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Stability of flow of a generalized Newtonian fluid down an inclined plane

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Introduction

The stability of a layer of triply non-linear fluid, power law fluid, and a second order fluid flowing down an inclined plane has been considered by Yih [1], Fan [2], and Gupta [3] respectively. Gupta found that the critical Reynolds number $(\mathrm{Re})_{\mathrm{cr}}$ depend on the elastic properties of the fluid, but the second order fluid had a constant viscosity and hence the effect of variation of apparent on $(\mathrm{Re})_{\mathrm{cr}}$ was not considered. In this paper, the same flow problem is considered for a generalized Newtonian fluid.

Yih's [4] perturbation technique is used in the following analysis.

Differential system governing stability

A layer of a generalized Newtonian fluid of thickness d flows down a plane (Fig. 1) inclined at an angle β to the horizontal. The steady primary flow is taken parallel to the x_1-axis with the x_2-axis normal to the plate downwards, the origin being taken at the undisturbed free surface.

The equation of momentum and continuity are

$$\varrho(\partial u_i/\partial t_1 + u_j\,\partial u_i/\partial x_j) = \partial \tau_{ij}/\partial x_j + \varrho X_i\,, \tag{1}$$

$$\partial u_j/\partial x_j = 0\,, \tag{2}$$

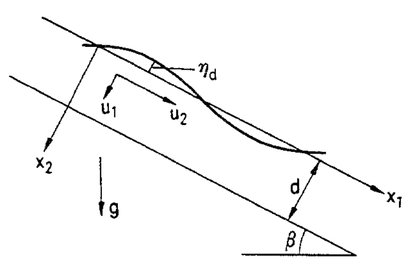

Figure 1
Definition sketch

where ϱ is density, t_1 is time and X_i are the components of force due to gravity.

For a generalized Newtonian fluid, the relationship between the stress tensor τ_{ij} and the rates of deformation $\dot{\gamma}_{ij}$ is

$$\tau_{ij} = -p_1 \delta_{ij} + f(\tfrac{1}{2} \dot{\gamma}_{st} \dot{\gamma}_{ts}) \dot{\gamma}_{ij} \tag{3}$$

where $\dot{\gamma}_{ij} = \partial u_j/\partial x_i + \partial u_i/\partial x_j$, and δ_{ij} is Kronecker delta, p_1 is pressure, f is an arbitrary function with continuous derivatives. (Several examples are given in Fig. 2.)

The primary flow is steady and unidirectional and the velocity depends on x_2 only. Using a bar to denote various quantities for this flow, (1) gives

$$f(\bar{u}'^2)\, \bar{u}' = -\varrho g \sin\beta\, x_2 , \tag{4}$$

$$-\varrho g \cos\beta = \bar{p} . \tag{5}$$

We now superimpose small disturbances on the main flow and write

$$u_1 = \bar{u} + \tilde{u}, \quad u_2 = \tilde{v}, \quad p_1 = \bar{p} + \tilde{p} , \tag{6}$$

where the tilde denotes various perturbed quantities.

Substituting (6) into (3), expanding f in a Taylor series about $\bar{u}'^2$, and neglecting quadratic terms in the perturbed quantities, we obtain

$$\tau_{11} = -p_1 + 2f\, \partial\tilde{u}/\partial x_1 ; \qquad \tau_{22} = -p_1 + 2f\, \partial\tilde{v}/\partial x_2 ;$$

$$\tau_{12} = (f + 2\bar{u}'^2 f')(\partial\tilde{v}/\partial x_1 + \partial\tilde{u}/\partial x_2) + \bar{u}' f , \tag{7}$$

where f, f' are respectively $f(\bar{u}'^2)$, $\mathrm{d}f(\bar{u}'^2)/\mathrm{d}(\bar{u}'^2)$.

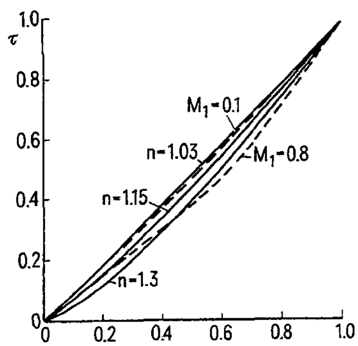

Figure 2
The relationship between the stress τ and the rates of deformation $\dot{\gamma}$ for the power-law fluid and the power series fluid.

