})\,, \tag{8b}$$

where k is the absolute value of wave vector $\mathbf{k}$. By (7) and (8), after some manipulation, (4) becomes

$$\left(\frac{d}{dt}+\nu k^2\right)W(\mathbf{k})=H\sum_{\mathbf{p},\mathbf{r}}P(\mathbf{p},\mathbf{r})\,W(\mathbf{p})W(\mathbf{r})\delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}\,, \tag{9}$$

where $\delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}$ is the 2-D Kronecker symbol, $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{r}$ are discrete 2-D wave vectors, and

$$P(\mathbf{p},\mathbf{r})=\left(\frac{1}{p^2}-\frac{1}{r^2}\right)\frac{r_1p_2-p_1r_2}{2}\,. \tag{10}$$

Let

$$W(\mathbf{k})=W^{(1)}(\mathbf{k})+iW^{(2)}(\mathbf{k}), \tag{11}$$

i.e., $W^{(1)}(\mathbf{k})$ and $W^{(2)}(\mathbf{k})$ are the real and imaginary parts of $W(\mathbf{k})$, respectively, the vorticity equation (9) becomes

$$\left(\frac{d}{dt}+\nu k^2\right)W^{(\alpha)}(\mathbf{k})=H\sum_{\mathbf{p},\mathbf{r}}\sum_{\beta,\gamma}P(\mathbf{p},\mathbf{r})C^{\alpha\beta\gamma}W^{(\beta)}(\mathbf{p})W^{(\gamma)}(\mathbf{r})\delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}\,. \tag{12}$$

Here the coefficients $C^{\alpha\beta\gamma}$ are defined by Table I.

III. MODAL PARAMETERS AND MODAL DYNAMIC EQUATION

Since the vorticity is a real quantity, $W(-\mathbf{k})=W^*(\mathbf{k})$, i.e.,

$$W^{(1)}(-\mathbf{k})=W^{(1)}(\mathbf{k})\,, \tag{13a}$$

TABLE I. Definition of $C^{\alpha\beta\gamma}$.

α	1	1	1	1	2	2	2	2
β	1	1	2	2	1	1	2	2
γ	1	2	1	2	1	2	1	2
$C^{\alpha\beta\gamma}$	1	0	0	-1	0	1	1	0

$$W^{(2)}(-\mathbf{k})=-W^{(2)}(\mathbf{k})\,. \tag{13b}$$

Hence $W^{(\alpha)}(\mathbf{k})$ are not independent modal parameters. In order to get a complete set of independent real modal parameters of a 2-D turbulence, the discrete countable wave vector $\mathbf{k}$ and the real-imaginary-part index α are combined together to form a single index i in the following way:

$$i=(\alpha,\mathbf{k}),\quad -i=(\alpha,-\mathbf{k}), \tag{14a}$$

$$i>0,\quad\text{if }(k_2>0),\quad\text{or }(k_1>0,\,k_2=0). \tag{14b}$$

The new modal parameters X_i are defined as follows:

$$X_i=S(i)\,W^{(\alpha)}(\mathbf{k})\,, \tag{15a}$$

here

$$S(i)=-1,\quad\text{if }(i<0,\,\alpha=2),\quad\text{otherwise }S(i)=1. \tag{15b}$$

Therefore we have $S(i)S(i)=1$. By using (14) and (15), (12) becomes

$$\left(\frac{d}{dt}+\nu_i\right)X_i=\sum_{j,m}M_{ijm}X_jX_m\,. \tag{16}$$

Here $\nu_i=\nu k^2$, $j=(\beta,\mathbf{p})$, $m=(\gamma,\mathbf{r})$, and

$$M_{ijm}=HS(i)S(j)S(m)P(\mathbf{p},\mathbf{r})C^{\alpha\beta\gamma}\delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}\,. \tag{17}$$

