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Stability of flow of a generalized second order fluid down an inclined plane

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Introduction

Yih [1] investigated the stability of a triply nonlinear fluid flowing down an inclined plane. Fan [2] studied the same stability problem for a generalized Newtonian fluid which exhibits a variation in viscosity with rate of shear. Gupta [3], [4] studied the same stability problem for a second order fluid. He found that the stability characteristics of the flow are influenced by its elastic properties, but the second order fluid has a constant viscosity and hence the effect of variation of apparent viscosity on critical Reynolds number was not considered. In this paper, the same flow problem is considered for a generalized second order fluid. It may be expected that stability characteristics of the flow of such a fluid will be influenced by both its elastic properties and viscosity which varies with rate of shear.

Differential system governing stability and solution

The constitutive equation of a generalized second order fluid is

$$S_{ij} + p_i\, g_{ij} = \eta_0\,(\Pi)\, A_{ij} + \beta\,(\Pi)\, A_i^k\, A_{kj} + \gamma\,(\Pi)\, \delta\, A_{ij}/\delta\, t_1, \tag{1}$$

where S_{ij} is the stress tensor, and g_{ij} is the metric tensor, p_i is a scalar, Π is the second invariant of Rivlin-Ericksen tensor A_{ij}. The parametes η_0, β, γ are given as a function of Π. A_{ij}, Π and $\delta\, A_{ij}/\delta\, t_1$ are given by

$$\begin{aligned} A_{ij} &= V_{i,j} + V_{j,i} \\ \Pi &= \tfrac{1}{2}\, A_t^s\, A_s^t \\ \frac{\delta\, A_{ij}}{\delta\, t_1} &= \frac{\partial A_{ij}}{\partial t_1} + v^s \frac{\partial A_{ij}}{\partial x^s} + \frac{\partial v^s}{\partial x^i} A_{js} + \frac{\partial v^s}{\partial x^j} A_{is} \end{aligned} \tag{2}$$

where "," denotes a covariant derivative.

A layer of a generalized second order fluid of thickness h flows down a plane inclined at an angle β_0 to the horizontal. In a rectangular system (x_1, x_2, x_3), the steady primary flow is taken parallel to the x_1-axis with the x_2-axis normal to the plate downwards, the origin being taken at the undisturded free surface. As is usual in solving plane three dimensional disturbances problems we introduce a new coordinate system (y_1, y_2, y_3) by rotating x_1, x_3, through an angle $\theta = \tan^{-1} \hat{\beta}/\hat{\alpha}$, keeping x^2 axis fixed, where $\hat{\alpha}$ and $\hat{\beta}$ are the wave-number along x_1 and x_3 axes. The relation between the coordinate x_i and y_i are:

$$y_1 = x_1 \cos\theta + x_3 \sin\theta, \qquad y_2 = x_2, \qquad y_3 = x_3 \cos\theta - x_1 \sin\theta\,.$$

We introduce the following dimensionless quantities:

$$\begin{aligned}
&u = u_1/v_1^{0'}(h)\,h, \quad v = u_2/v_1^{0'}(h)\,h, \quad w = u_3/v_1^{0'}(h)\,h,\\
&x = y_1/h, \quad y = y_2/h, \quad t = t_1\,v_1^{0'}(h), \quad M = \gamma/h^2\varrho, \quad N = \beta/h^2\varrho,\\
&R = \varrho\,v_1^{0'}(h)\,h^2/\eta_0([v_1^{0'}(h)]^2),
\end{aligned} \tag{3}$$

where $v_1^0(y_2)$ is the velocity of steady flow and $v_1^{0'}(h)$ denotes $dv_1^0(y_2)/dy_2$ at $y_2 = h$.

We introduce a dimensionless stream function ψ defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ and we assume:

$$\begin{aligned}
&\psi = \varphi(y)\exp[i\alpha(x - \mathrm{ct})]\\
&w = \xi(y)\exp[i\alpha(x - \mathrm{ct})].
\end{aligned} \tag{4}$$

After some rather lengthy calculations, we obtain the final equation for this stability problem as follows

