THE INFLUENCE OF FLUCTUATION FIELDS ON THE FORCE-FREE FIELD

II. *The Local Expansion*

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Abstract. In Paper I (Hu, 1982), we discussed the the influence of fluctuation fields on the force-free field for the case of conventional turbulence and demonstrated the general relationships. In the present paper, by using the approach of local expansion, the equation of average force-free field is obtained as $(1 + b)\nabla \times \mathbf{B}_0 = (\alpha + a)\mathbf{B}_0 + \mathbf{a}^{(1)} \times \mathbf{B}_0 + \mathbf{K}$. The average coefficients a, $\mathbf{a}^{(1)}$, b, and $\mathbf{K}$ show the influence of the fluctuation fields in small scale on the configurations of magnetic field in large scale. As the average magnetic field is no longer parallel to the average electric current, the average configurations of force-free fields are more general and complex than the usual ones. From the view point of physics, the energy and momentum of the turbulent structures should have influence on the equilibrium of the average fields. Several examples are discussed, and they show the basic features of the fluctuation fields and the influence of fluctuation fields on the average configurations of magnetic fields. The astrophysical environments are often in the turbulent state, the results of the present paper may be applied to the turbulent plasma where the magnetic field is strong.

1. Introduction

The approximation of force-free field is applied to the region where the magnetic field is strong enough that the magnetic pressure is much larger than the thermodynamical pressure and the kinetic pressure of plasma, and in this case, the local current will be nearly parallel to the local magnetic field, it results in the smaller Lorentz force (the review, see, for example, Hu, 1983a). The force-free field is expressed, approximately, as

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0\,, \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0\,. \tag{1.2}$$

The theory of force-free field is extensively applied to the astrophysical environments, especially, to the solar atmosphere. For example, the magnetic field in the sunspots and active regions are force-free fields which dominate the equilibrium and activities.

Usually, the theory of force-free field is laminar – that is, the magnetic field $\mathbf{B}$ in Equations (1.1) and (1.2) is considered as average field. In a lot of astrophysical environments, the plasma is in the turbulent state, the fluctuation fields of velocity and magnetic field will have influence on the average fields. In this case, the fluctuation velocity field couples with the fluctuating magnetic field, hence; the fluctuation Lorentz force has contributions in the momentum equilibrium. Lerche (1970) and Hu (1982)

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discussed the case of turbulent force-free field, and Hu (1983b) analyzed the case of magnetostatic one.

In considering the fluctuation of magnetic field and the state of turbulent plasma, we denote the magnetic field as

$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}\,. \tag{1.3}$$

In the case of conventional turbulence, the fluctuation current is parallel to the average magnetic field (see Hu, 1982)

$$\nabla \times \delta\mathbf{B} = \delta b \mathbf{B}_0 \tag{1.4}$$

and the average equation of force-free field is

$$\nabla \times \mathbf{B}_0 + \mathbf{L} = \alpha(t, r)\mathbf{B}_0 + \mathbf{K}\,, \tag{1.5}$$

where

$$\mathbf{K} = -\iint \langle \delta b(t, \mathbf{r})\mathbf{B}_r(\xi)\rangle \times \nabla_r G(\mathbf{r}, \xi)\, \mathrm{d}\sigma_\xi \tag{1.6}$$

and

$$\mathbf{L} = \iiint \langle \delta b(t, \mathbf{r})\,\delta b(t, \xi)\rangle\, \nabla_r G(\mathbf{r}, \xi) \times \mathbf{B}_0(t, \xi)\, \mathrm{d}\tau_\xi\,. \tag{1.7}$$

$G(\mathbf{r}, \xi)$ is the Green's function of Poisson equation, and $\mathbf{B}_r(\xi)$ the boundary value of fluctuation vector magnetic potential function. The terms $\mathbf{L}$ and $\mathbf{K}$ give the influence of the fluctuation fields, and Equation (1.5) is reduced to the usual force-free equation if both terms are zero. The terms $\mathbf{K}$ and $\mathbf{L}$ may be determined if the correlation terms are given, and then, we can analyze the detailed influence of the fluctuating fields on the average fields.

In the next section, we shall demonstrate the average equation of force-free field by using the approach of local expansion. In Section 3, we shall discuss the magnetic induction equation, and then, the consistent conditions of the equations. Three examples are given in Section 4, and they show the influence of fluctuations fields and the features of turbulent force-free field in detail. The last section is devoted to discussions and conclusions.

