Journal of the Physical Society of Japan
Vol. 51, No. 6, June, 1982, pp. 2024–2028

The Wave Model of Mesothermal Plasma near Wakes and Korteweg-de Vries Equation

C. SHEN and V. C. LIU*

Institute of Mechanics, The Chinese Academy of Sciences, Peking, China

(Received March 25, 1981)

The stationary two-dimensional (x, z) near wakes behind a flat-based projectile which moves at a constant mesothermal speed (V_∞) along a z-axis in a rarefied, fully ionized, plasma is studied using the wave model previously proposed by one of the authors (VCL). One-fluid theory is used to depict the free expansion of ambient plasma into the vacuum produced behind a fast-moving projectile. This nonstationary, one-dimensional (x, t) flow which is approximated by the K-dV equation can be transformed, through substitution, $t=z/V_\infty$, into a stationary two-dimensional (x, z) near wake flow seen by an observer moving with the body velocity (V_∞). The initial value problem of the K-dV equation in (x, t) variables is solved by a specially devised numerical method. Comparisons of the present numerical solution for the asymptotically small and large times with available analytical solutions are made and found in satisfactory agreements.

§1. Introduction

When a projectile or a spacecraft moves rapidly in a rarefied, ionized medium (*e.g.*, the upper ionosphere), the plasma disturbances around the moving body can be studied conveniently as three separate problems: (i) the sheath which refers to the zone of disturbances wrapping up the front end of the body, (ii) the far wake which designates the flow region behind the body beyond a distance of several body diameters and (iii) the near wake which represents the plasma flow immediately behind the body. This division was made[1] in considering the physical and mathematical characteristics of the flow field herein. The plasma sheath problem dates back 50 years to Langmuir's electric static probe. The recent contributions to the plasma sheath of a moving body are given in ref. 1. It is of interest to note the novel approach to the plasma sheath using the Schroedinger equation (with WKBJ-approximation) for mathematical advantages in treating the self-consistent field-particle interactions.[1,2]

In the far wake where the disturbances of both particle and field distributions have been, in general, considerably reduced in magnitudes, the linear perturbation techniques are applicable in treating the field-particle coupling effects. As a result, the physics of the plasma far wake behind a rapidly moving body are also well understood.[1] On the other hand, the elucidation of the plasma near wake, where the nonlinear coupling of particle and field distributions must be admitted, has not been effective. Earlier works on plasma near wakes had been formulated predominately on the basis of numerical iterations of the particle trajectories for self-consistent fields.[1] Computations are cumbersome and plagued with mathematical instability problems.[1]

An alternative approach to treat the plasma near wakes, using one-fluid theory, was conceptually proposed[3] to circumvent these difficulties. The purpose of the present note is to complete such a fluid model pertaining to the simplest geometric configuration.

In order to sharpen our focus on the essential physics of the plasma near wakes, it is postulated that the magnetic field effect is absent. It is noted that an electrically conducting body situated in a plasma will acquire an equilibrium negative surface potential.[1] In view of the above discussion, the wake-filling fluxes may be attributed to two primary causes: (i) deflection of the free stream plasma due to a negatively biased surface potential field which tends to produce an ion density peak on the axis of

* Permanent address: Department of Aerospace Engineering, The University of Michigan, Ann Arbor, Michigan (48109), U.S.A.

the wake. The magnitude of the peak depends, among other factors, upon the intensity of surface potential, and (ii) free expansion of the ambient plasma into the wake-cavity. During the expansion the ambient electrons, because of their higher thermal speed than the ions, move ahead of the ions initially until an electrical field of charge separation acts to equalize their motions. The advance of the coupled electron-ion wave front is thus ambipolar in nature.

To expedite plasma wake study, models have been constructed such that one of the two contributing causes overwhelms the other. For instance, with a sufficiently low surface potential, the free expansion process becomes predominant.[1] This will be assumed in the present study.

It is admitted that a kinetic analysis by means of the self-consistent Boltzmann-Vlasov equation and Poisson equation for the particle and field distributions in a rarefied plasma appears most reasonable. It poses, however, many problems in computational instabilities.[1] It is noted that the fluid mechanical approach has been shown to be effective because of the quasi-continuum nature of rarefied plasma.[4] A wave model for the mesothermal plasma near wakes using the fluid approach was developed[3,5] in which an analogy is drawn between the stationary near wake flow referred to by the body-fixed coordinates and the free expansion flow seen by an observer moving with a constant body speed (V_∞). The initial study[3] was made for the simplest geometrical configuration, namely a two-dimensional wake (x, z) which is related to the one-dimensional free expansion, approximated by the K-dV equation in (x, t) variables with $t=z/V_\infty$. The solution of the initial value problem of the K-dV equation was, however, obtained only for small t which is appropriate only for a partial near wake closest to the base. The extended study of the full near wake which ends where the inward moving plasma streams meet on the wake axis is undertaken herein.