$$\begin{aligned}
&i\alpha R\,[(U\cos\theta - c)(\varphi'' - \alpha^2\varphi) - U''\varphi\cos\theta]\\
&= 4\,i\alpha D\,\{[(y/U') + i\alpha R M\,(U\cos\theta - c)]\,i\alpha\varphi' - R M U'(\varphi''\cos\theta - \xi'\sin\theta)\\
&\qquad - (\varphi''\cos\theta - \xi'\sin\theta + \alpha^2\cos\theta)(U')^2 M' R/2\,U''\} + (D^2 + \alpha^2)\\
&\quad\cdot\{[y/U' + i\alpha R M\,(U\cos\theta - c)](\varphi'' + \alpha^2\varphi)\\
&\quad + 2\,i\alpha R M\,(\varphi'\cos\theta - \xi\sin\theta)\,U' - i\alpha R M U''\varphi\cos\theta\\
&\quad + (1/U'' - y/U')\cos\theta\,(\varphi''\cos\theta - \xi'\sin\theta + \alpha^2\varphi\cos\theta)\}\\
&\quad + i\alpha R D\,[2\,N U'\xi'\sin\theta - (\varphi''\cos\theta - \xi'\sin\theta + \alpha^2\varphi\cos\theta)\\
&\quad\cdot\sin^2\theta\,(U')^2 N'/U''] - (D^2 + \alpha^2)\,[i\alpha R N U'\xi\sin\theta]
\end{aligned} \tag{5}$$

$$\begin{aligned}
&i\alpha R\,[(U\cos\theta - c)\,\xi + U'\varphi\sin\theta] = -\alpha^2(y/U')\,\xi\\
&\quad + D\,[(y/U')\,\xi - (1/U'' - y/U')\sin\theta\,(\varphi''\cos\theta - \xi'\sin\theta + \alpha^2\varphi\cos\theta)]\\
&\quad + i\alpha R N\,[U'(\varphi''\sin\theta + \xi'\cos\theta - \alpha^2\varphi\sin\theta) + U''(\xi\cos\theta + 2\,\varphi'\sin\theta)]\\
&\quad + i\alpha R N' U'\,[(\xi\cos\theta + 2\,\varphi'\sin\theta)\\
&\quad - (\varphi''\cos\theta - \xi'\sin\theta + \alpha^2\varphi\cos\theta)\cos\theta\sin\theta\,(U')/U'']\\
&\quad - i\alpha^3 R M\,[(U\cos\theta - c)\,\xi + U'\varphi\sin\theta]\\
&\quad + D\,\{M D\,[(U\cos\theta - c)\,\xi + U'\varphi\sin\theta]\}\,i\alpha R
\end{aligned} \tag{6}$$

where D, D^2 denote $\partial/\partial y$, $\partial^2/\partial y^2$ respectively.

Equation (5) and (6) with the boundary conditions (i.e. at the free surface the tangential stress must vanish and the normal stress must balance the normal stress induced by surface tensor; at the bottom no slip condition) constitute an eigenvalue problem. We may solve it using Yih's perturbation technique [5]. After some rather lengthy calculations, we obtain the critical Reynolds number as follow:

$$\begin{aligned}
R(\theta) = {}&\cot\beta_0\,[1 + 2\int_0^1 U\,dy/U'(1) - \{1 + 3\int_0^1 U\,dy/U'(1)\}\sin^2\theta] \div\\
&\{\cos^2\theta\,\{[U'(1)]^2 + 4\,U'(1)\int_0^1 U\,dy + 2\int_0^1 U^2\,dy + \int_0^1 U'^2\,y\,dy - \int_0^1\int_0^y U'^2\,y\,dy\,dy\}\\
&+ \int_0^1\int_0^y \{M\,[U''(2\,U' - y\,U'')\cos^2\theta + 2(U')^2\sin^2\theta/y] + N(U')^2\sin^2\theta/y\\
&+ [(U'' - U'/y)/U'(1)]\int_0^y N U' U''\,dy\,(\sin^2\theta\cos^2\theta)\}\,dy\,dy\\
&- \int_0^1 \{M\,[U''(2\,U' - y\,U'')\cos^2\theta + 2(U')^2\sin^2\theta/y] + N(U')^2\sin^2\theta/y\\
&+ [(U'' - U'/y)/U'(1)]\int_0^y N U' U''\,dy\,(\sin^2\theta\cos^2\theta)\}\,dy\}.
\end{aligned} \tag{7}$$

Discussion

Case 1. Generalized Newtonian fluid. $M = N = 0$.

Substituting $\theta = 0$ and $M = N = 0$ into (7), we obtain the critical Reynolds number R_2 with respect to two-dimensional disturbances as follow:

$$R_2 = \frac{\left[1 + 2\int_0^1 U\,dy/U'(1)\right]\cot\beta_0}{[U'(1)]^2 + 4\,U'(1)\int_0^1 U\,dy + 2\int_0^1 U^2\,dy + \int_0^1 U'^2\,y\,dy - \int_0^1\int_0^y U'^2\,y\,dy\,dy}\,.$$

This result agrees with that of Fan [2].

Substituting $M = N = 0$ into equation (7), we obtain the critical Reynolds number R_3 with respect to three-dimensional disturbances as follow:

$$R_3 = \left[1 - \tan^2\theta \int_0^1 U\,dy/U'(1)\right] R_2$$

where $U'(1) = -1$, $\int_0^1 U\,dy > 0$, so $R_2 > R_3$. Thus Squire's theorem is valid.