2. The Local Expansion

The local expansion method is extensively used in the dynamic theory to demonstrate the induced turbulent electrical field (see, for example, Roberts, 1971). Similarly, this method may be applied to relationships (1.6) and (1.7) to give the connection between the average magnetic field $\mathbf{B}_0(t, \mathbf{r})$ and the correlation terms $\mathbf{K}$ and $\mathbf{L}$. In the relationships (1.6) and (1.7), we expand the magnetic field $\mathbf{B}_0(t, \xi)$ at position $\mathbf{r}$. The correlation term $\langle \delta b(t, \mathbf{r})\,\delta b(t, \xi)\rangle$ is a function with small typical length scale l_0, and decays rapidly to zero when $|\mathbf{r} - \xi| > l_0$. On the other hand, the k-order derivation of the average magnetic field is $O(B_0)/L_0^k$ in order of magnitude, where L_0 is the typical length on large scale. Therefore, relationship (1.7) may be written in order of magnitude as

$$|\mathbf{L}| = \sum_{k=0}^{\infty} O(B_0) l_0 \left(\frac{l_0}{L_0}\right)^k . \tag{2.1}$$

Relationship (2.1) implies that the contribution of the term is more important if the term associated with smaller index k.

Expanding $\mathbf{B}_0(t, \xi)$ in Taylor series about $\mathbf{r}$, relationship (1.7) may be written in the components forms as

$$L_i = -a_{ij} B_{0j} + b_{ijk} \frac{\partial B_{0k}}{\partial x_j} + c_{ijkl} \frac{\partial^2 B_{0l}}{\partial x_j \partial x_k} + \cdots , \tag{2.2}$$

where

$$a_{ij} = -\int\int \varepsilon_{inj} \langle \delta b(t, \mathbf{r}) \delta b(t, \xi) \rangle \frac{\partial G(\mathbf{r}, \xi)}{\partial x_n} \, \mathrm{d}\tau_\xi , \tag{2.3}$$

$$b_{ijk} = \int\int \varepsilon_{ink} \langle \delta b(t, \mathbf{r}) \delta b(t, \xi) \rangle (x_j - \xi_j) \frac{\partial G(\mathbf{r}, \xi)}{\partial x_n} \, \mathrm{d}\tau_\xi , \tag{2.4}$$

etc. for the coefficients of higher orders. We retain the first two terms in relationship (2.2), and obtain

$$L_i = -a_{ij} B_{0j} + b_{ijk} \frac{\partial B_{0k}}{\partial x_j} , \tag{2.5}$$

where (a_{ij}) and (b_{ijk}) are, respectively, the second and third tensors depending on the properties of the correlation function. Substituting (2.5) into Equation (1.5), the average equation of force-free field is given as

$$(\varepsilon_{ijk} + b_{ijk}) \frac{\partial B_{0k}}{\partial x_j} = \alpha B_{oi} + a_{ij} B_{0j} + K_i . \tag{2.6}$$

If K_i, a_{ij}, and b_{ijk} are zeros, Equation (2.6) is reduced to the usual force-free equation.

The influence of the turbulent fields in small structure on the average fields in large scale are given by the coefficients a_{ij}, b_{ijk}, and K_i. The influence may be discussed in detail after the coefficients are given. Similarly as in the kinetic theory or the turbulent theory, these coefficients may be determined by either the demonstrations of statistical approach or the macroscopic analyses. We will use the latter approach and avoid to treat the detailed structure of fluctuation fields at the first step.

The first term $\nabla \times \mathbf{B}_0$ in the left-hand side of Equation (1.5) is a pseudovector; hence, other terms in Equation (1.5), such as vector $\mathbf{K}$ and $\mathbf{L}$, should be pseudovectors, and $\alpha(t, \mathbf{r})$ is pseudoscale. In consideration of relationship (2.2), (a_{ij}) and (b_{ijk}) are pseudotensors of the second and third orders, respectively. Generally, a_{ij} may be expressed as

$$a_{ij} = a\delta_{ij} + a_k^{(1)}\varepsilon_{ijk} + a^{(2)}A_i^{(1)}A_j^{(2)}, \tag{2.7}$$

where a is a pseudoscalar, $\mathbf{a}^{(1)}$ is a pseudovector, and $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are of the same types if $a^{(2)}$ is a scale and of different types if $a^{(2)}$ is a pseudoscalar. Similarly, b_{ijk} may be expressed as

$$\begin{aligned} b_{ijk} = {} & b\varepsilon_{ijk} + b_1\beta_i^{(1)}\delta_{jk} + b_2\beta_j^{(2)}\delta_{ki} + b_3\beta_k^{(3)}\delta_{ij} + \\ & + b_4\varepsilon_{jkl}\beta_l^{(4)}\beta_i^{(5)} + b_5\varepsilon_{kil}\beta_l^{(6)}\beta_j^{(7)} + \\ & + b_6\varepsilon_{ijl}\beta_l^{(8)}\beta_k^{(9)} + b_7\beta_i^{(10)}\beta_j^{(11)}\beta_k^{(12)}, \end{aligned} \tag{2.8}$$

where b is a scalar; b_i and $\beta^{(i)}$ are of the same type for $i = 1, 2, 3$; b_4, b_5, and b_6 are scalars if $\beta^{(i)}$ and $\beta^{(i+1)}$ are of the same type, otherwise, b_4, b_5, and b_6 are pseudoscalars for $i = 4, 6, 8$; and b_7 is a pseudoscalar if $\beta^{(i)}$ $(i = 10, 11, 12)$ are all pseudovectors, or b_7 is a scalar if they are all axial vectors.