By (15), (13) is equivalent to

$$X_{-i}=X_i\,. \tag{18}$$

Hence $(X_i,\ i\geqslant 0)$ forms a complete set of independent real modal parameters of a 2-D turbulence. From (16), the modal dynamic equation for the independent real modal parameters is

$$\left(\frac{d}{dt}+\nu_i\right)X_i=\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}A_{ijm}X_jX_m=\sum_{j,m}A_{ijm}X_jX_m\quad(i,j,m\geqslant 0)\,, \tag{19}$$

where

$$A_{ijm}=M_{ijm}+M_{ij-m}+M_{i-jm}+M_{i-j-m}\,. \tag{20}$$

Afterwards, the summations over i, j, and m are restricted to over their nonnegative values. The useful properties of A_{ijm} are

$$A_{ijm}=A_{imj}\,, \tag{21a}$$

$$A_{ijm}=0,\quad\text{if any two of }i,j,m\text{ are equal.} \tag{21b}$$

For the study of high-wavenumber-range dynamics, the model of stationary homogeneous turbulence is assumed. Actually, the space homogeneity and time stationarity are conflict conditions for any turbulence; a homogenous turbulence is at the same time a decaying turbulence. In order to maintain stationarity one must assume that external force is continuously supplying energy to the turbulence at low wavenumbers to prevent it from decaying. The particular structure of this force is not relevant; for simplicity it is supposed that it has form $\nu_i'X_i$. Hence the modal dynamic equation (19) becomes

$$\frac{d}{dt}X_i=-(\nu_i-\nu_i')X_i+\sum_{j,m}A_{ijm}X_jX_m\,. \tag{22}$$

The ν_i' is different from zero only at low wavenumbers; at the later stage of the analysis of high-wavenumber range, it can be omitted.

IV. LIOUVILLE EQUATION AND ITS SOLUTION

By classical statistical mechanics, all possible dynamic states of the turbulence, or all possible sets (X_i, $i \geqslant 0$), constitute a phase space. The Liouville equation for the probability distribution in the phase space is[9-11]

$$\frac{\partial}{\partial t} P + \overline{L} P = 0 , \tag{23a}$$

where

$$\overline{L} = -\sum_i \left((\nu_i - \nu_i') \frac{\partial}{\partial X_i} X_i - \sum_{j,m} A_{ijm} X_j X_m \frac{\partial}{\partial X_i} \right), \tag{23b}$$

is the Liouville operator. According to the Langevin–Fokker–Planck model described in Ref. 9, we have

$$\sum_{j,m} A_{ijm} X_j X_m \simeq -\zeta_i X_i + f_i , \tag{24}$$

and

$$\overline{L} \simeq \overline{L}^{(f)} . \tag{25}$$

Here,

$$\overline{L}^{(f)} = -\sum_i \eta_i \left(\frac{\partial}{\partial X_i} X_i + \phi_i \frac{\partial^2}{\partial X_i^2} \right) \tag{26}$$

is the Fokker–Planck operator, corresponding to the Langevin equation

$$\frac{d}{dt} X_i = -\eta_i X_i + f_i, \quad \eta_i = \zeta_i + \nu_i - \nu_i' . \tag{27}$$

The Liouville operator of the Langevin equation is the Fokker–Planck operator. The f_i in (24) and (27) is a random force of the type of white noise. For the inertial range, $\nu_i = \nu_i' = 0$, so we have

$$\eta_i = \zeta_i . \tag{28}$$

For stationary turbulence, the Liouville equation (23) can be written as follows:

$$[\overline{L}^{(f)} + (\overline{L} - \overline{L}^{(f)})] P = 0. \tag{29}$$

According to (24)–(27), $(\overline{L} - \overline{L}^{(f)})$ is to be considered a small perturbation operator. The perturbation solution of (29) is[9]

$$P = \left(1 - \sum_i (\nu_i - \nu_i') \frac{X_i^2 - \phi_i}{2\eta_i \phi_i} + \sum_{ijm} \frac{A_{ijm} X_i X_j X_m}{\phi_i (\eta_i + \eta_j + \eta_m)} \right) P^{(0)} , \tag{30}$$

where $P^{(0)}$ is the approximate solution of (29) at zero order, i.e.,

$$\overline{L}^{(f)} P^{(0)} = 0 . \tag{31}$$

Equation (31) is the Fokker–Planck equation; its solution is the Gaussian density function

$$P^{(0)} = \prod_i (2\pi\phi_i)^{-1/2} \exp[-X_i^2/(2\phi_i)] . \tag{32}$$

By (30), the average modal intensity is

$$\langle X_i^2 \rangle = \int X_i^2 P \prod dX_i = \phi_i \left(1 - \frac{\nu_i - \nu_i'}{\eta_i} \right). \tag{33}$$

In the inertial range, $\nu_i = \nu_i' = 0$, and (33) becomes

$$\langle X_i^2 \rangle = \phi_i . \tag{34}$$

Hence the ϕ_i is the average intensity of mode i. By (24)–(28), the η_i is the dynamic damping coefficient. For isotropic turbulence, ϕ_i and η_i are functions of k only.