Case 2. Generalized second order fluid.

We assume power law relationship and thus write

$$\eta_0 = K\,\Pi^{(n-1)/2}, \qquad 1 \geqslant n > 0.6; \qquad \gamma = b\,\Pi^{m/2}; \qquad \beta = a\,\Pi^{l/2} \tag{8}$$

where K, a and b are constants.

Thus in viscometric flows, the viscosity function, the first and second normal stress differences as defined in Walter's book [6] are given respectively by

$$\eta(\dot\gamma) = K\,\dot\gamma^{n-1}$$
$$\nu_1(\dot\gamma) = -\,2\,b\,\dot\gamma^{m+2}$$
$$\nu_2(\dot\gamma) = 2\,b\,\dot\gamma^{m+2} + a\,\dot\gamma^{l+2}$$

where $\dot\gamma$ is the shear rate.

Substituting (3) (2) into (8) we obtain:

$$M = B\,y^{m/n}; \qquad N = A\,y^{l/n}; \qquad R = (h/g \sin\beta_0)\,(-\,\varrho\,g\,h \sin\beta_0/K)^{2/n} \tag{9}$$

where $B = (b/\varrho\,h^2)\,(-\,\varrho\,g\,h\sin\beta_0/K)^{m/n}$; $\quad A = (a/\varrho\,h^2)\,(-\,\varrho\,g\,h\sin\beta_0/K)^{l/n}$.

Because the second order fluid is a slightly viscoelastic fluid, so the Weissenberg number should be much less than one. Using an order of magnitude comparison we may deduce that $|BR| \ll 1$; $m \geqslant 2(n-1)$. For Newtonian fluid the critical Reynolds number is 2.5 and thus $|B|$ should not exceed about 0.1.

Walters [6] notes that a careful scrutiny of the more reputable measurements has led to a consensus that the second normal stress difference $-2M + N$ is usually much smaller than first normal stress difference $-2M$ (with $|2M + N| < 0.2\,|2M|$) and also of opposite sign. Because the first normal stress difference $-2M$ is positive, So $M < 0$. From the above we have $N > 0$ and

$$-\,1.6\,M < N < -\,2.2\,M\,. \tag{10}$$

Substituting $\theta = 0$ into (7), we obtain the critical Reynolds number R_2 with respect to two-dimensional disturbance:

$$R_2 = \frac{\dfrac{1}{1+2n}\cot\beta_0}{\dfrac{2}{3n+2}\,\dfrac{1}{1+2n} + \dfrac{1-2n}{n(m+n+2)}\,B}\,.$$

We note that the critical two dimensional Reynolds number is independent of A.

The critical Reynolds number R_3 with respect to three-dimensional disturbance is:

$$R_3 = \cot\beta_0\,[1 + (n-1)\sin\theta] \div \left[\left(\frac{2}{3n+2}\,\frac{2}{1+2n} + \frac{1-2n}{n(m+n+2)}B\right)\cos^2\theta - \left(\frac{2nB}{m+2+n} + \frac{2nA}{l+n+2}\right)\sin^2\theta - \frac{1-n}{(l+2)(l+3+n)}\sin^2\theta\cos^2\theta\right].$$

For case of $n = 1$; $l = m = 0$, this result agrees with that of Gupta [4]. Differentiating equation (7) with respect to θ, we obtain:

$$R'(\theta) = 2\sin\theta\cos\theta\cot\beta_0\left\{\frac{2n}{3n+2} + \frac{(1+2n)B}{m+2+n} + \frac{A(1+2n)}{l+3+n} \cdot \left\{\frac{n}{l+2+n} + \frac{n(l+n+1)}{l+2} + (1-n)\{(1-n)\cos^4\theta + 2n\cos^2\theta]\}\right\}. \tag{11}$$

For case of $1 \geqslant n > 0.6$; $|B| < 0.1$; $A > 0$ the value in the bracket of equation (11) is greater than zero for all θ. If $\theta = \frac{\pi}{2}$, the critical Reynolds number $R(\theta)$ is more than