For simplicity, in the expansion formulas of a_{ij} and b_{ijk} we neglect the tensors with order equal to or higher than 2. Therefore, the first two terms in the right-hand side of Equation (2.7) and the first term in the right-hand side of Equation (2.8) are retained, and Equation (2.6) is reduced into

$$(1 + b)\nabla \times \mathbf{B}_0 = (\alpha + a)\mathbf{B}_0 + \mathbf{a}^{(1)} \times \mathbf{B}_0 + \mathbf{K}. \tag{2.9}$$

On the other hand, the average relationship of condition (1.2) is of the form

$$\nabla \cdot \mathbf{B}_0 = 0. \tag{2.10}$$

Equations (2.9) and (2.10) describe the average properties of the force-free field. The coefficients a, $\mathbf{a}^{(1)}$, b, and $\mathbf{K}$ reflect the the influence of turbulent structure in small-scale and may depend on the average magnetic field $\mathbf{B}_0$. Relationships (1.6) and (1.7) require that

$$\nabla \cdot \mathbf{K} = 0, \qquad \nabla \cdot \mathbf{L} = 0. \tag{2.11}$$

Using conditions (2.11), we find that Equation (1.5) gives

$$(\mathbf{B}_0 \cdot \nabla)\alpha = 0, \tag{2.12}$$

and Equation (2.9) requires that

$$(\nabla \times \mathbf{B}_0) \cdot \nabla b = \mathbf{B}_0 \cdot \nabla a + \nabla \cdot (\mathbf{a}^{(1)} \times \mathbf{B}_0). \tag{2.13}$$

The condition (2.12) shows that the force-free factor α keeps constant along a magnetic force lines. This requirement is the same as in the case of usual force-free field. Condition (2.13) is the requirement for the coefficients of fluctuation fields, and the connections between the fluctuation fields and average fields.

In comparison with Equation (2.9), force-free field equation (1.1) may be written, as usual,

$$\nabla \times \mathbf{B} = \alpha(t, r)\mathbf{B} . \tag{2.14}$$

Equation (2.9) is formally reduced into Equation (2.14) if the coefficients a, $\mathbf{a}^{(1)}$, b, and $\mathbf{K}$ are zeros. Generally, average Equation (2.9) does not require that the current should be parallel to the average magnetic field. The average current consists of three parts: one component is parallel to $\mathbf{B}_0$, one component is perpendicular to $\mathbf{B}_0$ and one component is contributed by the turbulent boundary value $\mathbf{K}$. We can introduce the idea of effective force-free factor α_*, defined by

$$\alpha_* = \frac{\alpha + a}{1 + b} . \tag{2.15}$$

Therefore, parallel to average magnetic field $\mathbf{B}_0$, the current component is $\alpha_*\mathbf{B}_0$; and it is similar to the case of usual force-free field. However, α_* could be non-zero when $\alpha = 0$. In the usual theory of force-free field, the force-free factor α is a fundamental quantity, and $\alpha = 0$ corresponds to the potential field without current. In the present case, in consideration of the contributions of the turbulent energy, the state of zero α is no longer corresponding to the state with lowest energy. The turbulent energy may play an important role in equilibrium and conservation of the energies, and introduce additional mechanism in the astrophysical processes. On the other hand, in the consideration of the magnetic field configurations, the average magnetic field is no longer required to be parallel to the average current, and has more general structure.

3. Magnetic Induction Equation

The magnetic induction equation including the influence of fluctuation fields gives the connections between the fluctuation velocity and the fluctuation magnetic field. Then, the average magnetic induction equation may be given (see Hu, 1982) as

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{M} + \mathbf{N}) + \eta \Delta \mathbf{B}_0 + \nabla \times (\mathbf{v}_0 \times \mathbf{B}_0) , \tag{3.1}$$

where

$$M_i = -\frac{1}{\eta} \int\int\int \varepsilon_{ijk}\,\varepsilon_{kba}\,\varepsilon_{lmn} \frac{\partial G^*_{al}(\mathbf{r}; t; \xi)}{\partial x_b} \times$$
$$\times \langle \delta v_j(\mathbf{r}, t) \delta v_m(\xi, t) \rangle \, B_{0n}(\xi, t) \, \mathrm{d}\tau_\xi , \tag{3.2}$$

$$N_i = -\frac{1}{\eta} \int\int\int \varepsilon_{ijl}\,\varepsilon_{lmk} \frac{\partial G^*_{kl}(\mathbf{r}, t; \xi)}{\partial x_m} \times$$
$$\times \langle \delta v_j(\mathbf{r}, t) B_{\Gamma l}(\xi, t) \rangle \, \mathrm{d}\sigma_\xi . \tag{3.3}$$