§2. The Wave Model

Consider a flat plate of width L (in x-direction) and of infinite length (in y-direction) moving along its normal (in z-direction) at

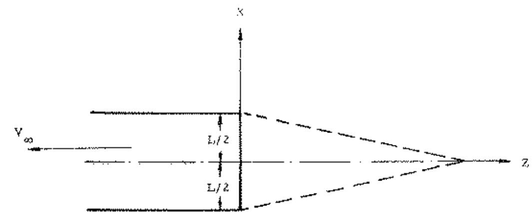

Fig. 1. Near wake model.

speed V_∞ through a fully ionized plasma (Fig. 1). The speed V_∞ is mesothermal, i.e., much greater than the ion thermal speed and yet much less than the electron thermal speed, $(\sqrt{kT_i/m_i} \ll V_\infty \ll \sqrt{kT_e/m_e})$. Immediately behind the plate, a cylindrical region, with a cross section equal to the area of the plate, becomes momentarily ion-free. The nonstationary free expansion of ambient plasma into the cylindrical cavity proceeds until collision occurs between the two plasma fluxes filling the cavity from opposite directions. In the coordinate system fixed to the body, the near wake flow is stationary and can be approximated[5] and resolved into a transverse expansion and a longitudinal drift with velocity V_∞. In other words, the near wake structure (*e.g.*, the ion density $n(x, z)$), in successive cross sections at a different distance (z; downstream from the plate, corresponds to successive stages of nonstationary filling of the cavity by the ambient plasma, $n(x, t)$, where $t=z/V_\infty$. It has been shown[3,6] that the ion wave disturbances moving with Mach number (defined relative to the ion-acoustic speed) slightly greater than unity in a uniform, magnetic field-free and collision-free plasma can be represented in terms of the Korteweg-de Vries equation:[6]

$$\frac{\partial \eta}{\partial t}+\eta\frac{\partial \eta}{\partial x}+\frac{\partial^3 \eta}{\partial x^3}=0, \qquad (1)$$

where η depicts ion density (n), electric potential (ϕ) or ion velocity (v). Note that the simplified from (1) with coefficients of unity on front of each term can always be obtained by rescaling with $x \to xb^{1/3}$ and $\eta \to \eta/ab^{-1/3}$ where a and b denote the coefficients of the convective and dispersive terms, respectively.[6] The quantities x, t, n, ϕ and v have been normalized with Debye length $\lambda_D=(kT_e/4\pi e^2 n_\infty)^{1/2}$, plasma period $t_i=\lambda_D/C_i$, ambient ion density n_∞, electron potential $\phi_e=kT_e/e$ and ion acoustic speed $C_i=(kT_e/m_i)^{1/2}$, respectively.

In the present near wake problem in which the focusing effect of the free stream plasma due to surface potential is neglected, the flow field bounded between the plane and the downstream station where the inward ion waves from the opposite edges of the plate meet can be formulated as the initial value problem of eq. (1) with the following conditions:

$$t=0, \quad \eta=\begin{cases} 0, & \text{for } x<0, \\ 1, & \text{for } x>0, \end{cases} \tag{2}$$

which depicts the initial ion density step at the start of implosion. It is of interest to estimate the time elapsed from the start to the reflection of the ion waves at the wake axis, namely $L/2C_i$, which in dimensionless variables becomes $\Delta t \sim L/2\lambda_D$. For a typical artificial earth satellite in the upper ionosphere, Δt is of the order of 50 which indicates the time range of interest in the solution of the above mentioned initial value problem. A special numerical algorithm which can be considered as an extension of MacCormack's method[7,8] is presented herein (see §5) to treat the problem of interest.

