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Strain gradient theory based on a new framework of non-local model

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Abstract A new framework of non-local model for the strain energy density is proposed in this paper. The global strain energy density of the representative volume element is treated as a non-local variable and can be obtained through a special integral of the local strain energy density. The local strain energy density is assumed to be dependent on both the strain and the rotation-gradient. As a result of the non-local model, a new strain gradient theory is derived directly, in which the first and second strain gradients, as well as the triadic and tetradic stress, are introduced in the context of work conjugate. For power law hardening materials, size effects in thin metallic wire torsion and ultra-thin cantilever beam bend are investigated. It is found that the result predicted by the theoretical model is well consistent with the experimental data for the thin wire torsion. On the other hand, the calculation result for the micro-cantilever beam bend clearly shows the size effect.

1 Introduction

The classical elasticity and plasticity theories follow the local assumption that the stress at a given material point is determined only by the strain, the history of deformation, and temperature at that point. The first gradient of the displacement field and Cauchy stress tensor are treated as the constitutive variables. However, in practice, both the nature materials and man-made materials have complicated internal structures whose sizes range over many orders of magnitude. When the macroscopic characteristic length of a material is much larger than the internal characteristic length, the material can be treated as a simple material and the classical theories are accurate enough to describe the mechanical behaviors of simple materials. Otherwise, the classical theories need to be enriched so as to capture the real processes more adequately and it is often more effective to use various forms of generalized continuum formulations (non-local theories), to deal with materials that are non-simple or polar, or both.

Many attempts on this subject have been done, which can be classified into three families according to Bazant and Jirasek [1]. The first one is the generalized Cosserat theories proposed first by Cosserat and Cosserat [2], which were developed by many researchers into couple-stress theory, theory of elasticity with microstructure, micropolar theory [3–6]. The second one is the non-local theories of the gradient type, in which the displacement field is the only independent kinematic field, but the gradients of strain are incorporated into

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the constitutive equations [7–12]. The third one consists of nonlocal theories of the integral type, also called strongly nonlocal theories [13–15]. The common characteristic of these theories, which does not exist in the classic theories, is that at least an intrinsic length is engaged in the constitutive relation.

In recent years, many experiments have shown that materials display strong size effects when the characteristic length scale associated with non-uniform plastic deformation is in the order of microns. For an aluminum-silicon matrix reinforced by silicon carbide particles, Lloyd [16] observed a substantial strength increase when the particle diameter was reduced from 16 to 7.5 microns with the particle volume fraction fixed at 15%. In the experiments of measuring micro-indentation hardness of metallic materials, the square of hardness increased linearly as the depth of indentation decreased [17–20]. In the experiment of thin copper torsion, Fleck et al. [21] observed that the scaled shear strength increased by a factor of 3 as the thin copper wire diameter decreased from 170 to 12 microns, while the increase of work-hardening in simple tension was negligible. In the experiment of ultra thin beam bend [22], a significant increase in the normalized bend hardening was observed as the beam thickness decreased from 50 to 12.5 microns, while data from simple tension displayed no size dependence.

The classical elasticity and plasticity theories cannot predict this size dependence of material behavior at the micron scale because their constitutive models possess no internal length scale.

In order to explain the size effects, developing strain gradient theories for micron level, which are classified into the non-local theories of the gradient type, is needed. Fleck and Hutchinson [7] and Fleck et al. [21] developed a phenomenological theory of strain gradient plasticity and a material length scale was introduced for dimensional grounds. From these theoretical developments and consequent attempts at explaining experimental findings of indentation [18, 19, 23] and fracture [24, 25], it has been found necessary to introduce two length parameters [8]. One length, l_R , refers to rotational gradients. The other, l_S , refers to stretch gradients. The latter is needed to rationalize length scale phenomena found in indentation and fracture. In 1998, Nix and Gao [17] started from the Taylor relation and gave out one kind of hardening law for gradient plasticity. Motivated by the indentation hardening law, Gao et al. [9] and Huang et al. [26] proposed a mechanism-based theory of strain gradient plasticity (MSG) based on a multiscale framework linking the microscale notion of statistically stored and geometrically necessary dislocations to the mesoscale notion of plastic strain and strain gradient.

The non-local elastic theories of the integral type were proposed first by Rogula [27], and were refined by many researchers [13, 28, 29] in the early 1970s. Those theories were developed in order to describe mechanical behaviors of materials in the microscopic scales which can be comparable to the size of the lattice parameter. For example, the non-local elastic theories were applied to explain the propagation of high-frequency waves and the deviation of short waves in micro-polar media [30].

The non-local plastic theories of the integral type were proposed first by Eringen [14, 31]. He derived non-local formulations of plasticity deformation theory, plasticity flow theory, and strain-space plasticity theory. Eringen's theories were proposed to describe interacting dislocations and non-local effects on the distribution of stress around the crack tip. The first non-local formulation of softening plasticity was proposed by Bazant and Lin [32]. It was applied in finite element analysis of the stability of unlined excavation of a subway tunnel in a grouted soil. Through introducing higher order derivatives, such as strain gradient, in constitutive relations, some non-local plastic theories of the gradient type were derived and advanced by Aifantis [33], Muhlhaus and Aifantis [34], de Borst et al. [35], and Marotti de Sciarra [36].

The non-local view is also engaged in damage theories. In order to achieve objective and properly convergent numerical solutions for localized damage, Bazant [37] and Bazant and Chang [38] proposed formulations of a non-local strain-softening theory. The non-local formulation of Mazars' model was advanced by di Prisco and Mazars [39]. Kennedy and Nahan [40] proposed a non-local anisotropic damage model. Fish and Yu [41] derived a non-local damage model for composites.

As mentioned above, strain gradient theories belong to the family of non-local theories. According to the classification in Bazant and Jirasek [1], the non-local theories can be categorized into strongly non-local theories and weakly non-local theories. In the weakly non-local theories, the strain gradient and intrinsic length are introduced in the constitutive relations, such as the theories proposed by Fleck and Hutchinson [7, 8]. Generally speaking, the strongly non-local integral approach consists in replacing a certain variable by its non-local counterpart obtained by weighted averaging over a spatial neighborhood of each point under consideration. If $f(\xi)$ is some local field in a solid body occupying a domain V , the corresponding nonlocal field is defined by

$$\bar{f}(\mathbf{x}) = \int_V \alpha(\mathbf{x}, \xi) f(\xi) d\xi, \quad (1)$$

where $\alpha(\mathbf{x}, \xi)$ is a weight function.

In many non-local theories, only stress-strain relations take the non-local form, while the equilibrium and kinematic equations and the corresponding boundary conditions retain their standard form. For an example, the constitutive equation of the non-local elasticity can be written in the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V \mathbf{D}_e(\mathbf{x}, \boldsymbol{\xi}) \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \mathbf{D}_e \int_V \alpha(\mathbf{x}, \boldsymbol{\xi}) \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad (2)$$

where $\tilde{\boldsymbol{\varepsilon}}$ is the local strain tensor, $\mathbf{D}_e$ is the elastic stiffness.