It can be seen that both **M** and **N** are pseudovectors. Using the local expansion method, we expand $\mathbf{B}_0(\xi, t)$ as a Taylor series at $(\mathbf{r}, t)$, and relationship (3.2) is reduced to

$$M_i = d_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0k}}{\partial x_j}, \tag{3.4}$$

where only the first two terms are remained. Similarly, the coefficients α_{ij} and β_{ijk} may be generally written (see Roberts, 1971) as

$$\alpha_{ij} = \alpha_0 \delta_{ij} + \alpha_k^{(1)} \varepsilon_{ijk} + \cdots, \tag{3.5}$$

$$\beta_{ijk} = -\beta_0 \varepsilon_{ijk} + \cdots, \tag{3.6}$$

where α_0 is a pseudoscalar; β_0, scalar; and $\boldsymbol{\alpha}^{(1)}$, a pseudovector. In relationships (3.5) and (3.6), we neglect the tensors with order equal to and larger than 2. In this case, Equation (3.1) is reduced to

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\alpha_0 \mathbf{B}_0 + \boldsymbol{\alpha}^{(1)} \times \mathbf{B}_0 - \beta_0 \nabla \times \mathbf{B}_0) + \eta \Delta \mathbf{B}_0 + + \nabla \times (\mathbf{v}_0 \times \mathbf{B}_0) + \nabla \times \mathbf{N}. \tag{3.7}$$

If α_0 and β_0 are constants, Equation (3.7) may be written as

$$\frac{\partial \mathbf{B}_0}{\partial t} = \alpha_0 \nabla \times \mathbf{B}_0 + (\eta + \beta_0) \Delta \mathbf{B}_0 + \nabla \times (\mathbf{v}_0 \times \mathbf{B}_0) + + \nabla \times \mathbf{N} + \nabla \times (\boldsymbol{\alpha}^{(1)} \times \mathbf{B}_0), \tag{3.8}$$

where the terms with α_0 and β_0 correspond, respectively, to the α effect and β effect in the usual dynamo theory. In the turbulent dynamo theory, people pay attentions mainly to the magnetic induction equation, and the momentum equation is not considered in detail, that is, the evolution of the average magnetic field is analyzed by assuming that the average and fluctuation velocity are given. In principle, the kinetic theory of turbulent dynamo is inconsistent.

In the static problem, it is required that $\partial/\partial t = 0$ and $v_0 = 0$, and Equation (3.7) is reduced into

$$\nabla \times [\alpha_0 \mathbf{B}_0 + \boldsymbol{\alpha}^{(1)} \times \mathbf{B}_0 - (\eta + \beta_0) \nabla \times \mathbf{B}_0 + \mathbf{N}] = 0, \tag{3.9}$$

or it may be rewritten as

$$\nabla \times \mathbf{B}_0 = \left(\frac{\alpha_0}{\eta + \beta_0}\right) \mathbf{B}_0 + \left(\frac{1}{\eta + \beta_0}\right) \boldsymbol{\alpha}^{(1)} \times \mathbf{B}_0 + \frac{1}{\eta + \beta_0} (\mathbf{N} + \nabla N_1), \tag{3.10}$$

which allows us to express Equation (2.9) in the form

$$\nabla \times \mathbf{B}_0 = \frac{\alpha + a}{1 + b} \mathbf{B}_0 + \frac{1}{1 + b} \mathbf{a}^{(1)} \times \mathbf{B}_0 + \frac{1}{1 + b} \mathbf{K}. \tag{3.11}$$

It can be seen that Equations (3.10) and (3.11) are similar in form, although they are demonstrated, respectively, from the induction equation and the momentum equation. The consistent conditions require that

$$\frac{\alpha_0}{\eta + \beta_0} = \frac{\alpha + a}{1 + b}, \qquad \frac{\boldsymbol{\alpha}^{(1)}}{\eta + \beta_0} = \frac{\mathbf{a}^{(1)}}{1 + b}, \qquad \frac{1}{\eta + \beta_0} (\mathbf{N} + \nabla N_1) = \frac{\mathbf{K}}{1 + b}. \tag{3.12}$$

In this case, the consistent problem is described by Equations (2.9) and (2.10).

Generally, the consistent problem should discuss Equations (2.9), (2.10), and (3.7). Induction equation (3.7) is a parabolic differential equation, and momentum equation (2.9) is an elliptic differential equation. The consistency condition requires that the magnetic field must satisfy some special conditions, and the problem is complex even for the case of usual kinetic force-free field (Hu, 1977). On the other hand, if we consider a special approximate case where the process is quasi-static ($v_0 \simeq 0$, $\partial/\partial t \to 0$) and the magnetic Reynold number is much larger than 1, then, magnetic induction equation (3.7) is approximately satisfied if the turbulent coefficients satisfy the conditions similar to (3.12). The problem is described approximately by Equation (2.9) and (2.10).

4. The Average Magnetic Field

We analyze the configurations of average magnetic fields which are described by Equation (2.9) and (2.10). There are three terms in the right-hand side of Equation (2.9): the first term gives the current component which is parallel to $\mathbf{B}_0$, and the second term is perpendicular to $\mathbf{B}_0$. We discuss three examples, which show the basic features of the turbulent force-free field.