§3. Special Solution at $t \ll 1$

Washimi and Taniuti[9] obtained solution of the linearized K-dV equation, which is appropriate at $t \ll 1$,

$$\frac{\partial \eta}{\partial t}+\frac{\partial^3 \eta}{\partial x^3}=0, \tag{3}$$

with the initial conditions (2) as follows:

$$\eta(x, t)=\int_{x(3t)^{-1/3}}^{\infty} Ai(\xi)\, \mathrm{d}\xi, \tag{4}$$

where $Ai(\xi)$ is the Airy function

$$Ai(\xi)=\frac{1}{\pi}\int_0^{\infty} \cos\left(\alpha\xi+\frac{\alpha^3}{3}\right) \mathrm{d}\alpha. \tag{5}$$

An elementary analysis of the comparative order of magnitude of the terms of the K-dV equation can be made to establish the validity range of the linear solution (4). Let $\Delta t \sim O(\varepsilon)$, Δx must be $O(\varepsilon^{1/3})$ in order that the first and the third terms of eq. (1) have the same order of magnitude. If η is $O(1)$, the order of the nonlinear term in eq. (1) becomes $O(\varepsilon^{-1/3})$ which is smaller than that of the first or the third term. It is thus concluded that the solution of Washimi and Taniuti[9] is a good approximation provided t is small. The results of our numerical integration (see §5) for both $t=0.15$ and $t=0.60$ of eq. (3) are shown in Fig. 2 with dots and crosses, respectively as well as those of eq. (1), with circles and triangles, respectively, along with the solid curve plot of eq. (4). The deviations of the solid curve for $t=0.60$ from the numerical solution of eq. (1) are apparent.

§4. Asymptotic Solution at $t \gg 1$

Gurevich and Pitaevsky[10] obtained the asymptotic solution of the initial value problem

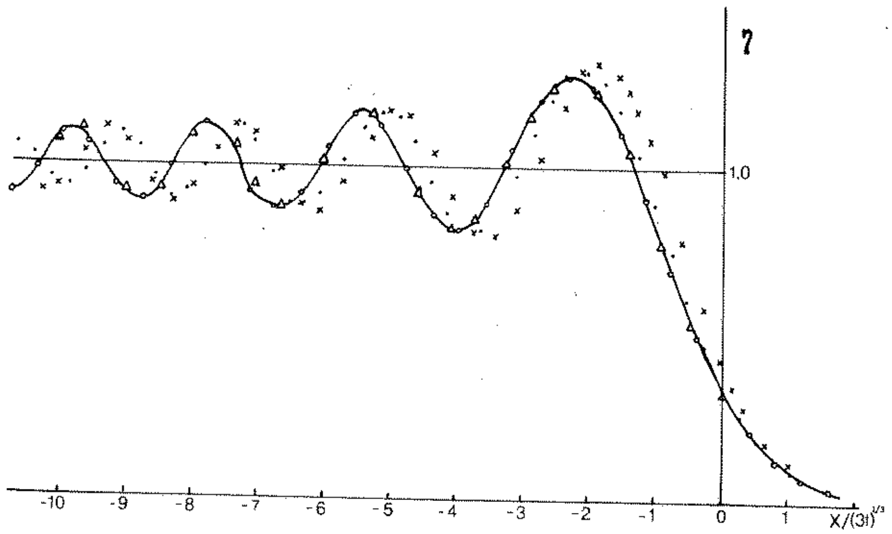

Fig. 2. Solutions of K-dV equation at small t. —— Analytical solution (4), ○ ($t=1.125$), Δ($t=0.600$) numerical solution [eqs. (3) and (2)], • ($t=0.150$), × ($t=0.600$) numerical solution [eqs. (1) and (2)].

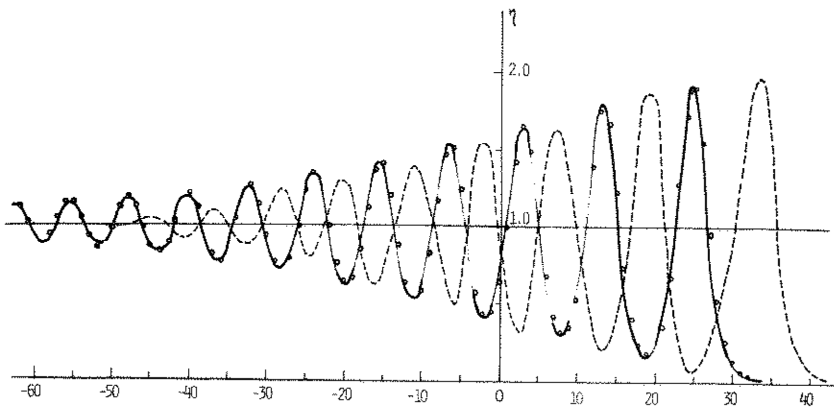

Fig. 3. Solutions of K-dV equation at $t=50$. —— Numerical solution [eqs. (1) and (2)] (mesh sizes: $\Delta x=0.5$, $\Delta t=0.0075$), ○ Numerical solution [eqs. (1) and (2)] (mesh sizes: $\Delta x=0.25$, $\Delta t=0.00375$), ----- Asymptotic solution [eq. (6)].