According to Muhlhaus and Aifantis [34], the local strain $\tilde{\boldsymbol{\varepsilon}}(\mathbf{x} + \boldsymbol{\xi})$ can be expanded into a Taylor series about $\boldsymbol{\xi} = 0$. We have

$$\tilde{\varepsilon}_{ij} = \varepsilon_{ij} + \varepsilon_{ij,k} \xi_k + \frac{1}{2} \varepsilon_{ij,kl} \xi_k \xi_l. \quad (3)$$

In view of the assumed isotropy, the weight function depends only on the distance between points $\mathbf{x}$ and $\mathbf{x} + \boldsymbol{\xi}$ [1], and we can write

$$\alpha(\mathbf{x}, \boldsymbol{\xi}) = \alpha(\|\boldsymbol{\xi}\|). \quad (4)$$

Equation (2) can be written as

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V \mathbf{D}_e(\mathbf{x}, \boldsymbol{\xi}) \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \mathbf{D}_e \int_V \alpha(\mathbf{x}, \boldsymbol{\xi}) \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \int_V \alpha(\|\boldsymbol{\xi}\|) \tilde{\boldsymbol{\sigma}}(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad (5)$$

where $\tilde{\boldsymbol{\sigma}}(\boldsymbol{\xi})$ is the local stress tensor.

The non-local model of this type (as shown in Eq. (2) for the nonlocal elasticity) does not provide any formula for the high-order stress tensor, which is the work conjugate tensor of the gradient of strain tensor. This paper tries to construct a bridge between the nonlocal model of the integral type and the strain gradient theory. Our attention is focused on finding a direct way to derive a new strain gradient theory from an energy non-local framework.

In the present paper, we propose a new non-local framework in which both the stress and the high-order stress tensors are introduced and the first and second gradients of the strain tensors are taken into account in the constitutive equations. In the non-local framework, the global strain energy density of the representative volume element is taken as a non-local variable (shown in Fig. 1), i.e.,

$$w = \frac{1}{V_c} \int_{V_c} \tilde{w}(\mathbf{x}, \boldsymbol{\xi}) dV_c, \quad (6)$$

where V_c is the volume of the representative cubic volume element with each boundary length l_c , $\boldsymbol{\xi}$ is the local coordinate as shown in Fig. 1, with the original point at the center of cubic volume. $\mathbf{x}$ is the global coordinate, w is the global strain energy density and $\tilde{w}(\mathbf{x}, \boldsymbol{\xi})$ is the local one.

2 The new framework of non-local theory

2.1 Constitutive equation of the new non-local framework

Equation (6) shows clearly that the energy density w is a non-local variable. We assume the local strain energy density $\tilde{w}$ is a function of the local strain $\tilde{\varepsilon}_{ij}$ and the local rotation gradient $\tilde{\chi}_{ij}$. The first variation of the global strain energy density w can be written as

$$\delta w = \frac{1}{V_c} \int_{V_c} \delta \tilde{w}(\mathbf{x}, \boldsymbol{\xi}) dV_c = \frac{1}{V_c} \int_{V_c} (\tilde{\sigma}_{ij}(\mathbf{x}, \boldsymbol{\xi}) \delta \tilde{\varepsilon}_{ij}(\mathbf{x}, \boldsymbol{\xi}) + \tilde{m}_{ij}(\mathbf{x}, \boldsymbol{\xi}) \delta \tilde{\chi}_{ij}(\mathbf{x}, \boldsymbol{\xi})) dV_c, \quad (7)$$

where $\tilde{\varepsilon}_{ij}$ is the local strain, $\tilde{\chi}_{ij}$ is the local rotation gradient, $\tilde{\sigma}_{ij}$ is the local stress tensor, and $\tilde{m}_{ji}$ is the local couple stress.

Fig. 1 The representative volume element

For homogeneous materials, it is reasonable to take the value of l_c in the range of sub-micrometers to micrometers. For a proper value of l_c , we have

$$\tilde{\varepsilon}_{ij} = \varepsilon_{ij} + \varepsilon_{ij,k} \xi_k + \frac{1}{2} \varepsilon_{ij,kl} \xi_k \xi_l \quad \tilde{\chi}_{pj} = \chi_{pj} + \chi_{pj,l} \xi_l = e_{pki} \varepsilon_{ij,k} + e_{pki} \varepsilon_{ij,kl} \xi_l, \quad (8)$$

where ε_{ij} , $\varepsilon_{ij,k}$ and $\varepsilon_{ij,kl}$ are the strain, the first gradient, and the second strain gradient, respectively. Substituting Eq. (8) into Eq. (7) yields

$$\begin{aligned} \delta w &= \frac{1}{V_c} \int_{V_c} \tilde{\sigma}_{ij}(\mathbf{x}, \boldsymbol{\xi}) \delta \varepsilon_{ij}(\mathbf{x}) dV_c + \frac{1}{V_c} \int_{V_c} \tilde{\sigma}_{ij}(\mathbf{x}, \boldsymbol{\xi}) \delta \varepsilon_{ij,k}(\mathbf{x}) \xi_k dV_c + \frac{1}{2V_c} \int_{V_c} \tilde{\sigma}_{ij}(\mathbf{x}, \boldsymbol{\xi}) \delta \varepsilon_{ij,kl}(\mathbf{x}) \xi_k \xi_l dV_c \\ &\quad + \frac{1}{V_c} \int_{V_c} \tilde{m}_{pj}(\mathbf{x}, \boldsymbol{\xi}) e_{pki} \delta \varepsilon_{ij,k}(\mathbf{x}) dV_c + \frac{1}{V_c} \int_{V_c} \tilde{m}_{pj}(\mathbf{x}, \boldsymbol{\xi}) e_{pki} \delta \varepsilon_{ij,kl}(\mathbf{x}) \xi_l dV_c \\ &= \sigma_{ij} \delta \varepsilon_{ij} + \hat{\tau}_{ijk} \delta \varepsilon_{ij,k} + \hat{t}_{lkji} \delta \varepsilon_{ij,kl}, \end{aligned} \quad (9)$$

where σ_{ij} , $\hat{\tau}_{ijk}$ and $\hat{t}_{lkji}$ are the stress, triadic stress, and tetradic stress, respectively, and

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}, \quad \hat{\tau}_{ijk} = \frac{\partial w}{\partial \varepsilon_{ij,k}}, \quad \hat{t}_{lkji} = \frac{\partial w}{\partial \varepsilon_{ij,kl}}, \quad (10)$$

$\hat{\tau}_{ijk}$ consists of a symmetric part τ_{ijk} and an asymmetric one τ_{ijk}^A , and

$$\tau_{ijk} = \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,k}} + \frac{\partial w}{\partial \varepsilon_{ji,k}} \right) \quad \tau_{ijk}^A = \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,k}} - \frac{\partial w}{\partial \varepsilon_{ji,k}} \right), \quad (11)$$

$\hat{t}_{lkji}$ consists of a symmetric part t_{lkji} and an asymmetric one t_{lkji}^A , and

$$t_{lkji} = \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,kl}} + \frac{\partial w}{\partial \varepsilon_{ji,kl}} \right) \quad t_{lkji}^A = \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,kl}} - \frac{\partial w}{\partial \varepsilon_{ji,kl}} \right). \quad (12)$$

Noting $\tau_{ijk}^A \delta \varepsilon_{ij,k} = 0$ and $t_{lkji}^A \delta \varepsilon_{ij,kl} = 0$, Eq. (9) can be written as

$$\delta w = \sigma_{ij} \delta \varepsilon_{ij} + \hat{\tau}_{ijk} \delta \varepsilon_{ij,k} + \hat{t}_{lkji} \delta \varepsilon_{ij,kl} = \sigma_{ij} \delta \varepsilon_{ij} + \tau_{ijk} \delta \varepsilon_{ij,k} + t_{lkji} \delta \varepsilon_{ij,kl}. \quad (13)$$

Then, the stress σ_{ij} , triadic stress τ_{ijk} , and tetradic stress t_{lkji} can be expressed as