Substituting (6), (7) into (1), cancelling out the terms corresponding entirely to the primary flow, and neglecting quadratic terms in the perturbed quantities, we obtain

$$\varrho\left(\frac{\partial\tilde{u}}{\partial t_1}+\bar{u}\,\frac{\partial\tilde{u}}{\partial x_1}+\bar{u}'\,\tilde{v}\right)=-\frac{\partial\tilde{p}}{\partial x_1}-(f-2\bar{u}'^2 f')\frac{\partial^2\tilde{u}}{\partial x_1^2}$$
$$+(f+2\bar{u}'^2 f')\frac{\partial^2\tilde{u}}{\partial x_2^2}+\left(\frac{\partial\tilde{v}}{\partial x_1}+\frac{\partial\tilde{u}}{\partial x_2}\right)\frac{\mathrm{d}}{\mathrm{d}x_2}(f+2\bar{u}'^2 f'), \quad (8)$$

$$\varrho\left(\frac{\partial\tilde{v}}{\partial t_1}+\bar{u}\,\frac{\partial\tilde{v}}{\partial x_1}\right)=-\frac{\partial\tilde{p}}{\partial x_2}-(f-2\bar{u}'^2 f')\,\frac{\partial^2\tilde{u}}{\partial x_1\,\partial x_2}$$
$$+(f+2\bar{u}'^2 f')\frac{\partial^2\tilde{v}}{\partial x_1^2}+\frac{\mathrm{d}f}{\mathrm{d}x_2}\,\frac{\partial\tilde{v}}{\partial x_2}. \quad (9)$$

We introduce the following dimensionless variables

$$x=x_1/d;\qquad y=x_2/d;\qquad t=t_1\,\bar{u}'(d);$$
$$u=\tilde{u}/\bar{u}'(d)\,d;\quad v=\tilde{v}/\bar{u}'(d)\,d;\quad p=\tilde{p}/\varrho[\bar{u}'(d)]^2\,d^2 \quad (10\text{a})$$

and the following dimensionless quantities

$$\mathrm{Re}=\varrho\,\bar{u}'(d)\,d^2/f([\bar{u}'(d)]^2),\qquad U=\bar{u}/\bar{u}'(d)\,d. \quad (10\text{b})$$

By substituting (10a), (10b) into (4), we obtain

$$f=f([\bar{u}'(d)]^2)\,(y/U'),\quad f([\bar{u}'(d)]^2)=(f+2\bar{u}'^2 f')\,U'', \quad (11)$$

where the primes of U', U'' denote differentiation with respect to y. Further, (2) reduces to

$$\partial u/\partial x+\partial v/\partial y=0. \quad (12)$$

We now introduce the stream function ψ and write

$$u=\partial\psi/\partial y;\qquad v=-\,\partial\psi/\partial x \quad (13)$$

and write

$$\psi=\varphi(y)\exp[i\,\alpha(x-c\,t)], \quad (14)$$

where α $(=2\pi\,d/\lambda)$ is perturbation wave number, λ is perturbation wave-length, c $(=c_r+i\,c_i)$ is wave velocity, i is imaginary number.

Substituting (10a), (10b)–(14) into (8), (9), and eliminating p from (8), (9), we obtain finally

$$i\,\alpha\,\mathrm{Re}[(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi]$$
$$=(\varphi''/U'')''+2\alpha^2[(-2y/U'+1/U'')\,\varphi']'+(1/U'')\,\alpha^4\varphi+(1/U'')''\,\alpha^2\varphi. \quad (15)$$

This equation is true for all f and hence is valid for any generalized Newtonian fluid.

The boundary conditions at the bottom of the layer are:

$$u=0; \quad v=0 \quad \text{at} \quad y=1$$

or

$$\varphi'(1)=0, \qquad \varphi(1)=0\,. \tag{16}$$

The boundary conditions on the free surface are more complicated, since they must be applied on the free surface, not merely at $y=0$. Let $\eta\, d$ be the dimensional deviation of the free surface from its mean position, so that

$$\partial\eta/\partial t + U(0)\,\partial\eta/\partial x = v = -\,i\,\alpha\,\varphi(0)\,\exp[i\,\alpha(x-c\,t)]$$

or

$$\eta = [\varphi(0)/\{c-U(0)\}\,\exp[i\,\alpha(x-c\,t)]\,. \tag{17}$$

At the free surface the shear stress must vanish, and the normal stress must balance with the normal stress induced by surface tension, thus we have