V. ENSTROPHY EQUATION

By the Liouville equation (23), we have

$$\frac{d}{dt} \langle X_i^2 \rangle = \int X_i^2 \left(\frac{\partial}{\partial t} P \right) \prod dX_i = -\int X_i^2 \overline{L} P \prod dX_i$$
$$= -2(\nu_i - \nu_i') \langle X_i^2 \rangle + 2 \sum_{j,m} A_{ijm} \langle X_i X_j X_m \rangle . \tag{35}$$

The last step is based upon an integration by parts. From (30) and (34) the triple correlation is

$$\langle X_i X_j X_m \rangle = 2B_{ijm}/(\eta_i + \eta_j + \eta_m), \tag{36a}$$

$$B_{ijm} = A_{ijm} \phi_j \phi_m + A_{jmi} \phi_m \phi_i + A_{mij} \phi_i \phi_j . \tag{36b}$$

Let

$$q(k) = H \langle X_i^2 \rangle = H\phi_i , \tag{37}$$

and using (7), (8), and (A9) of the Appendix, we have

$$\left(\frac{1}{2}\right) \langle u_1^2 + u_2^2 \rangle = \int_0^\infty E(k) dk, \quad E(k) = \frac{2\pi q(k)}{k} , \tag{38}$$

$$\left(\frac{1}{2}\right) \langle w^2 \rangle = \int_0^\infty S(k) dk, \quad S(k) = k^2 E(k) = 2\pi k q(k). \tag{39}$$

$E(k)$ is the energy spectrum, and $S(k)$ is the enstrophy spectrum.

Afterwards, ν_i' is omitted. Let L approach infinity and H approach zero. By (35)–(39) and the formulas in the Appendix, finally we obtain

$$\left(\frac{d}{dt} + 2\nu k^2 \right) S(k) = T(k) = \iint^{\Delta} dp\, dr\, S(k \mid p,r) , \tag{40a}$$

$$S(k \mid p,r) = \frac{8\pi k b(p,r)}{\eta(k) + \eta(p) + \eta(r)} \{ a(p,k,r)\, q(r) \times [q(p) - q(k)] - a(r,k,p)\, q(p) [q(r) - q(k)] \} . \tag{40b}$$

Here

$$b(p,r) = r/p - p/r, \tag{41}$$

and $a(k,p,r)$ is defined by (A7) in the Appendix. $S(k \mid p,r) = S(k \mid r,p)$.

According to Kraichnan, the enstrophy transfer function is[8,10]

$$A(k) = \int_k^\infty dk' \iint^{\Delta}_{(p,r<k)} dp\, dr\, S(k' \mid p,r) - \int_0^k dk' \iint^{\Delta}_{(p,r>k)} dp\, dr\, S(k' \mid p,r) . \tag{42}$$

The first integral on the right-hand side is the total net input into all wavenumbers $>k$ from interaction with p and r, both $<k$, while the second integral is the total net loss to

wavenumbers $<k$ from interaction with p and r, both $>k$. These two types of triad interactions are mutually exclusive and exhaust the interactions which can transfer energy across the boundary at k. By the enstrophy-cascade model, in the inertial range,

$$\Lambda(k) = \chi \text{ (const)}, \tag{43}$$

χ is the dissipation rate of enstrophy.