(1)–(2) for $t \gg 1$ in the form of a quasi-stationary wave:

$$\eta=\frac{2a}{s^2}\,\mathrm{d}n_s^2\left[\left(\frac{a}{6s^2}\right)^{1/2}(x-Vt)\right]+\gamma, \qquad (6)$$

where $\mathrm{d}n_s$ is the elliptic Jacobi delta function with modulus s; a, γ and V are slowly varying functions of x and t through variables s:

$$a=s^2, \quad V=\frac{1}{3}(1+s^2), \quad \gamma=-(1-s^2), \qquad (7)$$

$$\frac{x}{t}=\frac{1+s^2}{3}-\frac{2}{3}\frac{s^2(1-s^2)K(s)}{E(s)-(1-s^2)K(s)}, \qquad (8)$$

with K and E denoting the full elliptic integrals of the first and the second kind, respectively.

The comparison of our numerical solution (see §5) with the asymptotic solution[9] for $t=50$ is shown in Fig. 3 with the former in a solid line and the latter in a dotted line. The apparent variance between them suggests that $t=50$ is not large enough to comply with the asymptotic condition. In the numerical integrations, mesh sizes of both $\Delta x=0.25$ (circles) and $\Delta x=0.50$ (solid line) are used with no noticeable differences in results.

§5. Numerical Solution at Intermediate Values of *t*

It has been shown (see §2) that in the present modeling study of the plasma near wakes using the K-dV equation, the typical range of interest for the t-variable is 0–50. The previous solutions[3,9,10] of the K-dV equation are, therefore, not adequate for the present purpose. The extended MacCormack method[7,8,11] is herein further modified to treat the present initial value problem (1)–(2). The predictor and corrector in the MacCormack method can be written as follows:

$$\eta_j^{\overline{n+1}}=\eta_j^n-\alpha\frac{\Delta t}{\Delta x}\eta_j^n(\eta_{j+1}^n-\eta_j^n)-\beta\frac{\Delta t}{(\Delta t)^3}(\eta_{j+2}^n+3\eta_j^n-3\eta_{j+1}^n-\eta_{j-1}^n)+\mu\frac{\Delta t}{(\Delta x)^2}(\eta_{j+1}^n+\eta_{j-1}^n-2\eta_j^n), \quad (9)$$

$$\eta_j^{\overline{\overline{n+2}}}=\eta_j^{\overline{n+1}}-\alpha\frac{\Delta t}{\Delta x}\eta_j^{\overline{n+1}}(\eta_j^{\overline{n+1}}-\eta_{j-1}^{\overline{n+1}})-\beta\frac{\Delta t}{(\Delta x)^3}(\eta_{j+1}^{\overline{n+1}}+3\eta_{j-1}^{\overline{n+1}}-3\eta_j^{\overline{n+1}}-\eta_{j-2}^{\overline{n+1}})$$
$$+\mu\frac{\Delta t}{(\Delta x)^2}(\eta_{j+1}^{\overline{n+1}}+\eta_{j-1}^{\overline{n+1}}-2\eta_j^{\overline{n+1}}), \qquad (10)$$

and

$$\eta_j^{n+1}=\frac{1}{2}(\eta_j^{\overline{n+1}}+\eta_j^{\overline{\overline{n+2}}}), \qquad (11)$$

respectively.

The effectiveness of the present numerical algorithm using (9), (10) and (11) has been tested in some limited extent in §3 and 4 and also, with slight variance, used in ref. 8 for various flow problems. The numerical solution for $t=1.125$ using spatial mesh sizes: $\Delta x=$ 0.2, 0.25 and 0.5 are shown in Fig. 4. The excellent agreement between the results from different mesh sizes reveals the convergent property of the present numerical approach.