4.1. Parallel configurations with $\alpha_* = 0$

When $\alpha_* = 0$, Equation (2.9) is reduced to

$$\nabla \times \mathbf{B}_0 = \frac{1}{1 + b} \mathbf{a}^{(1)} \times \mathbf{B}_0 + \frac{1}{1 + b} \mathbf{K}, \tag{4.1}$$

where b is a constant. We discuss the parallel configuration of average magnetic field, for example, $\mathbf{B}_0$ is parallel to $\mathbf{e}_x$, and denote

$$\mathbf{B}_0 = B_0(x, y, z)\mathbf{e}_x. \tag{4.2}$$

By use of (4.2), Equation (4.1) may be written in the component forms as

$$0 = K_x, \tag{4.3}$$

$$(1 + b) \frac{\partial B_0}{\partial z} = a_z^{(1)} B_0 + K_y, \tag{4.4}$$

$$(1 + b) \frac{\partial B_0}{\partial y} = a_y^{(1)} B_0 - K_z. \tag{4.5}$$

According to condition (4.3), the first relationship of condition (2.11) leads to the definition

$$\mathbf{K} = \left(0, \frac{\partial B_*}{\partial z}, -\frac{\partial B_*}{\partial y}\right), \tag{4.6}$$

where B_* is a function of (x, y, z) and has the dimension of magnetic field. By using (4.6), we write Equations (4.4) and (4.5) as

$$\frac{\partial}{\partial z}\,[(1+b)B_0 - B_*] = a_z^{(1)} B_0\,, \tag{4.7}$$

$$\frac{\partial}{\partial y}\,[(1+b)B_0 - B_*] = a_y^{(1)} B_0\,. \tag{4.8}$$

On the other hand, the condition (2.13) yields

$$(1+b)B_0\left(\frac{\partial a_z^{(1)}}{\partial y} - \frac{\partial a_y^{(1)}}{\partial z}\right) = a_z^{(1)}\,\frac{\partial B_*}{\partial z} - a_y^{(1)}\,\frac{\partial B_*}{\partial y}\,. \tag{4.9}$$

It is easy to see that this latter condition (4.9) may be obtained from Equation (4.7) and (4.8), and is automatically satisfied by the solution of the turbulent force-free field.

The consistency of Equations (4.7) and (4.8) requires that

$$a_z^{(1)}\,\frac{\partial}{\partial y}\,[(1+b)B_0 - B_*] - a_y^{(1)}\,\frac{\partial}{\partial z}\,[(1+b)B_0 - B_*] = 0\,. \tag{4.10}$$

The characteristic equation of (4.10) is of the form

$$a_y^{(1)}\,\mathrm{d}y - a_z^{(1)}\,\mathrm{d}z = 0\,. \tag{4.11}$$

We denote the integral relationship of Equation (4.11) as

$$\zeta(x, y, z) = \text{constant}\,; \tag{4.12}$$

then the solution of Equation (4.10) becomes

$$B_0(x, y, z) = \frac{1}{1+b}\,[B_* + f(\zeta)]\,. \tag{4.13}$$

In the infinite space, the average configuration of magnetic field at infinite requires

$$B_* \to 0 \quad \text{and} \quad f(\zeta) \to 0\,, \quad \text{when} \quad r \to \infty\,.$$

Therefore, we obtain an important conclusion, that the average configurations of force-field could persist by their own electric currents if the influence of fluctuation fields is included. The reason of this conclusion is that the energy of fluctuation fields will contribute to the energy conservation. In the usual force-free theory, the field cannot persist by its own current, and it is one of the fundamental features.

If $a_y^{(1)}$ and $a_z^{(1)}$ are constants, we have

$$\zeta(x, y, z) = a_y^{(1)} y + a_z^{(1)} z + c\,, \tag{4.14}$$

where c is a constant. The condition (4.9) gives

$$B_* = B_*(a_y^{(1)} y + a_z^{(1)} z, x)\,. \tag{4.15}$$

Furthermore, Equations (4.7) and (4.8) give

$$B_0(x, y, z) = f_1(a_y^{(1)} y + a_z^{(1)} z, x)\,. \tag{4.16}$$

Both (4.15) and (4.16) satisfy the relation of general solution (4.12). In detail, we express especially solution (4.16) in the form

$$B_0(x, y, z) = \frac{f_0}{x^2 + x_0^2} \exp[(a_y^{(1)} y + a_z^{(1)} z)^2]\,, \tag{4.17}$$

where x_0 is a constant. Then, the current may be written as

$$j_{0y} = -\frac{c}{4\pi} \frac{f_0 a_z^{(1)}}{x^2 + x_0^2} (a_y^{(1)} y + a_z^{(1)} z) \exp[-(a_y^{(1)} y + a_z^{(1)} z)^2]\,, \tag{4.18}$$

$$j_{0z} = \frac{c}{4\pi} \frac{f_0 a_y^{(1)}}{x^2 + x_0^2} (a_y^{(1)} y + a_z^{(1)} z) \exp[-(a_y^{(1)} y + a_z^{(1)} z)^2]\,. \tag{4.19}$$

It shows that the average current is perpendicular everywhere to the average magnetic field, and both current and field tend to zero at infinite. This simple example describes one of the features which associated with the influence of turbulent fields, and this feature is obviously different from the properties of usual force-free fields.