VI. EQUILIBRIUM SPECTRUM OF INVISCID TURBULENCE

For the equilibrium ensemble of an inviscid turbulence, $(d/dt)S(k) = 0$ and $\nu = 0$, from (40) we must have

$$a(p,k,r)q(r)[q(p) - q(k)] - a(r,k,p)q(p)[q(r) - q(k)] = 0\,, \tag{44}$$

which expresses the principle of detailed balance of statistical mechanics applied to the 2-D turbulence. By (A7), (44) means

$$p^2(r^2 - k^2)q(r)[q(p) - q(k)] = r^2(p^2 - k^2)\,q(p)[q(r) - q(k)]\,. \tag{45}$$

It is easy to prove that the general solution of (45) is

$$q(k) = C_1 k^2/(C_2 + k^2)\,. \tag{46}$$

From (38) and (46), we have

$$E(k) = 2\pi[C_1 k/(C_2 + k^2)]\,, \tag{47}$$

which is the well-known equilibrium spectrum of inviscid 2-D turbulence and has been studied by many authors.[12–15]

VII. η EQUATION

The enstrophy equation (40) contains two unknown functions: $q(k)$ and $\eta(k)$. Another equation of $q(k)$ and $\eta(k)$ is needed to solve the closure problem of 2-D turbulence. The validity and error of the perturbation solution (30) depend on the "smallness" of the operator $(\overline{L} - \overline{L}^{(f)})$, i.e., depend on the validity and error of the approximation (24). We treat ζ_i or η_i as optimum control parameters to minimize the error of the approximation (24). The "optimum" is interpreted statistically in the mean-square sense,[9] i.e., we choose η_i in such a way that

$$\sum_i \left\langle \left(\sum_{j,m} A_{ijm} X_j X_m - (-\zeta_i X_i) \right)^2 \right\rangle$$

is the minimum. This leads to the following equation for optimum η (Ref. 9):

$$\eta_i = 2 \sum_{j,m} \frac{B_{ijm}[\eta_j A_{jmi} + \eta_m A_{mij} - (\eta_j + \eta_m)A_{ijm}]}{\phi_i(\eta_i + \eta_j + \eta_m)^2}\,, \tag{48}$$

where B_{ijm} is given by (36b). Similar to the derivation of (40), by using the formulas of the Appendix, (48) becomes

$$\eta(k) = 4 \iint^{\Delta} dp\,dr \frac{a(p,k,r)[2\eta(p) + \eta(r)]}{q(k)[\eta(k) + \eta(p) + \eta(r)]^2} \times \{(r/k)b(k,p)q(p)[q(r) - q(k)] - (p/k)b(k,r)q(r)[q(p) - q(k)]\}\,. \tag{49}$$

Equations (40) and (49) form a closed set of equations for q (or ϕ) and η. As soon as ϕ and η are obtained by solving (40) and (49), the probability distribution (30), which contains two parameters ϕ and η, is determined completely, thereby in principle solving the problem of nonequilibrium statistical mechanics of a 2-D turbulence. As an example of applications, the enstrophy equation and the η equation will be solved to obtain the inertial-range spectrum according to a enstrophy-cascade model.

VIII. LOCALIZATION FACTOR

The enstrophy-cascade inertial-range model is an enstrophy source at very low wavenumbers and a sink at very high wavenumbers, with an enstrophy flow across the spectrum at the constant rate χ. From (42) and (43), we have

$$\chi = \int_k^\infty dk' \iint_{(p,r<k)}^{\Delta} dp\,dr\,S(k'|p,r) - \int_0^k dk' \iint_{(p,r>k)}^{\Delta} dp\,dr\,S(k'|p,r)\,. \tag{50}$$

Suppose that $q(k)$ and $\eta(k)$ are of the type of power function

$$q(k) = C_q k^m\,, \quad \eta(k) = C_\eta k^n\,. \tag{51}$$

Substitute these into (49) and (50), we obtain

$$m - 2n + 2 = 0, \quad 2m - n + 4 = 0. \tag{52}$$

Hence $m = -2$ and $n = 0$. Letting $C_q = (C/2\pi)\chi^{2/3}$ and $C_\eta = D\chi^{1/3}$, (51) becomes

$$q(k) = (C/2\pi)\chi^{2/3}k^{-2}\,, \tag{53a}$$

$$\eta(k) = D\chi^{1/3}\,. \tag{53b}$$

By (38) and (53a), the energy spectrum is

$$E(k) = C\chi^{2/3}k^{-3}\,. \tag{54}$$

Unfortunately, when we substitute (53) into (50) the first integral of (50) diverges logarithmically. This is related to the nonlocalness of the enstrophy transfer in a 2-D turbulence. The dimensionless constant C would not be a universal constant. Kraichnan[6,8] suggested that the simple power-law spectrum be altered by a logarithmic correction term, but neither (2) nor (3) satisfies (49) and (50).