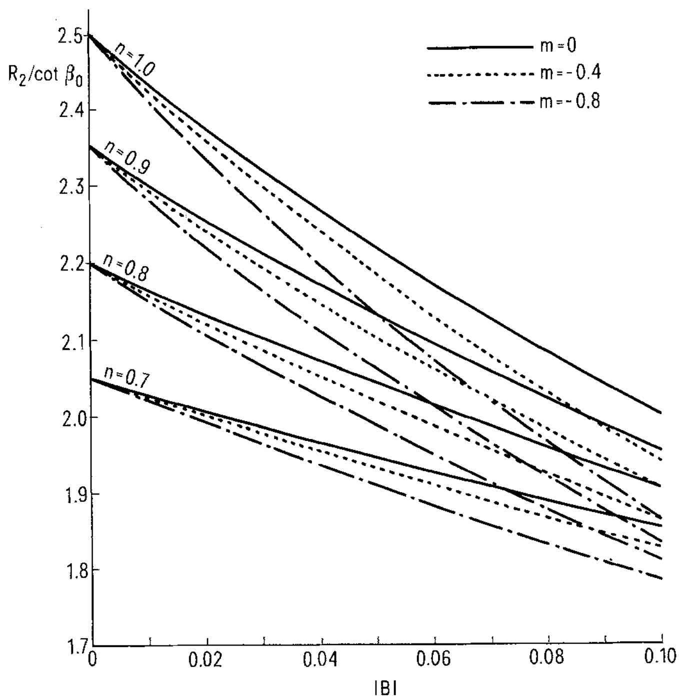

Figure 1
The critical Reynolds number $R_2/\cot\beta_0$ as a function of elastic parameter $|B|$ for various values of n and m ($\leqslant 0$)

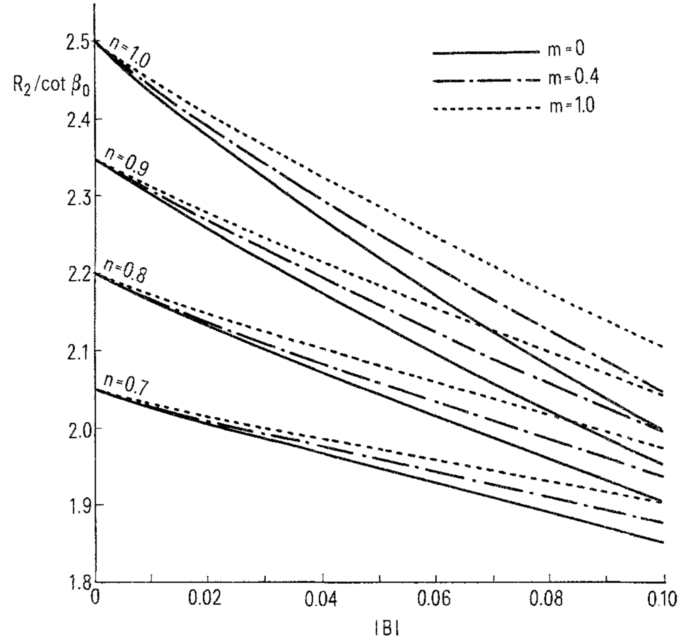

Figure 2
The critical Reynolds number $R_2/\cot\beta_0$ as a function of elastic parameter $|B|$ for various values of n and m ($\geqslant 0$)

$2.5 \cot \beta_0$. The Reynolds number $R(\theta)$ is a minimum when $\theta = 0$. Thus Squire's theorem is valid. This result contradicts Gupta's result. For case of $2M + N = 0$: $n = 1$ and $l = m = 0$, this result agrees with that of Lockett [7].

Gupta's critical Reynolds number R_c and R_{ct} are correct, but he said: "if $-M > 2/5 + N/2$, in which case oblique disturbances will be more unstable than the two-dimensional ones" [4]. From the inequality (10) and $|M| < 0.1$ we can show that the inequality stated by Gupta is impossible. The data used by Gupta was based of those given by Markovitz & Coleman [8] and their data were obtained prior to the discovery of pressure hole error.

From the above discussions we note that Squire's theorem is valid and thus we shall consider only two-dimensional disturbance. It will be observed from Figs. 1 and 2 that the critical Reynolds number $R_2/\cot\beta_0$ decreases as the elastic parameter $|B|$ increases or as the values of n decreases. Hence, the elastic property and the shear-thinning properties of the fluid tend to destabilize the flow. It can be seen from Fig. 1. and 2 that for a fixed value of $|B|$ the critical Reynolds number $R_2/\cot\beta_0$ increases as m increases.

The model we have chosen may be considered to be only appropriate for a slightly viscoelastic fluid and thus we limit our results to small values of elastic parameters and to the values of n near 1.

When $|B|$ is large and n is much less than one then we note that the flow is very unstable. This might be due to the model chosen. It is known that the second order fluid may exhibit undesirable instability characteristics [9].

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Summary

This investigation concerns the stability of flow of a generalized second-order fluid down an inclined plane with respect to three-dimensional disturbances. The critical Reynolds number is given as a function of dimensionless steady flow velocity $U(y)$, material parameters and the slope of the plane. In this case, for long wave disturbances, Squire's theorem is valid. This result contradicts that of Gupta.

Résumé

On considère le problème de stabilité d'un fluide de second ordre sur un plan incliné. On a examiné des perturbations tridimensionelles et sous certaines conditions le théorème de Squire est valable. Ce résultat contredit le résultat obtenu par Gupta.

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