4.2. Two-dimensional magnetic field with $\alpha_* = 0$

We discuss Equation (4.1) with constant b and constant vector $\mathbf{a}^{(1)}$. According to the assumption of two-dimensional configuration, the magnetic field may be expressed as

$$B_0(x, y, z) = \left(\frac{\partial \psi}{\partial z},\ 0,\ -\frac{\partial \psi}{\partial y}\right). \tag{4.20}$$

If we substitute (4.20) in (4.1), the components equations of average force-free field are

$$\frac{\partial}{\partial x}\left[(1 + b)\frac{\partial \psi}{\partial y} - a_y^{(1)} \psi\right] = -K_x\,, \tag{4.21}$$

$$(1 + b)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}\right) - \left(a_x^{(1)} \frac{\partial \psi}{\partial x} + a_z^{(1)} \frac{\partial \psi}{\partial z}\right) = K_y\,, \tag{4.22}$$

$$\frac{\partial}{\partial z}\left[(1 + b)\frac{\partial \psi}{\partial y} - a_y^{(1)} \psi\right] = -K_z\,. \tag{4.23}$$

By virtue of condition (2.13), Equation (2.9) requires that

$$\mathbf{a}^{(1)} \cdot \mathbf{K} = 0 . \tag{4.24}$$

Equations (4.21) and (4.23) require also that the consistency condition

$$\frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x} = 0 , \tag{4.25}$$

be satisfied. We introduce a typical magnetic field B^*, which is defined as

$$K_x = -\frac{\partial B^*}{\partial x} , \qquad K_z = -\frac{\partial B^*}{\partial z} . \tag{4.26}$$

For the case $a_y^{(1)} = 0$, by use of the definition (4.26), Equation (4.24) admits of the solution

$$B^* = B^*(a_x^{(1)} z - a_z^{(1)} x) . \tag{4.27}$$

Substituting (4.27) in the first equation of (2.11), we obtain

$$a_x^{(1)} = 0 , \qquad a_z^{(1)} = 0 \quad \text{and} \quad B^* = \text{constant} .$$

In this case, Equations (4.21)–(4.23) are reduced to the Poisson equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{K_y}{1 + b} ; \tag{4.28}$$

the solution of which is of the form

$$\psi(x, y, z) = \int\int\int G(\mathbf{r}, \xi) \, \frac{K_y(\xi)}{1 + b} \, \mathrm{d}\tau_\xi , \tag{4.29}$$

where $G(\mathbf{r}, \xi)$ is the Green's function of the Poisson equation. The influence of fluctuation fields in this example is the existence of an average current component in the direction which is parallel to the y-axis and perpendicular to the average magnetic field.

For the case of $a_y^{(1)} \neq 0$ and $K_y \neq 0$, if we use the condition (4.24), Equation (2.11) may be written as

$$\frac{\partial^2 B^*}{\partial x^2} + \frac{\partial^2 B^*}{\partial z^2} - \frac{a_x^{(1)}}{a_y^{(1)}} \frac{\partial^2 B^*}{\partial x \, \partial y} - \frac{a_z^{(1)}}{a_y^{(1)}} \frac{\partial^2 B^*}{\partial y \, \partial z} = 0 . \tag{4.30}$$

The solution of Equation (4.30) in half-plane $y > 0$ is

$$B^*(x, y, z) = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} (c_{nm}^{\mathrm{I}} e^{s_1 x} + c_{nm}^{\mathrm{II}} e^{s_2 x}) (c_{nm}^{\mathrm{III}} e^{s_3 z} + c_{nm}^{\mathrm{IV}} e^{s_4 z}) \right] e^{-\theta_n^2 y} , \tag{4.31}$$

where c_{nm} are constants, γ_{nm} and θ_n are characteristic values and the indices

$$s_{1,2} = -\frac{1}{2}\left[\frac{a_x^{(1)}}{a_y^{(1)}}\,\theta_n^2 \pm \sqrt{\left(\frac{a_x^{(1)}}{a_y^{(1)}}\right)^4 \theta_n^4 - 4\gamma_{nm}^2}\right],$$

$$s_{3,4} = -\frac{1}{2}\left[\frac{a_z^{(1)}}{a_y^{(1)}}\,\theta_n^2 \pm \sqrt{\left(\frac{a_z^{(1)}}{a_y^{(1)}}\right)^2 \theta_n^4 + 4\gamma_{nm}^2}\right]. \tag{4.32}$$

Integrations of Equations (4.21) and (4.23) may be written as

$$(1+b)\,\frac{\partial\psi}{\partial y} - a_y^{(1)}\psi - B^* = c_0(y), \tag{4.33}$$

where $c_0(y)$ is an arbitrary function. Using (4.33), we can reduce Equation (4.22) to the form