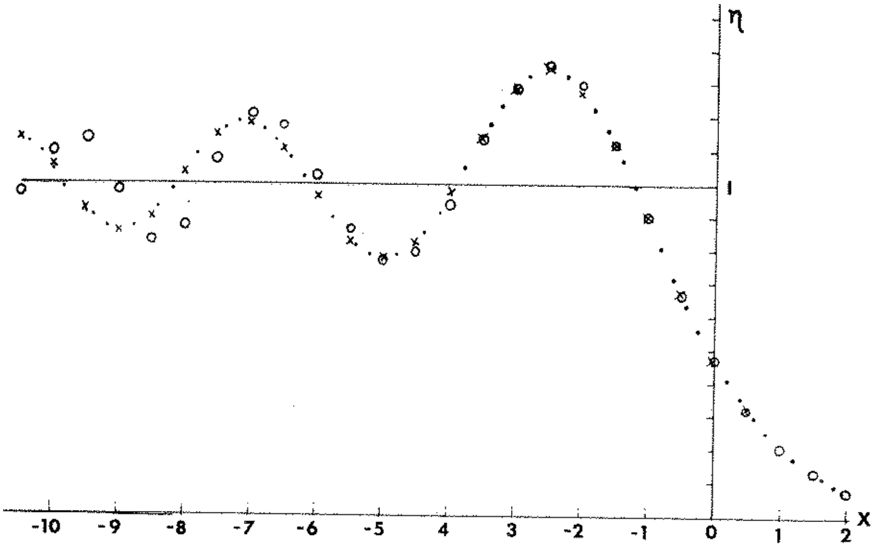

Fig. 4. Numerical solutions of K-dV equation at $t=1.125$ with different mesh sizes. • Solution [eqs. (1) and (2)] ($\Delta x=0.2$, $\Delta t=0.0015$), ○ Solution [eqs. (1) and (2)] ($\Delta x=0.25$, $\Delta t=0.00375$), × Solution [eqs. (1) and (2)] ($\Delta x=0.5$, $\Delta t=0.0075$).

The profile of ion density $n(x, z)$ in the near wake can be identified with $\eta(x, t)$ with t replaced by z/V_∞. The profiles of electrical potential $\phi(x, z)$ and ion velocity in the near wake can be calculated in terms of the ion density using the formulations provided in ref. 6.

It is of interest to reiterate that the main purpose of the present paper is to give a thorough discussion on the conceptual innovation of wave model of the mesothermal near wakes—a fundamental ionospheric aerodynamic problem yet to be elucidated. This, of course, stems from the inherent difficulties of treating the nonlinear coupling of plasma particles and field. It is a fortunate coincidence that the KdV equation is shown to be suitable for the wave analogy of near wakes. Since a solution to KdV equation for an extensive time domain is needed for the present purpose, a numerical algorithm of proven convergency[8] is used. It goes without saying that there are many alternative schemes for the numerical solutions of the KdV equation in the literature from which we would like to cite one[12] that gives rigorous mathematical considerations of stability criteria for the numerical schemes advanced therein.

Acknowledgements

This work was done when V. C. Liu was a Guest Professor at the Institute of Mechanics, The Chinese Academy of Science, Peking. He would like to express his sincere thanks to Professor T. C. Lin and the Fluid Mechanics Faculties at the Institute for their hospitality during his visit.

References

1) V. C. Liu: Space Science Rev. **9** (1969) 423.
2) V. C. Liu: Prog. Aerospace Sci. **16** (1975) 273.
3) V. C. Liu: Geophys. Research Lett. **2** (1975) 485.
4) V. C. Liu: Adv. Appl. Mech. **12** (1972) 195.
5) V. C. Liu and S. J. Ying: AIAA J. **18** (1980) 1268.
6) R. C. Davidson: *Methods in Nonlinear Plasma Theory* (Academic Press, 1972) Chap. 2.
7) R. W. MacCormack: AIAA Paper 69-354 (1969).
8) S. J. Ying and V. C. Liu: Int. J. Computers and Fluids **6** (1978) 173.
9) H. Washimi and T. Taniuti: Phys. Rev. Lett. **17** (1966) 996.
10) A. V. Gurevich and L. P. Pitaevsky: J.E.T.P. Lett. **17** (1973) 268.
11) R. W. MacCormack and H. Lomax: Ann. Rev. Fluid Mech. **11** (1979) 289.
12) K. Goda: J. Phys. Soc. Jpn. **39** (1975) 229.