$$[(f+2\bar{u}'^2 f')(\partial\tilde{v}/\partial x_1 + \partial\tilde{u}/\partial x_2) + \bar{u}' f]_{y=\eta} = 0 \tag{18a}$$

and

$$-\frac{\bar{p}}{\varrho[\bar{u}'(d)]^2\,d^2} - \tilde{p} + \frac{f}{f([\bar{u}'(d)]^2\,\mathrm{Re}}\,\frac{\partial v}{\partial y} + S\,\frac{\partial^2\eta}{\partial x^2} = 0\,, \tag{18b}$$

where $S = Td/\varrho[\bar{u}'(d)]^2$, T being the surface tension.

Equation (18a) and (18b) may be written in term of φ as

$$\varphi''(\eta) + \alpha^2\,\varphi(\eta) + U''(\eta)\,\varphi(0)/[c-U(0)] = 0 \tag{19a}$$

and

$$\begin{aligned}&\{[\alpha(\cot\beta + \alpha^2\,S\,\mathrm{Re})]\,\varphi(0)/[c-U(0)] - \alpha\,\mathrm{Re}(U-c)\,\varphi' + \alpha\,\mathrm{Re}\,U'\varphi\\ &\quad - i\,[(\varphi''/U'')' + (1/U'')'\,\alpha^2\,\varphi - (4y/U' - 1/U'')\,\alpha^2\,\varphi']\}_{y=\eta} = 0\,.\end{aligned} \tag{19b}$$

Solution for long wave

Because we only consider the case of long waves, it is convenient to use Yih's method [4] and expand the eigenfunction φ and the eigenvalue c in power series of the small parameter α, thus

$$\begin{aligned}\varphi &= \varphi_0 + \alpha\,\varphi_1 + \alpha^2\,\varphi_2 + \dots\\ c &= c_0 + \alpha\,c_1 + \alpha^2\,c_2 + \dots\,.\end{aligned} \tag{20}$$

Substituting the foregoing series into equation (15) and boundary conditions at the free surface (19a), (19b) and collecting terms to the zerothorder in α, we have the following differential equation

$$(\varphi_0''/U'')''=0 \tag{21}$$

and the boundary conditions at the free surface are

$$U''(\eta)+U''(\eta)[U(0)+U'(1)]/[c_0-U(0)]=0$$

or

$$c_0=-U'(1) \tag{22}$$

and

$$[\varphi_0''(\eta)/U'']'=0\,. \tag{23}$$

The final result of equation (21) and the boundary conditions (16), (23) is

$$\varphi_0(y)=U(y)-U'(1)\,y+U'(1)\,, \tag{24}$$

where a multiplicative constant can be chosen to be unity without loss of generality.

The first-order approximation is obtained by collecting terms of order α, which yields the following governing differential equation

$$(\varphi_1''/U'')''=i\,\mathrm{Re}[(U-c_0)\,\varphi_0''-U''\,\varphi_0]\,. \tag{25}$$

Since only terms of first order in α are retained in the differential system, boundary condition (19b) becomes

$$\cos\beta\,\varphi_0(0)/[c_0-U(0)]-\mathrm{Re}\,(U-c_0)\,\varphi_0'+\mathrm{Re}\;U'\varphi+i(\varphi_1''/U'')'=0\,. \tag{26}$$

As to boundary condition (19a) care must be taken that c suffers a change in the second approximation, so that the proper form of (19a) is now

$$\varphi_0''(\eta_1)+\alpha\,\varphi_1''(\eta_1)+U''(\eta_1)[\varphi_0(0)+\alpha\,\varphi_1(0)]/[c_0-U(0)+\alpha\,c_1]=0\,. \tag{27}$$

For a first order approximation $\eta_1 \ll 1$ the terms of order η_1 are negligible, and so are the terms of order α^2 in boundary condition (27). The final result of equation (25) with its boundary conditions (26), (27), (16) is

$$c_1=i\,\{\cot\beta-\mathrm{Re}\,U'(1)[U'(1)+U(0)]\}\left[U'(1)+2\int_0^1 U\,\mathrm{d}y\right] + i\,\mathrm{Re}[G'(1)-G(1)]\,, \tag{28}$$

where

$$G(y)=U'(1)\int_0^y\int_0^y\left[U''\int_0^y\int_0^y y\,U''\,\mathrm{d}y\,\mathrm{d}y\right]\mathrm{d}y\,\mathrm{d}y\,.$$