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} = \frac{1}{V_c} \int_{V_c} \tilde{\sigma}_{ij}(\mathbf{x}, \boldsymbol{\xi}) dV_c, \quad (14)$$

$$\begin{aligned} \tau_{ijk} &= \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,k}} + \frac{\partial w}{\partial \varepsilon_{ji,k}} \right) \\ &= \frac{1}{V_c} \int_{V_c} \left\{ \tilde{\sigma}_{ij}(\mathbf{x}, \boldsymbol{\xi}) \xi_k + \frac{1}{2} [\tilde{m}_{pj}(\mathbf{x}, \boldsymbol{\xi}) e_{pki} + \tilde{m}_{pi}(\mathbf{x}, \boldsymbol{\xi}) e_{pkj}] \right\} dV_c, \end{aligned} \quad (15)$$

$$\begin{aligned} t_{lkji} &= \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,kl}} + \frac{\partial w}{\partial \varepsilon_{ji,kl}} \right) \\ &= \frac{1}{2V_c} \int_{V_c} \left\{ \tilde{\sigma}_{ij}(\mathbf{x}, \boldsymbol{\xi}) \xi_k \xi_l + [\tilde{m}_{pj}(\mathbf{x}, \boldsymbol{\xi}) e_{pki} + \tilde{m}_{pi}(\mathbf{x}, \boldsymbol{\xi}) e_{pkj}] \xi_l \right\} dV_c. \end{aligned} \quad (16)$$

Equations (14–16) provide a set of integral type constitutive equations, which are similar to the Eq. (2) (or Eq. (5)). Equation (14) is essentially the same as the Eq. (5). The local stress and the local couple stress appear in the constitutive equations (14–16). We should emphasize that the stress σ_{ij} , triadic stress τ_{ijk} , and tetradic stress t_{lkji} are also the non-local variables.

2.2 Equilibrium equations and boundary conditions

Similar to Mindlin [42], the equilibrium equations and boundary conditions can be derived. Total potential energy Π can be written as

$$\Pi = \int_V w dV - \int_V f_i u_i dV - \int_S \bar{p}_i u_i dS - \int_S \bar{r}_i D u_i dS - \int_S \bar{q}_i D^2 u_i dS, \quad (17)$$

where V is the volume, S is the boundary of V , f_i is the force per unit volume, u_i is the displacement, D is defined in the following Eq. (21), $\bar{p}_i$, $\bar{r}_i$ and $\bar{q}_i$ are generalized surface tractions.

The variation of the total potential energy is

$$\begin{aligned} \delta \Pi &= \int_V \delta w dV - \int_V f_i \delta u_i dV - \int_S \bar{p}_i \delta u_i dS - \int_S \bar{r}_i D \delta u_i dS - \int_S \bar{q}_i D^2 \delta u_i dS \\ &= \int_V (\sigma_{ij} \delta \varepsilon_{ij} + \tau_{ijk} \delta \varepsilon_{ij,k}) dV + \int_V t_{lkji} \delta \varepsilon_{ij,kl} dV - \int_V f_i \delta u_i dV - \int_S \bar{p}_i \delta u_i dS \\ &\quad - \int_S \bar{r}_i D \delta u_i dS - \int_S \bar{q}_i D^2 \delta u_i dS. \end{aligned} \quad (18)$$

According to divergence theorem, Eq. (18) can be written as

$$\begin{aligned} \delta \Pi &= - \int_V (\sigma_{ij} - \tau_{ijk,k})_{,j} \delta u_i dV + \int_S (\sigma_{ij} - \tau_{ijk,k}) n_j \delta u_i dS + \int_S \tau_{ijk} n_k \delta \varepsilon_{ij} dS + \int_V t_{lkji} \delta \varepsilon_{ij,kl} dV \\ &\quad - \int_V f_i \delta u_i dV - \int_{S_\sigma} \bar{p}_i \delta u_i dS - \int_{S_\tau} \bar{r}_i D \delta u_i dS - \int_S \bar{q}_i D^2 \delta u_i dS. \end{aligned} \quad (19)$$

The spatial gradient operator $\nabla = \{\partial_i\}$ can be decomposed at the boundary surface S into its tangential and normal components, that is

$$\begin{aligned}\nabla &= \bar{\nabla} + nD, \\ \partial_i &= \bar{\partial}_i + n_i D,\end{aligned}\tag{20}$$

$$\begin{aligned}\bar{\nabla} &= (I - nn) \cdot \nabla \quad D = n \cdot \nabla, \\ \bar{\partial}_i &= (\delta_{ij} - n_i n_j) \partial_j \quad D = n_m \partial_m,\end{aligned}\tag{21}$$

where I is the unit dyadic, n is the unit outward normal to S .

Thus, we have

$$\begin{aligned}\Phi : \nabla u &= \Phi : \bar{\nabla} u + n \cdot \Phi \cdot Du, \\ \phi_{ij} \partial_i u_j &= \phi_{ij} \bar{\partial}_i u_j + n_i \phi_{ij} Du_j,\end{aligned}\tag{22}$$

where Φ is a dyadic.

Furthermore, Eq. (22) can be written as

$$\begin{aligned}\Phi : \nabla u &= \bar{\nabla} \cdot (\Phi \cdot u) - (\bar{\nabla} \cdot \Phi) \cdot u + n \cdot \Phi \cdot Du, \\ \phi_{ij} \partial_i u_j &= \bar{\partial}_i (\phi_{ij} u_j) - (\bar{\partial}_i \phi_{ij}) u_j + n_i \phi_{ij} Du_j.\end{aligned}\tag{23}$$

Using the surface divergence theorem [43] for a smooth and closed surface, the integrals

$$\begin{aligned}\int_S \bar{\nabla} \cdot (\Phi \cdot u) dS &= \int_S (\bar{\nabla} \cdot n) n \cdot \Phi \cdot u dS, \\ \int_S \bar{\partial}_i (\phi_{ij} u_j) dS &= \int_S (\bar{\partial}_k n_k) n_i \phi_{ij} u_j dS,\end{aligned}\tag{24}$$

lead to

$$\begin{aligned}\int_S \Phi : \nabla u dS &= \int_S (\bar{\nabla} \cdot n) n \cdot \Phi \cdot u dS - \int_S (\bar{\nabla} \cdot \Phi) \cdot u dS + \int_S n \cdot \Phi \cdot Du dS, \\ \int_S \phi_{ij} \partial_i u_j dS &= \int_S (\bar{\partial}_p n_p) n_i \phi_{ij} u_j dS - \int_S (\bar{\partial}_i \phi_{ij}) u_j dS + \int_S n_i \phi_{ij} Du_j dS.\end{aligned}\tag{25}$$

If $\phi_{ij} = \tau_{ijk} n_k$, the third integral in the right hand side of Eq. (19) can be written as

$$\begin{aligned}\int_S \tau_{ijk} n_k \delta \varepsilon_{ij} dS &= \int_S \tau_{ijk} n_k \partial_i \delta u_j dS \\ &= \int_S (\bar{\partial}_p n_p) n_i \tau_{ijk} n_k \delta u_j dS - \int_S (\bar{\partial}_i \tau_{ijk} n_k) \delta u_j dS + \int_S n_i \tau_{ijk} n_k D \delta u_j dS.\end{aligned}\tag{26}$$

According to the divergence theorem, the forth integral in the right hand side of Eq. (19) can be written as

$$\int_V t_{lkji} \delta \varepsilon_{ij,kl} dV = \int_V \left((t_{lkji} \delta \varepsilon_{ij,k})_{,l} - t_{lkji,l} \delta \varepsilon_{ij,k} \right) dV = \int_S t_{lkji} n_l \delta \varepsilon_{ij,k} dS - \int_V t_{lkji,l} \delta \varepsilon_{ij,k} dV.\tag{27}$$