When k approaches zero, the $q(k)$ in (53a) approaches infinity as k^{-2}, making the first integral of (50) divergent. If the solutions of (49) and (50) are not restricted to the simple power function (53), there would be no divergence difficulty. For example,[16] if the actual spectrum is bounded at zero wavenumber and is approximated by the power function (53) for very high wavenumbers, then all integrals in (49) and (50) will be convergent. But it is a difficult task to solve (49) and (50) to get solutions which are expressed by more complicated functions, because (49) and (50) are nonlinear integral equations.

In order to get out of this dilemma, a localization procedure is introduced. Let

$$S_g(k\,|\,p,r) = \begin{cases} S(k\,|\,p,r), & \text{if } \dfrac{\max(k,p,r)}{\min(k,p,r)} \leqslant g, \\ 0, & \text{otherwise}\,. \end{cases} \tag{55}$$

When g approaches infinity, $S_g(k\,|\,p,r)$ approaches $S(k\,|\,p,r)$.

The integrand $S(k \mid p,r)$ in (50) is replaced by $S_g(k \mid p,r)$, and the resulting integral equation is called the localized form of (50). The localized form of (49) is obtained in a similar way, although (49) is convergent for the power function (53). This localization is equivalent to omitting the contribution of any triad interaction of (k,p,r) for which the ratio of the maximum wavenumber of (k,p,r) to the minimum wavenumber is greater than g. This is called the localization factor. It can be proved that the simple power-law spectrum (53) does satisfy the localized forms of (49) and (50) when g is not too small and there is no divergence problem. The localized forms of (49) and (50) are to be solved to get C and D of (53). The C and D obtained in this way depend upon the localization factor g. Finally, the asymptotic behavior of C and D with g approaching infinity is to be studied.

The actual spectrum of a 2-D turbulence, in the numerical simulation[3] or in the modeling of the atmospheric turbulence, is cut off at both low and high wavenumbers. This cutoff is roughly corresponding to the above-mentioned localization of triad interactions among modes, although not exactly.

IX. CALCULATION OF INERTIAL-RANGE SPECTRUM

The steps from (51) to (54) are still valid for the localized forms of (49)–(50), if g is not too small. Substituting (53) into the localized form of (50), and letting $p = k'p$, $r = k'z$, and $k' = ku$, we have

$$\frac{D}{C^2} = \frac{4}{3\pi}\left[\int_{g^{-1}}^{1} dy \int_{z^*}^{1} dz\, F(y,z)\ln\left(\frac{1}{z}\right) - \int_{1}^{g} dy \int_{y}^{y+1} dz\, F(y,z)\ln y\right], \tag{56a}$$

where $z^* = \max(y, 1-y)$, and

$$F(y,z) = a(1,y,z)b(1,y)b(1,z)b(y,z)/(yz). \tag{56b}$$

Substituting (53) into the localized form of (49), and letting $p = ky$ and $r = kz$, we have

$$\frac{D^2}{C} = \frac{2}{3\pi}\int_{g^{-1}}^{g} dy \int_{z^*}^{y+1} dz\, G(y,z), \tag{57a}$$

$$G(y,z) = a(1,y,z)b(1,y)b(1,z)(y^2 - z^2)/(yz)^2. \tag{57b}$$

The integrals in (56)–(57) are numerically evaluated for different g. The resultant C is given in Fig. 1.

By careful calculation of the asymptotic behavior of the integrals in (56)–(57), we have

$$\lim_{g\to\infty}(D^2/C) = 0.0416, \quad \lim_{g\to\infty}[D/(C^2 \ln g)] = 0.083; \tag{58}$$

therefore,

$$C \simeq 1.82(\ln g)^{-2/3}, \quad (\ln g \gg 1). \tag{59}$$

Actually, the numerical results for C in Fig. 1 can be fitted by the following curve:

$$C = 1.82(\ln g - 1.23)^{-2/3}, \quad (g \geqslant 10). \tag{60}$$

When $\ln g \gg 1$, (60) becomes (59). From (54) and (60) the inertial-range spectrum is

$$E(k) = 1.82(\ln g - 1.23)^{-2/3}\chi^{2/3}k^{-3}, \quad (g \geqslant 10). \tag{61}$$