The critical Reynolds number is

$$(\mathrm{Re})_{\mathrm{cr}}=\frac{\left[1+2\int_0^1 U\,\mathrm{d}y/U'(1)\right]\cot\beta}{[U'(1)]^2+4\,U'(1)\int_0^1 U\,\mathrm{d}y+2\int_0^1 U^2\,\mathrm{d}y+\int_0^1 U'^2\,y\,\mathrm{d}y-\int_0^1\int_0^y U'^2\,y\,\mathrm{d}y\,\mathrm{d}y}\,. \tag{29}$$

If $\mathrm{Re} > (\mathrm{Re})_{\mathrm{cr}}$, c_i is positive, and the flow is unstable. If $\mathrm{Re} < (\mathrm{Re})_{\mathrm{cr}}$, c_i is negative, and the flow is stable.

Calculation and disnussion

In this paper, two kinds of generalized Newtonian fluid are considered.

1. Power series

f is given by $f = \mu + \mu_3 \dot{\gamma}^2 + \mu_5 \dot{\gamma}^4 + \mu_7 \dot{\gamma}^6$.
The dimensionless velocity gradient is

$$U' + M_1 (U')^3 + M_2 (U')^5 + M_3 (U')^7 = -y\,,$$

where

$$M_1 = \mu_3 (\varrho g \sin\beta)^2 d^2/\mu^3;$$
$$M_2 = \mu_5 (\varrho g \sin\beta)^4 d^4/\mu^5;$$
$$M_3 = \mu_7 (\varrho g \sin\beta)^6 d^6/\mu^7\,.$$

$U(y)$ has not, in general, analytical expression, numerical computation is needed.

For $|M_1| < 0.2$; $M_2 = M_3 = 0$, we obtain an approximate value as follows

$$(\mathrm{Re})_{\mathrm{cr}} = (-1 + M_1 - 3M_1^2)^2 (5/2 + 46 M_1/7 - 2399 M_1^2/294) \cot\beta\,.$$

The result agrees with that of Yih [1].

Numerical computation has been carried out. The critical values of $(\mathrm{Re})_{\mathrm{cr}}/\cot\beta$ for various values of M_1, M_2, M_3 are given in Table 1 and illustrated in Fig. 3.

Fig. 2 shows that constitutive equation of power law fluid of $n = 1-1.3$ bear similarity to that of power series fluid of $M_1 = 0-0.8$; $M_2 = M_3 = 0$. It

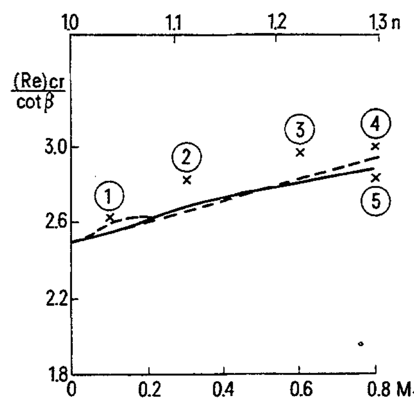

Figure 3
The critical Reynolds number $(\mathrm{Re})_{\mathrm{cr}}/\cot\beta$ as e function of dimensionless parameter M.
M_1, M_2, M_3 are respectively ① 0.1, 0.05, 0.05; ② 0.3, 0.3, 0; ③ 0.6, 0.6, 0; ④ 0.8, 0.4, 0.2; ⑤ 0.8, −0.2, 0.
---- values of power-law fluid,
- - - - - approximate values of Yih,
——— various values of M_1, with $M_2 = M_3 = 0$.

Table 1
Values of $(\mathrm{Re})_{cr}/\cot\beta$ for various values of M_1, M_2, M_3.

M_1	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.10	0.30	0.60	0.80	0.80
M_2	0	0	0	0	0	0	0	0	0.05	0.30	0.60	0.40	−0.20
M_3	0	0	0	0	0	0	0	0	0.05	0	0	0.20	0
$\frac{(\mathrm{Re})_{cr}}{\cot\beta}$	2.55	2.63	2.69	2.75	2.79	2.83	2.86	2.90	2.63	2.83	2.98	3.00	2.84

can also be seen from Fig. 3 that there exists a similarity between the stability characteristic of power law fluid of $n=1-1.3$ and that of power series fluid of $M_1=0-0.8$; $M_2=M_3=0$.