Similar to

$$\begin{aligned}\int_V \tau_{ijk} \delta \varepsilon_{ij,k} dV &= \int_V (\tau_{ijk,k})_{,j} \delta u_i dV + \int_S ((-\tau_{ijk,k} + (\bar{\partial}_p n_p) \tau_{ijk} n_k) n_j - \bar{\partial}_j \tau_{ijk} n_k) \delta u_i dS \\ &\quad + \int_S n_j n_k \tau_{ijk} \delta Du_i dS,\end{aligned}\tag{28}$$

the second integral in the right hand side of Eq. (27) can be written as

$$\begin{aligned} - \int_V t_{lkji,l} \delta \varepsilon_{ij,k} dV &= \int_V (-t_{lkji,lk})_{,j} \delta u_i dV + \int_S ((t_{lkji,lk} - (\bar{\partial}_p n_p) t_{lkji,l} n_k) n_j + \bar{\partial}_j t_{lkji,l} n_k) \delta u_i dS \\ &\quad - \int_S n_j n_k t_{lkji,l} \delta D u_i dS. \end{aligned} \quad (29)$$

According to Mindlin [42], if Ψ is a triadic

$$\begin{aligned} \int_S \Psi : \nabla \Phi dS &= \int_S (\mathbf{L} \cdot \Psi) : \Phi dS + \int_S \mathbf{n} \cdot \Psi : D \Phi dS, \\ \int_S \psi_{ijk} \partial_i \phi_{jk} dS &= \int_S (n_i \bar{\partial}_p n_p - \bar{\partial}_i) \psi_{ijk} \phi_{jk} dS + \int_S n_i \psi_{ijk} D \phi_{jk} dS, \end{aligned} \quad (30)$$

where

$$\begin{aligned} \mathbf{L} &= \mathbf{n} (\bar{\nabla} \cdot \mathbf{n}) - \bar{\nabla}, \\ l_i &= n_i \bar{\partial}_p n_p - \bar{\partial}_i. \end{aligned} \quad (31)$$

If $\Phi = \nabla \delta u$ then Eq. (30) can be written as

$$\begin{aligned} \int_S \Psi : \nabla \nabla \delta u dS &= \int_S (\mathbf{L} \cdot \Psi) : \nabla \delta u dS + \int_S \mathbf{n} \cdot \Psi : D \nabla \delta u dS, \\ \int_S \psi_{ijk} \partial_i \partial_j \delta u_k dS &= \int_S (n_i \bar{\partial}_p n_p - \bar{\partial}_i) \psi_{ijk} \partial_j \delta u_k dS + \int_S n_i \psi_{ijk} D \partial_j \delta u_k dS. \end{aligned} \quad (32)$$

Similar to Eq. (25), the first integral in the right hand side of Eq. (32) can be written as

$$\begin{aligned} \int_S (\mathbf{L} \cdot \Psi) : \nabla \delta u dS &= \int_S \mathbf{L} \cdot (\mathbf{L} \cdot \Psi) \cdot \delta u dS + \int_S \mathbf{n} \cdot (\mathbf{L} \cdot \Psi) \cdot D \delta u dS, \\ \int_S (n_i \bar{\partial}_p n_p - \bar{\partial}_i) \psi_{ijk} \partial_j \delta u_k dS &= \int_S (n_j \bar{\partial}_p n_p - \bar{\partial}_j) (n_i \bar{\partial}_p n_p - \bar{\partial}_i) \psi_{ijk} \delta u_k dS \\ &\quad + \int_S n_j (n_i \bar{\partial}_p n_p - \bar{\partial}_i) \psi_{ijk} D \delta u_k dS. \end{aligned} \quad (33)$$

The second integral in the right hand side of Eq. (32) can be written as

$$\begin{aligned} \int_S \mathbf{n} \cdot \Psi : D \nabla \delta u dS &= \int_S \mathbf{n} \cdot \Psi : \mathbf{n} \cdot \nabla \nabla \delta u dS = \int_S \mathbf{n} \cdot \Psi : (\nabla (\mathbf{n} \cdot \nabla \delta u) - (\nabla \mathbf{n}) \cdot \nabla \delta u) dS, \\ \int_S n_i \psi_{ijk} D \partial_j \delta u_k dS &= \int_S n_i \psi_{ijk} n_m \partial_m \partial_j \delta u_k dS = \int_S n_i \psi_{ijk} (\partial_j (n_m \partial_m \delta u_k) - (\partial_j n_m) \partial_m \delta u_k) dS. \end{aligned} \quad (34)$$

Noting $\nabla (\mathbf{n} \cdot \nabla \mathbf{u}) = \nabla (D \mathbf{u}) = \bar{\nabla} D \mathbf{u} + \mathbf{n} D^2 \mathbf{u}$ and $(\nabla \mathbf{n}) \cdot \nabla \mathbf{u} = (\bar{\nabla} \mathbf{n}) \cdot \bar{\nabla} \mathbf{u}$, Eq. (34) can be written as

$$\begin{aligned} \int_S \mathbf{n} \cdot \Psi : D \nabla \delta u dS &= \int_S \mathbf{n} \cdot \Psi : (\bar{\nabla} D \delta u + \mathbf{n} D^2 \delta u - (\bar{\nabla} \mathbf{n}) \cdot \bar{\nabla} \delta u) dS, \\ \int_S n_i \psi_{ijk} D \partial_j \delta u_k dS &= \int_S n_i \psi_{ijk} (\bar{\partial}_j D \delta u_k + n_j D^2 \delta u_k - (\bar{\partial}_j n_m) \bar{\partial}_m \delta u_k) dS. \end{aligned} \quad (35)$$

Using the surface divergence theorem, Eq. (35) can be expressed as

$$\begin{aligned} \int_S \mathbf{n} \cdot \Psi : D \nabla \delta \mathbf{u} dS &= - \int_S \mathbf{L} \cdot ((\bar{\nabla} \mathbf{n}) \cdot (\mathbf{n} \cdot \Psi)) \cdot \delta \mathbf{u} dS + \int_S \mathbf{L} \cdot (\mathbf{n} \cdot \Psi) \cdot D \delta \mathbf{u} dS + \int_S \mathbf{nn} : \Psi \cdot D^2 \delta \mathbf{u} dS, \\ \int_S n_i \psi_{ijk} D \partial_j \delta u_k dS &= - \int_S (n_i \bar{\partial}_p n_p - \bar{\partial}_i) ((\bar{\partial}_i n_j) n_s \psi_{sjk}) \delta u_k dS \\ &\quad + \int_S (n_i \bar{\partial}_p n_p - \bar{\partial}_i) (n_s \psi_{sik}) D \delta u_k dS + \int_S n_i n_j \psi_{ijk} D^2 \delta u_k dS. \end{aligned} \quad (36)$$

Substituting Eqs. (33) and (36) into Eq. (32) and letting $\psi_{ijk} = t_{lijk} n_l$, we get

$$\begin{aligned} \int_S \Psi : \nabla \nabla \delta \mathbf{u} dS &= \int_S \mathbf{L} \cdot (\mathbf{L} \cdot \Psi) \cdot \delta \mathbf{u} dS + \int_S \mathbf{n} \cdot (\mathbf{L} \cdot \Psi) \cdot D \delta \mathbf{u} dS \\ &\quad - \int_S \mathbf{L} \cdot ((\bar{\nabla} \mathbf{n}) \cdot (\mathbf{n} \cdot \Psi)) \cdot \delta \mathbf{u} dS + \int_S \mathbf{L} \cdot (\mathbf{n} \cdot \Psi) \cdot D \delta \mathbf{u} dS + \int_S \mathbf{nn} : \Psi \cdot D^2 \delta \mathbf{u} dS, \\ \int_S t_{lijk} n_l \partial_i \partial_j \delta u_k dS &= \int_S (n_j \bar{\partial}_p n_p - \bar{\partial}_j) (n_i \bar{\partial}_p n_p - \bar{\partial}_i) t_{lijk} n_l \delta u_k dS \\ &\quad + \int_S n_j (n_i \bar{\partial}_p n_p - \bar{\partial}_i) t_{lijk} n_l D \delta u_k dS - \int_S (n_i \bar{\partial}_p n_p - \bar{\partial}_i) ((\bar{\partial}_i n_j) (n_s t_{lsjk} n_l)) \delta u_k dS \\ &\quad + \int_S (n_i \bar{\partial}_p n_p - \bar{\partial}_i) (n_s t_{lsik} n_l) D \delta u_k dS + \int_S n_i n_j t_{lijk} n_l D^2 \delta u_k dS. \end{aligned} \quad (37)$$