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{1+b}\left(a_x^{(1)}\,\frac{\partial\psi}{\partial x} + a_z^{(1)}\,\frac{\partial\psi}{\partial z}\right) = \frac{1}{1+b}\left(\frac{a_x^{(1)}}{a_y^{(1)}}\,\frac{\partial B^*}{\partial x} + \frac{a_z^{(1)}}{a_y^{(1)}}\,\frac{\partial B^*}{\partial z}\right). \tag{4.34}$$

The nonhomogeneous terms in the right-hand side of (4.34) are given by solution (4.31). Similar to (4.31), we expand solution $\psi(x, y, z)$ as

$$\psi(x, y, z) = \sum_{n=0}^{\infty} \psi_0(x, z)\, e^{-\theta_n^2 y}, \tag{4.44}$$

and Equation (4.34) reduces to

$$\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial z^2} - \frac{1}{1+b}\left(a_x^{(1)}\,\frac{\partial\psi_0}{\partial x} + a_z^{(1)}\,\frac{\partial\psi_0}{\partial z}\right) = \frac{1}{1+b}\left(\frac{a_x^{(1)}}{a_y^{(1)}}\,\frac{\partial B_0^*}{\partial x} + \frac{a_z^{(1)}}{a_y^{(1)}}\,\frac{\partial B_0^*}{\partial z}\right), \tag{4.45}$$

where the function $B_0^*(x, z)$ is the terms in square bracket of solution (4.31). Therefore, the solution of Equation (4.45) may be written as

$$\psi_0(x, z) = \sum_{m=0}^{\infty} (c_{nm}^{\mathrm{V}}\, e^{s_5 x} + c_{nm}^{\mathrm{VI}}\, e^{s_6 x})\,(c_{nm}^{\mathrm{VII}}\, e^{s_7 z} + c_{nm}^{\mathrm{VIII}}\, e^{s_8 z}) + \psi_*(x, z), \tag{4.46}$$

where $\psi_*(x, z)$ is a special solution of inhomogeneous equation (4.45), c_{nm} are constants, and the index s_i are defined as

$$s_{5,6} = \frac{1}{2}\left[\frac{a_x^{(1)}}{1+b} \pm \sqrt{\left(\frac{a_x^{(1)}}{(1+b)}\right)^2 - 4\gamma_{nm}^2}\right],$$

$$s_{7,8} = \frac{1}{2}\left[\frac{a_z^{(1)}}{1+b} \pm \sqrt{\left(\frac{a_z^{(1)}}{1+b}\right)^2 + 4\gamma_{nm}^2}\right]; \tag{4.47}$$

and the γ_{nm}'s are the characteristic values.

If we substitute the solutions (4.44) and (4.46) into (4.20), the two-dimensional configuration of magnetic field is then obtained. Furthermore, we obtain the electric current. In general cases, the magnetic field is neither parallel nor perpendicular to the electric current. This feature is different from the one of usual force-free field.

4.3. The configuration of magnetic field with $a^{(1)} = 0$

By use of the condition $a^{(1)} = 0$, Equation (2.9) may be written as

$$\nabla \times \mathbf{B}_0 = \alpha_* \mathbf{B}_0 + \frac{1}{1+b} \mathbf{K} . \tag{4.48}$$

Equation (4.48) is reduced into the usual equation of force-free field, formally, if $K = 0$, and in this special case, the average current is parallel to the average magnetic field. There is a component of current which is not parallel to $\mathbf{B}_0$, if vector $\mathbf{K}$ is not parallel to $\mathbf{B}_0$. According to the first condition of (2.11), Equation (4.48) requires

$$(\mathbf{B}_0 \cdot \nabla)\alpha_* = 0 ,$$

and the effective force-free factor keeps constant along a magnetic force line.

In considering the linear field, we assume that all quantities depend on x and z – i.e., $\partial/\partial y = 0$. In this case, the magnetic field may be written as

$$\mathbf{B}_0(x, z) = \left(\frac{\partial \psi}{\partial z},\ B_{0y},\ -\frac{\partial \psi}{\partial x}\right). \tag{4.49}$$

According to the condition (2.11), vector $\mathbf{K}$ may be written as

$$\mathbf{K} = \left(\frac{\partial h}{\partial z},\ K_y,\ -\frac{\partial h}{\partial x}\right). \tag{4.50}$$

Substituting definitions (4.49) and (4.50) into Equation (4.48), we have

$$\frac{\partial}{\partial z}\left(B_{0y} + \frac{h}{1+b}\right) = -\alpha_* \frac{\partial \psi}{\partial z} , \tag{4.51}$$

$$\Delta \psi = \alpha_* B_{0y} + \frac{1}{1+b} K_y , \tag{4.52}$$

$$\frac{\partial}{\partial x}\left(B_{0y} + \frac{h}{1+b}\right) = -\alpha_* \frac{\partial \psi}{\partial x} . \tag{4.53}$$

Both Equations (4.51) and (4.53) require that

$$B_{0y} + \frac{h}{1+b} = G(\psi) \tag{4.54}$$

and

$$\alpha_* = -\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}. \tag{4.55}$$