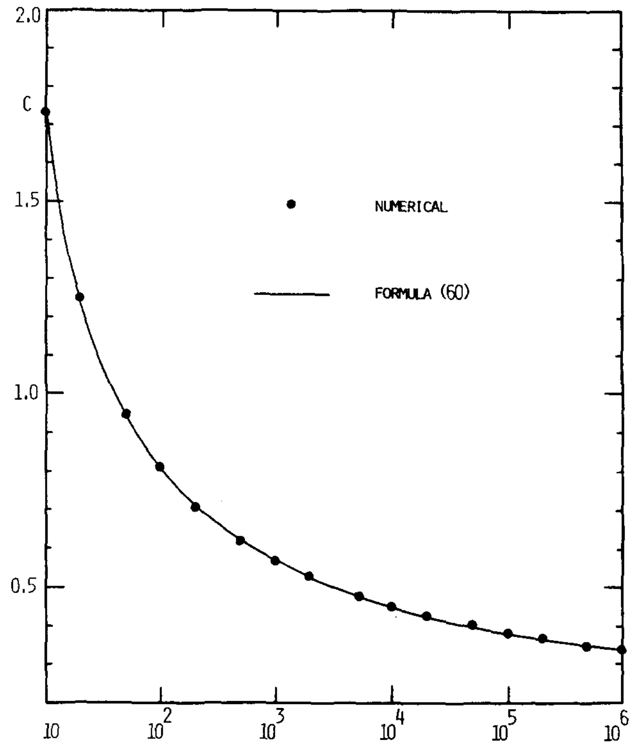

FIG. 1. Constant C in (54) versus localization factor g.

X. DISCUSSION

The localization factor g in (61) corresponds to (k/k_1) in (2) and (3). If the asymptotic formula for $E(k)$ is written as

$$E(k) \simeq C'(\ln g)^{-n}\chi^{2/3}k^{-3}, \quad (\ln g \gg 1), \tag{62}$$

$n = 1/3$ in Kraichnan's formula (2), $n = 2$ in Lilly's formula (3), and $n = 2/3$ in (61). Since the enstrophy transfer function $\Lambda(k)$ is divergent as $\ln g$ and the η equation is convergent when g approaches infinity, we have $n = 2/3$ and obtain (59)–(61).

The values of C shown in Fig. 1, or given by (60), are compatible with Lilly's numerical experiments, which give $C \sim 2$ and roughly correspond to the case of $g \sim 10$.

Although the localization procedure introduced in Sec. VIII is not realizable physically, it can be simulated in numerical experiments by simply discarding all the triad interactions of $(\mathbf{k},\mathbf{p},\mathbf{r})$ for which the ratio of maximum wavenumber of (k,p,r) to the minimum wavenumber is greater than the localization factor g. This is a way to test the validity of the theory proposed in this paper.

As Kraichnan[6,8] pointed out, besides the k^{-3} enstrophy-cascade inertial-range, there is another $k^{-5/3}$ energy-cascade inertial-range for a 2-D turbulence. The theory proposed in this paper can be applied to the study of the $k^{-5/3}$ range as well as the study of the k^{-3} range. For example, it can be proved by our theory that in the $k^{-5/3}$ energy-cascade range the direction of energy cascade is backward, from higher to lower wavenumbers. In the author's opinion, the k^{-3} enstrophy-cascade range is more characteristic of 2-D turbulence.

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APPENDIX: IMPORTANT FORMULAS FOR A_{ijm}

All the formulas needed for the derivations in Secs. V and VII are given below. Their detailed derivations are too long to be listed here. Suppose $f(k,p,r)$ is any function of k, p, and r, by using (17) and (20), we can prove that

$$\sum_{j,m} f(k,p,r) A_{ijm} A_{ijm} = C^* \sum_{\mathbf{p},\mathbf{r}} f(k,p,r) \times a(k,p,r) a(k,p,r) \delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}, \quad \text{(A1)}$$

$$\sum_{j,m} f(k,p,r) A_{ijm} A_{jmi} = C^* \sum_{\mathbf{p},\mathbf{r}} f(k,p,r) \times a(k,p,r) a(p,r,k) \delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}, \quad \text{(A2)}$$