By Eq. (37), the first integral in the right hand side of Eq. (27) can be written as

$$\begin{aligned} \int_S t_{lkji} n_l \partial_k \partial_j \delta u_i dS &= \int_S (n_j \bar{\partial}_p n_p - \bar{\partial}_j) (n_k \bar{\partial}_p n_p - \bar{\partial}_k) t_{lkji} n_l \delta u_i dS + \int_S n_j (n_k \bar{\partial}_p n_p - \bar{\partial}_k) t_{lkji} n_l D \delta u_i dS \\ &\quad - \int_S (n_k \bar{\partial}_p n_p - \bar{\partial}_k) ((\bar{\partial}_k n_j) (n_s t_{lsji} n_l)) \delta u_i dS \\ &\quad + \int_S (n_k \bar{\partial}_p n_p - \bar{\partial}_k) (n_s t_{lski} n_l) D \delta u_i dS + \int_S n_k n_j t_{lkji} n_l D^2 \delta u_i dS. \end{aligned} \quad (38)$$

Substituting Eqs. (29) and (38) into Eq. (27), then substituting Eqs. (26) and (27) into Eq. (19) and using $\delta \Pi = 0$, we can obtain equilibrium equations:

$$(\sigma_{ij} - \tau_{ijk,k} + t_{lkji,lk})_{,j} + f_i = 0 \quad \text{in } V. \quad (39)$$

In addition, the traction boundary conditions can be written as

$$\begin{aligned} (\sigma_{ij} - \tau_{ijk,k} + t_{lkji,lk}) n_j + (n_j \bar{\partial}_p n_p - \bar{\partial}_j) (\tau_{ijk} n_k - t_{lkji,l} n_k + (n_k \bar{\partial}_p n_p - \bar{\partial}_k) t_{lkji} n_l) \\ - (n_s \bar{\partial}_p n_p - \bar{\partial}_s) ((\bar{\partial}_s n_j) t_{lkji} n_k n_l) = \bar{p}_i \quad \text{on } S_\sigma, \end{aligned} \quad (40)$$

$$n_j n_k (\tau_{ijk} - t_{kji,l}) + n_j (n_k \bar{\partial}_p n_p - \bar{\partial}_k) (t_{lkji} n_l) + (n_j \bar{\partial}_p n_p - \bar{\partial}_j) (t_{lkji} n_k n_l) = \bar{r}_i \quad \text{on } S_\tau, \quad (41)$$

$$t_{lkji} n_j n_k n_l = \bar{q}_i \quad \text{on } S_t. \quad (42)$$

The governing equations of the present boundary problem consist of the integral type constitutive equations (14–16), the equilibrium equation (39), and the boundary conditions Eqs. (40–42). The local stress and the local couple stress appear in the constitutive equations (14–16). The boundary problem seems too complicated for engineering application. Further simplification is needed for deriving a strain gradient theory.

3 Constitutive equations for power law hardening materials

According to Fleck and Hutchinson [7] and Chen and Wang [10], the local strain energy-density is assumed to be the function of a local generalized effective strain $\tilde{E}_e$,

$$\tilde{w} = \int_0^{\tilde{E}_e} \tilde{\Sigma}_e d\tilde{E}_e + \frac{1}{2} K \tilde{\varepsilon}_v^2 = \int_0^{\tilde{E}_e} \sigma_Y \left(\frac{\tilde{E}_e}{\varepsilon_Y} \right)^n d\tilde{E}_e + \frac{1}{2} K \tilde{\varepsilon}_v^2 = \frac{\sigma_Y \varepsilon_Y}{1+n} \left(\frac{\tilde{E}_e}{\varepsilon_Y} \right)^{n+1} + \frac{1}{2} K \tilde{\varepsilon}_v^2, \quad (43)$$

where $\tilde{E}_e$ is the local generalized effective strain and

$$\tilde{E}_e^2 = \frac{2}{3} \tilde{\varepsilon}'_{ij} \tilde{\varepsilon}'_{ij} + l_{cs}^2 \tilde{\chi}_e^2 \quad \tilde{\chi}_e^2 = \frac{2}{3} \tilde{\chi}_{ij} \tilde{\chi}_{ij} \quad K = \frac{E}{3(1-2\nu)} \quad \tilde{\varepsilon}_v = \tilde{\varepsilon}_{11} + \tilde{\varepsilon}_{22} + \tilde{\varepsilon}_{33}, \quad (44)$$

here E is Young's modulus, ν is the Poisson ratio, l_{cs} is the intrinsic length for the rotation gradient, $\tilde{\varepsilon}'_{ij}$ is the local deviatoric strain, $\tilde{\chi}_{ij}$ is the local rotation gradient, and $\tilde{\varepsilon}_v$ is the local volume strain.

The local strain energy density $\tilde{w}$ can be approximately expressed as

$$\tilde{w} = w^0 + w_{,k}^0 \xi_k + \frac{w_{,kl}^0}{2} \xi_k \xi_l \quad \text{on } V_c, \quad (45)$$

where

$$w^0 = (\tilde{w})_{\xi=0} = \frac{\sigma_Y \varepsilon_Y}{1+n} \left(\frac{E_e}{\varepsilon_Y} \right)^{n+1} + \frac{1}{2} K \varepsilon_v^2, \quad (46)$$

$$\begin{aligned} w_{,kl}^0 = (\tilde{w}_{,kl})_{\xi=0} &= \frac{4(n-1)\sigma_Y}{9\varepsilon_Y^n} (E_e)^{n-3} \left(\varepsilon'_{ij} \varepsilon'_{ij,k} + l_{cs}^2 \chi_{ij} \chi_{ij,k} \right) (\varepsilon'_{mn} \varepsilon'_{mn,l} + l_{cs}^2 \chi_{mn} \chi_{mn,l}) \\ &+ \frac{2\sigma_Y}{3\varepsilon_Y^n} (E_e)^{n-1} \left(\varepsilon'_{ij,l} \varepsilon'_{ij,k} + \varepsilon'_{ij} \varepsilon'_{ij,kl} + l_{cs}^2 \chi_{ij,l} \chi_{ij,k} \right) + K \varepsilon_{v,l} \varepsilon_{v,k} + K \varepsilon_{v,l} \varepsilon_{v,kl}. \end{aligned} \quad (47)$$

Substituting Eq. (45) into Eq. (6) yields

$$w = \frac{1}{V_c} \int_{V_c} \left(w^0 + w_{,k}^0 \xi_k + \frac{w_{,kl}^0}{2} \xi_k \xi_l \right) dV_c = w^0 + A_k w_{,k}^0 + B_{kl} w_{,kl}^0, \quad (48)$$

where

$$A_k = \frac{1}{V_c} \int_{V_c} \xi_k dV_c, \quad B_{kl} = \frac{1}{2V_c} \int_{V_c} \xi_k \xi_l dV_c, \quad (49)$$