Substituting conditions (4.54) and (4.55) into (4.52), we obtain the basic equation in the form

$$\Delta\psi - G(\psi)\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi} = \frac{1}{1+b}\left(K_y - h\frac{\mathrm{d}G(\psi)}{\mathrm{d}\psi}\right). \tag{4.56}$$

For the linear field, Equation (4.56) reduces to

$$\Delta\psi + \alpha_*^2\psi = \frac{1}{1+b}(K_y + \alpha_* h). \tag{4.57}$$

In general, the boundary condition for Equation (4.54) is given as

$$\psi|_\Gamma = \psi_\Gamma. \tag{4.58}$$

It is easy to see that the homogeneous equation of (4.55) is formally the same as the usual equation of linear force-free field. That is, the turbulent force-field problem (4.57) and (4.58) may be divided into two parts, and one is the usual force-field problem: i.e.,

$$\begin{cases} \Delta\psi_1 + \alpha_*^2\psi_1 = 0, \\ \psi_1|_\Gamma = \psi_\Gamma; \end{cases} \tag{4.59}$$

and the other is a special solution

$$\begin{cases} \Delta\psi_2 + \alpha_*^2\psi_2 = \dfrac{1}{1+b}(K_y + \alpha_* h), \\ \psi_2|_\Gamma = 0. \end{cases} \tag{4.60}$$

The solution of turbulent field is obtained by the addition of both solutions, of the form

$$\psi = \psi_1 + \psi_2. \tag{4.61}$$

The influence of fluctuation fields introduces the term associated with the special solution ψ_2. We denote the average turbulent force-free field in the form

$$\mathbf{B}_0 = \mathbf{B}_{\text{effective}} + \mathbf{B}_{\text{fluctuation}}, \tag{4.62}$$

where the effective force-free field is

$$\mathbf{B}_{\text{effective}} = \left(\frac{\partial\psi_1}{\partial z},\ G(\psi),\ -\frac{\partial\psi_1}{\partial x}\right) \tag{4.63}$$

and the field corresponding to the fluctuation field is

$$\mathbf{B}_{\text{fluctuation}} = \left(\frac{\partial\psi_2}{\partial z},\ \frac{h}{1+b},\ -\frac{\partial\psi_2}{\partial x}\right). \tag{4.64}$$

Generally, according to the influence of fluctuation fields, the average current is no longer parallel to the average magnetic field. The basic equation (4.59) is formally the same as usual equation of force-free field, however, the force-free factor is α_* instead of α, which may be larger or smaller than α_*. The value of α_* associates with the transverse component of magnetic field and, than, the component of transverse magnetic energy, which is often considered as the energy sources of solar flare and activities. Furthermore, there is a transverse component of magnetic field in (4.64) associated with $\mathbf{K}$. All of these introduce new components of magnetic field, and then the sources and mechanism in energy conservations, it may have extensive applications in astrophysics.

It can be seen that the coefficient a in (2.7) should be zero in the approximation of conventional turbulence. However, $a \neq 0$ in the general cases, and it is retained in the present paper.

5. Discussion

In the present paper, the basic equation of average force-free field are demonstrated by the method of local expansion. As the average current is no longer parallel to the average magnetic field, the features of turbulent force-free field are different from those of usual one. For example, in the usual theory, the transverse component of magnetic field B_{0y} keeps constant in a magnetic surface $\psi =$ constant as shown in (4.63). The influence of turbulent fields introduce an additional component of magnetic field $h/(1 + b)$, which is generally not a constant in a magnetic surface. Therefore, the magnetic force line in a magnetic surface may be sheared, and then, the magnetic energy may be stored. Moreover, as the average current is generally not parallel to the average magnetic field, there is an average Lorentz force in the region of strong field, and the plasma may be driven to flow with the kinetic pressure smaller than the magnetic pressure. In the case of magnetostatics, the average Lorentz force is balanced by the turbulent forces, which associate with the turbulent energy and introduce additional components of magnetic field in the equilibrium configurations. Therefore, the average force-free field may have more general features than the usual ones.

In Section 4, the first two examples show that the magnetic field depends on three space variables (x, y, z) and has no topological invariance. We have discussed that the cosmical magnetic field may generally have equilibrium configuration without topological invariance (Hu, 1983c, d; Hu *et al.*, 1983). As the fluctuation fields are included, there are additional freedom in the equilibrium equations, and it is easy to construct the magnetic field without topological invariance.

The plasma in the astrophysical environments is often in the turbulent state, and so is the magnetic field. Paper I and the present paper discuss the configurations in the region where the magnetic field is relatively strong and can be considered as force-free field. Similar results may be obtained for the magnetostatic problems (Hu, 1983b). The applications of these results to the detailed processes in astrophysics will be interesting and important. In the view point of physics, the turbulent coefficients, which is introduced in the present paper by the local expansion, should be connected with special physical processes as discussed in the turbulent dynamo theory. These will be analyzed in future.

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