$$\sum_{j,m} f(k,p,r) A_{ijm} A_{mij} = C^* \sum_{\mathbf{p},\mathbf{r}} f(k,p,r) \times a(k,p,r) a(r,k,p) \delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}, \quad \text{(A3)}$$

$$\sum_{j,m} f(k,p,r) A_{jmi} A_{jmi} = C^* \sum_{\mathbf{p},\mathbf{r}} f(k,p,r) \times a(p,r,k) a(p,r,k) \delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}, \quad \text{(A4)}$$

$$\sum_{j,m} f(k,p,r) A_{mij} A_{mij} = C^* \sum_{\mathbf{p},\mathbf{r}} f(k,p,r) \times a(r,k,p) a(r,k,p) \delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}, \quad \text{(A5)}$$

$$\sum_{j,m} f(k,p,r) A_{jmi} A_{mij} = C^* \sum_{\mathbf{p},\mathbf{r}} f(k,p,r) \times a(p,r,k) a(r,k,p) \delta_{\mathbf{k},\mathbf{p}+\mathbf{r}}. \quad \text{(A6)}$$

$C^* = 2H^2$ and

$$a(k,p,r) = (1/p^2 - 1/r^2)[s(s-k)(s-p)(s-r)]^{1/2}. \quad \text{(A7)}$$

Here

$$s = (k+p+r)/2, \quad \text{(A8)}$$

is the semiperimeter of the triangle with sides k, p, and r.

When L approaches infinity and H approaches zero, the summations in (A1)–(A6) become integrals, and

$$(1/H)\delta_{\mathbf{k},\mathbf{p}+\mathbf{r}} \to \delta(\mathbf{k}-\mathbf{p}-\mathbf{r}), \quad \text{(A9)}$$

where $\delta(\mathbf{k}-\mathbf{p}-\mathbf{r})$ is 2-D Dirac delta function. The transform of the bipolar integral is

$$\int\int f(k,p,r)\delta(\mathbf{k}-\mathbf{p}-\mathbf{r}) d\mathbf{p}\, d\mathbf{r} = \int\int^{\Delta} \frac{f(k,p,r)}{|\sin(\mathbf{p},\mathbf{r})|} dp\, dr. \quad \text{(A10)}$$

Here Δ indicates that the integration is restricted to the following infinite slot in the first quarter of the p-r plane:

$$\Delta: (r \geqslant 0, |k-r| \leqslant p \leqslant k+r), \quad \text{or} \quad (p \geqslant 0, |k-p| \leqslant r \leqslant k+p). \quad \text{(A11)}$$

[1] J. G. Charney and M. E. Stern, J. Atmos. Sci. **19**, 159 (1962).
[2] J. B. Taylor and W. B. Thompson, in *Statistical Model and Turbulence* (Springer, Berlin, 1972), p. 280.
[3] D. K. Lilly, J. Fluid Mech. **45**, 395 (1971).
[4] J. R. Herring, S. A. Orszag, R. H. Kraichnan, and D. G. Fox, J. Fluid Mech. **66**, 417 (1974).
[5] G. K. Batchelor, Phys. Fluids Suppl. II **12**, 233 (1969).
[6] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[7] C. E. Leith, Phys. Fluids **11**, 671 (1968).
[8] R. H. Kraichnan, J. Fluid Mech. **47**, 525 (1971).
[9] J. Qian, Phys. Fluids **26**, 2098 (1983).
[10] D. C. Leslie, *Developments in The Theory of Turbulence* (Clarendon, Oxford, 1973).
[11] S. F. Edwards and W. D. McComb, J. Phys. A2: Gen. Phys. **157** (1969).
[12] G. S. Deem and N. J. Zabusky, Phys. Rev. Lett. **27**, 396 (1971).
[13] D. G. Fox and S. A. Orszag, Phys. Fluids **16**, 169 (1973).
[14] R. H. Kraichnan, J. Fluid Mech. **67**, 155 (1975).
[15] K. Yamamoto and I. Hosokawa, J. Phys. Soc. Jpn. **48**, 347 (1980).
[16] A. Fukuyu, J. Phys. Soc. Jpn. **51**, 3415 (1982).