ξ_k is an anti-symmetric function in V_c about center of the cubic cell, then

$$A_k = 0, \quad B_{kl} = B \delta_{kl} = \frac{1}{2V_c} \int_{V_c} (\xi_1)^2 dV_c \delta_{kl} = \frac{l_c^2}{24} \delta_{kl}, \quad (50)$$

$$w = w^0 + B_{kl} w_{,kl}^0 = w^0 + B w_{,kk}^0, \quad B = \frac{l_c^2}{24}. \quad (51)$$

The global stress σ_{ij} and high order stress τ_{ijk} can be written very easily as

$$\begin{aligned}\sigma_{ij} = & \frac{\partial w}{\partial \varepsilon_{ij}} = \frac{2\sigma_Y (E_e)^{n-1}}{3\varepsilon_Y^n} \varepsilon'_{ij} + K \varepsilon_v \delta_{ij} \\ & + \frac{8B(n-1)(n-3)\sigma_Y}{27\varepsilon_Y^n} (E_e)^{n-5} (\varepsilon'_{rs} \varepsilon'_{rs,l} + l_{cs}^2 \chi_{rs} \chi_{rs,l}) (\varepsilon'_{mn} \varepsilon'_{mn,l} + l_{cs}^2 \chi_{mn} \chi_{mn,l}) \varepsilon'_{ij} \\ & + \frac{8B(n-1)\sigma_Y}{9\varepsilon_Y^n} (E_e)^{n-3} (\varepsilon'_{mn} \varepsilon'_{mn,l} + l_{cs}^2 \chi_{mn} \chi_{mn,l}) \varepsilon'_{ij,l} \\ & + \frac{4B\sigma_Y(n-1)}{9\varepsilon_Y^n} (E_e)^{n-3} (\varepsilon'_{mn,l} \varepsilon'_{mn,l} + \varepsilon'_{mn} \varepsilon'_{mn,ll} + l_{cs}^2 \chi_{mn,l} \chi_{mn,l}) \varepsilon'_{ij} \\ & + \frac{2B\sigma_Y}{3\varepsilon_Y^n} (E_e)^{n-1} \varepsilon'_{ij,ll} + K B \varepsilon_{v,ll} \delta_{ij},\end{aligned}\quad (52)$$

$$\begin{aligned}\tau_{ijk} = & \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,k}} + \frac{\partial w}{\partial \varepsilon_{ji,k}} \right) = \frac{\sigma_Y (E_e)^{n-1} l_{cs}^2}{3\varepsilon_Y^n} (e_{pki} \chi_{pj} + e_{pkj} \chi_{pi}) + \frac{4B(n-1)(n-3)\sigma_Y}{27\varepsilon_Y^n} (E_e)^{n-5} \\ & \times (\varepsilon'_{rs} \varepsilon'_{rs,l} + l_{cs}^2 \chi_{rs} \chi_{rs,l}) (\varepsilon'_{mn} \varepsilon'_{mn,l} + l_{cs}^2 \chi_{mn} \chi_{mn,l}) l_{cs}^2 (e_{pki} \chi_{pj} + e_{pkj} \chi_{pi}) \\ & + \frac{8B(n-1)\sigma_Y}{9\varepsilon_Y^n} (E_e)^{n-3} (\varepsilon'_{mn} \varepsilon'_{mn,k} + l_{cs}^2 \chi_{mn} \chi_{mn,k}) \varepsilon'_{ij} \\ & + \frac{4B(n-1)\sigma_Y}{9\varepsilon_Y^n} (E_e)^{n-3} (\varepsilon'_{mn} \varepsilon'_{mn,s} + l_{cs}^2 \chi_{mn} \chi_{mn,s}) l_{cs}^2 (e_{pki} \chi_{pj,s} + e_{pkj} \chi_{pi,s}) \\ & + \frac{2B\sigma_Y(n-1)}{9\varepsilon_Y^n} (E_e)^{n-3} (\varepsilon'_{mn,l} \varepsilon'_{mn,l} + \varepsilon'_{mn} \varepsilon'_{mn,ll} + l_{cs}^2 \chi_{mn,l} \chi_{mn,l}) l_{cs}^2 (e_{pki} \chi_{pj} + e_{pkj} \chi_{pi}) \\ & + \frac{4B\sigma_Y}{3\varepsilon_Y^n} (E_e)^{n-1} \varepsilon'_{ij,k} + 2BK \varepsilon_{v,k} \delta_{ij},\end{aligned}\quad (53)$$

$$\begin{aligned}t_{lki} = & \frac{1}{2} \left(\frac{\partial w}{\partial \varepsilon_{ij,kl}} + \frac{\partial w}{\partial \varepsilon_{ji,kl}} \right) \\ = & \frac{4B(n-1)\sigma_Y}{9\varepsilon_Y^n} (E_e)^{n-3} (\varepsilon'_{mn} \varepsilon'_{mn,l} + l_{cs}^2 \chi_{mn} \chi_{mn,l}) l_{cs}^2 (e_{pki} \chi_{pj} + e_{pkj} \chi_{pi}) \\ & + \frac{4B\sigma_Y}{3\varepsilon_Y^n} (E_e)^{n-1} (\varepsilon'_{ij} \delta_{kl} + l_{cs}^2 \chi_{pj,l} e_{pki} + l_{cs}^2 \chi_{pi,l} e_{pkj}) + K B \varepsilon_{v,ij} \delta_{kl}.\end{aligned}\quad (54)$$

From the above equations, one can see that the constitutive relations for homogeneous power law hardening materials are derived directly from the new framework of the non-local model. The present strain gradient theory possesses clear physical meaning. Two length scales are involved: one is the boundary length scale l_c of the representative volume element and the other is the intrinsic length scale l_{cs} related with the rotation gradient.

4 Thin wire torsion

The size effect emerging in the experiment of thin-wire torsion [21] is analyzed using the present strain gradient theory. A Cartesian coordinate system (x_1, x_2, x_3) and a cylindrical polar coordinate system (r, θ, x_3) are introduced as shown in Fig. 2. The radius of the wire denotes as α . κ is the twist per unit length of the wire.

The displacement fields

$$u_1 = -\kappa x_2 x_3, \quad u_2 = \kappa x_1 x_3, \quad u_3 = 0 \quad (55)$$

Fig. 2 Coordinate systems (x_1, x_2, x_3) and (r, θ, x_3) on a thin wire

lead to the non-vanishing strain components, strain gradient, and rotation gradient components as

$$\varepsilon_{13} = \varepsilon_{31} = -\frac{1}{2}\kappa x_2, \quad \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\kappa x_1, \quad (56)$$

$$\varepsilon_{13,2} = \varepsilon_{31,2} = -\frac{1}{2}\kappa, \quad \varepsilon_{23,1} = \varepsilon_{32,1} = \frac{1}{2}\kappa, \quad (57)$$

$$\chi_{11} = \chi_{22} = -\frac{1}{2}\kappa, \quad \chi_{33} = \kappa. \quad (58)$$

Based on Eq. (55), all second-order strain gradient components are equal to zero. According to Eq. (44) and Eqs. (56–58), we have

$$E_e = \left(\tilde{E}_e \right)_{\xi=0} = \kappa \sqrt{\frac{r^2 + 3l_{cs}^2}{3}}, \quad (59)$$

$$\varepsilon_{mn}\varepsilon_{mn,k} = 2\varepsilon_{13}\varepsilon_{13,k} + 2\varepsilon_{23}\varepsilon_{23,k} = \kappa (\varepsilon_{23,k}x_1 - \varepsilon_{13,k}x_2), \quad (60)$$

$$\varepsilon'_{mn}\varepsilon'_{mn,k}\varepsilon'_{ij}\varepsilon'_{ij,l} = \kappa^2 (\varepsilon_{23,k}x_1 - \varepsilon_{13,k}x_2) (\varepsilon_{23,l}x_1 - \varepsilon_{13,l}x_2), \quad (61)$$

$$\varepsilon_{ij,l}\varepsilon_{ij,k} = 2\varepsilon_{13,l}\varepsilon_{13,k} + 2\varepsilon_{23,l}\varepsilon_{23,k}. \quad (62)$$