2. Solution and calculation for viscoplastic fluid

We write the constitutive equation of a viscoplastic fluid as

$$\tau_{ij}=-p\,\delta_{ij}+\{m\,|(\tfrac{1}{2}\,\dot\gamma_{st}\,\dot\gamma_{ts})^{1/2}|^{(n-1)/2}+\tau_0/|\tfrac{1}{2}\,\dot\gamma_{st}\,\dot\gamma_{ts}|^{1/2}\}\,\dot\gamma_{ij}$$
$$\text{for}\quad \tfrac{1}{2}\,\tau_{st}\,\tau_{ts}\geqq\tau_0^2\,, \tag{31 a}$$

$$\dot\gamma_{ij}=0 \qquad \text{for}\quad \tfrac{1}{2}\,\tau_{st}\,\tau_{ts}\leqq\tau_0^2\,. \tag{31 b}$$

where m is modulus index, n is power law index, τ_0 is the yield stress. Equation (31) has been found to describe adequately the rheological behavior of emulsion used in film coating and thus the present work might be of interest to the film industry.

As is usual in solving viscoplastic flow problems we need to consider two layers one where (31 a) is applicable and the other where (31 b) is applicable. The interface is where $\frac{1}{2}\,\tau_{st}\,\tau_{ts}=\tau_0^2$.

On non-dimensionalising all quantities, as above, we define the Reynolds number to be $\mathrm{Re}=[\varrho g\sin\beta d(1-\Theta)/m]^{2/n}d/g\sin\beta$, where $\Theta=\tau_0/\varrho g\sin\beta d$. The steady velocity distribution is

$$U(y)=\frac{n}{1+n}\,[1-\Theta-(y-\Theta)^{1/n+1}/(1-\Theta)^{1/n}] \quad \text{at} \quad y\geqq\Theta\,,$$

$$U(y)=\frac{n}{1+n}\,(1-\Theta) \qquad \text{at} \quad y\leqq\Theta\,.$$

Superimposing small disturbances of the form given above, using (31 b) we find that the interface is given by $y=\Theta+\eta$, where η is defined by kinematic condition to be $\eta=[\varphi(\Theta)/\{c-U(\Theta)\}]\exp[i\,\alpha(x-c\,t)]$.

At the interface we assume:

(a) The total velocity components must be continuous;

(b) The stresses must be continuous.

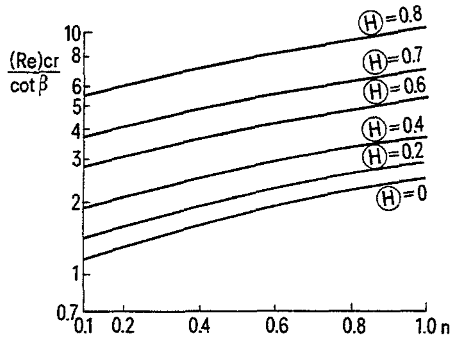

Figure 4
The critical Reynolds number $(\mathrm{Re})_{cr}/\cot\beta$ as a function of power law index n for various values of dimensionless yield stress Θ.

Using foregoing interface condition and bottom condition (16), proceeding as in previous section we obtain the critical Reynolds number to be

$$(\mathrm{Re})_{cr} = (1+n)(3n+2)(1+n+n\Theta)/(1-\Theta)[2(1+n)^2+(4n+3)\,n\Theta]\,.$$

Fig. 4 shows the critical Reynolds number as a function of power law index n for various values of dimensionless yield stress Θ. It can be seen from Fig. 4 that $(\mathrm{Re})_{cr}/\cot\beta$ increases (more stable) with increasing Θ for any fixed n. Thus the effect of plasticity is to stabilize the flow.

The analysis of this paper is restricted to the case of two dimensional long wave (wave number $\alpha \ll 1$), and small perturbation amplitude ($\eta \ll 1$).

Acknowledgement

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Reference

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Summary

The analogue of Orr-Sommerfeld equation is derived for a generalized Newtonian fluid. Based on this equation, the stability of such fluid flowing down an inclined plane under gravity is studied. The critical Reynolds number is given as a function of dimensionless steady flow velocity $U(y)$ and the slope of the plane, and is computed for several fluids.

Résumé

On a étudié le problème de stabilité de l'écoulement d'un fluide Newtonien généralisé sur un plan incliné.

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