The total strain energy for a unit length of the bar is given by

$$W = \int_V w dV = \int_V (w^0 + B_{kl}w_{,kl}^0) dV = \int_0^\kappa Q(\kappa) d\kappa \quad (63)$$

and

$$W = \int_V (w^0 + B_{kl}w_{,kl}^0) dV = \int_0^{2\pi} d\theta \int_0^a (w^0 + B_{kl}w_{,kl}^0) r dr. \quad (64)$$

Substituting Eq. (59) into Eq. (46) yields

$$w^0 = (\tilde{w})_{\xi=0} = \frac{\sigma_Y \varepsilon_Y}{1+n} \left(\frac{E_e}{\varepsilon_Y} \right)^{1+n} = \frac{\sigma_Y}{(1+n) \varepsilon_Y^n} \left(\frac{1}{3} \right)^{\frac{1+n}{2}} \kappa^{1+n} (r^2 + 3l_{cs}^2)^{\frac{1+n}{2}}. \quad (65)$$

Substituting Eqs. (59–62) into Eq. (47), we have

$$\begin{aligned} B_{kl} w_{,kl}^0 &= B \delta_{kl} w_{,kl}^0 \\ &= \frac{2B\sigma_Y}{3(\varepsilon_Y)^n} \left[\frac{2(n-1)(E_e)^{n-3} \varepsilon'_{mn} \varepsilon'_{mn,k} \varepsilon'_{ij} \varepsilon'_{ij,k}}{3} + (E_e)^{n-1} \varepsilon'_{ij,k} \varepsilon'_{ij,k} \right] \\ &= \frac{2B\sigma_Y}{3(\varepsilon_Y)^n} \left\{ \frac{2(n-1) \left[\frac{1}{3} \kappa^2 (r^2 + 3l_c^2) \right]^{\frac{n-3}{2}} \left(\frac{\kappa^4 r^2}{4} \right)}{3} + \left[\frac{1}{3} \kappa^2 (r^2 + 3l_c^2) \right]^{\frac{n-1}{2}} \kappa^2 \right\}. \end{aligned} \quad (66)$$

Then, it follows

$$\begin{aligned} W &= \int_V (w^0 + B_{kl} w_{,kl}^0) dV = \int_0^{2\pi} d\theta \int_0^a (w^0 + B_{kl} w_{,kl}^0) r dr \\ &= \frac{2\pi\sigma_Y}{(1+n)(3+n)\varepsilon_Y^n} \left(\frac{1}{3} \right)^{\frac{1+n}{2}} \kappa^{1+n} \left[(a^2 + 3l_{cs}^2)^{\frac{n+3}{2}} - (3l_{cs}^2)^{\frac{n+3}{2}} \right] \\ &\quad + \frac{4\pi B\sigma_Y \kappa^{n+1}}{3^{\frac{n+1}{2}} (\varepsilon_Y)^n} \left[\frac{(a^2 + 3l_{cs}^2)^{\frac{n+1}{2}}}{2} - \frac{3l_{cs}^2}{2} (a^2 + 3l_{cs}^2)^{\frac{n-1}{2}} \right]. \end{aligned} \quad (67)$$

Let $\beta = \frac{3l_{cs}^2}{a^2}$. Equation (67) can be rewritten as

$$\begin{aligned} W &= \frac{2\pi\sigma_Y \kappa^{n+1} a^{n+3}}{3^{\frac{n+1}{2}} (n+1)(n+3)(\varepsilon_Y)^n} \\ &\quad \times \left\{ \left[(1+\beta)^{\frac{n+3}{2}} - \beta^{\frac{n+3}{2}} \right] + \frac{B(1+n)(3+n)}{a^2} \left[(1+\beta)^{\frac{n+1}{2}} - \beta(1+\beta)^{\frac{n-1}{2}} \right] \right\}. \end{aligned} \quad (68)$$

The torque Q can be written as

$$Q = \frac{dW}{d\kappa} = \frac{(n+1)W}{\kappa}. \quad (69)$$

Combining Eq. (68) and Eq. (69) leads to

$$Q = \frac{2\pi\sigma_Y \kappa^n a^{n+3}}{(n+3)\varepsilon_Y^n 3^{\frac{n+1}{2}}} \left\{ \left[(1+\beta)^{\frac{n+3}{2}} - \beta^{\frac{n+3}{2}} \right] + \frac{B(n+1)(n+3)}{a^2} (1+\beta)^{\frac{n-1}{2}} \right\}. \quad (70)$$

Due to $l_c/a \ll 1$, Eq. (70) can be simplified as

$$Q = \frac{2\pi\sigma_Y \kappa^n a^{n+3}}{(n+3)\varepsilon_Y^n 3^{\frac{n+1}{2}}} \left[(1+\beta)^{\frac{n+3}{2}} - \beta^{\frac{n+3}{2}} \right]. \quad (71)$$

Furthermore, when $l_{cs}/a \ll 1$, Eq. (71) can be re-written as

$$\frac{Q}{a^3} = \frac{(n+1)W}{\kappa} = \frac{2\pi\sigma_Y}{(n+3)\varepsilon_Y^n} (\kappa a)^n \left(\frac{1}{3} \right)^{\frac{n+1}{2}}. \quad (72)$$

Fig. 3 Plots of torque against the surface strain for copper wires with different diameters. The solid lines denote the present theoretical results and the dotted lines denote the experiment ones [21]

Since the intrinsic lengths l_{cs} and l_c are much smaller than $170 \mu\text{m}$, we choose the torque experiment results of the thin wire when the diameter $2a$ is $170 \mu\text{m}$ as a calibration curve and obtain $\frac{2\pi\sigma_Y}{(3+n)\varepsilon_Y^n} \left(\frac{1}{3}\right)^{\frac{n+1}{2}} = 232.7 \text{ MPa}$, and $n = 0.21$, which is close to the tensile experimental result given by Fleck et al. [21].

According to Gao et al. [9] and Fleck et al. [21], we take $l_c = 0.1 \mu\text{m}$ and $l_{cs} = 3.7 \mu\text{m}$. The theoretical results compared with the experiment ones are shown in Fig. 3, where one can see that both are consistent very well with each other.

Begley and Hutchinson [23] pointed out that the rotation gradient and the intrinsic lengths l_{cs} play the most important role in thin wire torsion. Similarly, in our calculation, the intrinsic length l_{cs} is the most important parameter in thin wire torsion for the present theory. Begley and Hutchinson [23] also pointed out that the intrinsic length l_1 for the stretch gradient is the most important parameter in micro indentation; meanwhile the intrinsic lengths l_{cs} is clearly of secondary importance in micro indentation. For the couple stress theory of strain gradient plasticity, Huang et al. [44,45] and Xia and Hutchinson [46] obtained analytically the plane strain model I,II, and mixed asymptotic crack tip field. It is established that the stress level estimated by the couple strain theory of strain gradient plasticity near a model I crack tip is essentially the same as that in classical plasticity. Wei and Hutchinson [47] used SG theory to investigate the crack tip field. The numerical results have clearly shown that stretch gradients indeed remarkably elevate the stress level around a steadily propagating crack tip.

5 Ultra-thin cantilever beam

In the section, we will study the problem of an ultra-thin cantilever beam bend with different micro-meters thickness using the present strain gradient theory. The model is shown in Fig. 4. x, z is a Cartesian coordinate system, h is the thickness of the cantilever beams, b is the width, L is the length of the beam, and Δ is the deflection at the free end. The beam is subjected to plane strain deformation.

According to the beam bend theory, the non-vanishing strain components, strain gradient, and higher strain gradient components are

$$\begin{aligned} \varepsilon_x &= \frac{3\Delta}{l^3} z(x-L), & \varepsilon_{x,x} &= \frac{3\Delta}{l^3} z, & \varepsilon_{x,z} &= \frac{3\Delta}{L^3} (x-L), \\ \chi_{21} &= \frac{3\Delta}{L^3} (x-L), & \varepsilon_{x,xz} &= \frac{3\Delta}{L^3}, & \varepsilon_{x,zx} &= \frac{3\Delta}{L^3}, & \chi_{21,x} &= \frac{3\Delta}{L^3}. \end{aligned} \quad (73)$$

For incompressible materials, E_e is

$$E_e = \sqrt{\frac{2}{3} \left((\varepsilon'_x)^2 + l_{cs}^2 (\chi_{12})^2 \right)} = \sqrt{\frac{2}{3} (z^2 + l_{cs}^2)} \frac{3\Delta}{L^3} (L-x). \quad (74)$$

Fig. 4 Coordinate systems (x, z) on an ultra-thin beam

Using Eqs. (46), (47), (51) and (74), the non-local strain energy density w , for homogeneous power-law hardening materials, can be approximately expressed as

$$\begin{aligned}
 w = & \frac{\sigma_Y}{(n+1)\varepsilon_Y^n} \left(\frac{3}{L^3} \sqrt{\frac{2}{3}} (z^2 + l_{cs}^2) \right)^{n+1} (L-x)^{n+1} \Delta^{n+1} \\
 & + \frac{4(n-1)B\sigma_Y}{9\varepsilon_Y^n} \left(\frac{3}{L^3} \right)^{n+1} \left(\sqrt{\frac{2}{3}} (z^2 + l_{cs}^2) \right)^{n-3} (L-x)^{n-1} (z^4 + 2l_{cs}^2 z^2 + l_{cs}^4 + z^2(x-L)^2) \Delta^{n+1} \\
 & + \frac{2B\sigma_Y}{3\varepsilon_Y^n} \left(\frac{3}{L^3} \right)^{n+1} \left(\sqrt{\frac{2}{3}} (z^2 + l_{cs}^2) \right)^{n-1} (L-x)^{n-1} (z^2 + (x-L)^2 + l_{cs}^2) \Delta^{n+1}.
 \end{aligned} \quad (75)$$

The total strain energy W of the cantilever beam equals

$$W = \int_0^L \iint_S w dV. \quad (76)$$

Substituting Eq. (75) into Eq. (76), we have

$$\int_0^L \iint_S w dV = I \Delta^{n+1}, \quad (77)$$

where

$$\begin{aligned}
 I = & \frac{b(6)^{\frac{n+1}{2}} \sigma_Y h^{n+2}}{(n+1)(n+2)\varepsilon_Y^n L^{2n+1}} \int_{\frac{h}{2}}^{\frac{h}{2}} \left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n+1} d\left(\frac{z}{h} \right) + \frac{(n-1)bB(6)^{\frac{n+1}{2}} \sigma_Y h^n}{\varepsilon_Y^n L^{2n+3}} \\
 & \times \left(\frac{L^2}{n+2} \int_{\frac{h}{2}}^{\frac{h}{2}} \left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n-3} \left(\frac{2z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right) d\left(\frac{z}{h} \right) + 2 \frac{h^2}{n} \int_{\frac{h}{2}}^{\frac{h}{2}} \left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n+1} d\left(\frac{z}{h} \right) \right).
 \end{aligned} \quad (78)$$

The concentrate force P takes the form

$$P = \frac{\partial W}{\partial \Delta} = \frac{\partial (\int_0^L \iint_S w dV)}{\partial \Delta} = (n+1)I \Delta^n. \quad (79)$$

Fig. 5 The non-dimensional concentrate force, $\frac{P(n+2)\varepsilon_Y^n L^{n+1}}{(6)^{\frac{n+1}{2}} \sigma_Y h^{n+2} b}$, versus the normalized deflection at the free end, $\frac{\Delta}{L}$; the plastic work hardening exponent $n = 0.2, 0.5, 0.7$ and 1 ; the ratio of intrinsic material length l_{cs} to beam thickness h is 0.1 , and the ratio of the boundary length scale l_c to intrinsic material length l_{cs} is 0.1

The non-dimensional concentrate force can be written as

$$\begin{aligned} \frac{P(n+2)\varepsilon_Y^n L^{n+1}}{(6)^{\frac{n+1}{2}} \sigma_Y h^{n+2} b} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n+1} d\left(\frac{z}{h}\right) \left(\frac{\Delta}{L}\right)^n \\ &+ (n-1)(n+1) \frac{B}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n-3} \left(\frac{2z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right) \right) d\left(\frac{z}{h}\right) \left(\frac{\Delta}{L}\right)^n \\ &+ \frac{h^2}{L^2} \frac{2(n+2)(n-1)(n+1)B}{nh^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n+1} d\left(\frac{z}{h}\right) \left(\frac{\Delta}{L}\right)^n. \quad (80) \end{aligned}$$

Noting that $\frac{h^2}{L^2} \ll 1$, Eq. (80) can be written as

$$\begin{aligned} \frac{P(n+2)\varepsilon_Y^n L^{n+1}}{(6)^{\frac{n+1}{2}} \sigma_Y h^{n+2} b} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n+1} d\left(\frac{z}{h}\right) \left(\frac{\Delta}{L}\right)^n + (n-1)(n+1) \frac{B}{h^2} \\ &\times \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\left(\sqrt{\left(\frac{z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right)} \right)^{n-3} \left(\frac{2z^2}{h^2} + \frac{l_{cs}^2}{h^2} \right) \right) d\left(\frac{z}{h}\right) \left(\frac{\Delta}{L}\right)^n. \quad (81) \end{aligned}$$

The present calculation results are shown in Figs. 5 and 6. In Fig. 6, the result for $\frac{l_{cs}}{h} = 0$ is the classic result. From this figure, we can clearly see that the strain gradients have a remarkable effect on the non-dimensional concentrate force. It means that the non-dimensional concentrate force will significantly increase as the thickness h decreases.

Fig. 6 The non-dimensional concentrate force, $\frac{(n+2)P\epsilon_y^n L^{n+1}}{(6)^{\frac{n+1}{2}} \sigma_y h^{n+2} b}$, versus the normalized deflection at the free end, $\frac{\Delta}{L}$, for the plastic work hardening exponent $n = 0.2$ and various ratios of intrinsic material length l_{cs} to beam thickness h . The ratio of the boundary length scale l_c to intrinsic material length l_{cs} is 0.1, and the curve for $\frac{l_{cs}}{h} = 0$ corresponds to classical plasticity

6 Summary

A new framework of non-local model is proposed, in which the strain energy density of the representative volume element is taken as a non-local variable. The local strain is expressed as the second order Taylor series with the first and second strain gradients within the representative volume element. Based on the new non-local model, a new strain gradient theory is derived. The triadic and tetradic stresses are introduced in the new strain gradient theory, which are the work conjugates to the first and second strain gradients, respectively. Size effects in two typical examples, i.e., thin wire torsion and micro-cantilever beam bend, have been analyzed using the present strain gradient theory. It is found that the result predicted by the theoretical model is consistent well with the experimental data for the thin wire torsion. On the other hand, the calculation result for the micro-cantilever beam bend clearly shows the size effect.

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