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# Estimation of Transverse Thermal Conductivity of Doubly-periodic Fiber Reinforced Composites

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**Abstract**

For steady-state heat conduction, a new variational functional for a unit cell of composites with periodic microstructures is constructed by considering the quasi-periodicity of the temperature field and in the periodicity of the heat flux fields. Then by combining with the eigenfunction expansion of complex potential which satisfies the fiber-matrix interface conditions, an eigenfunction expansion-variational method (EEVM) based on a unit cell is developed. The effective transverse thermal conductivities of doubly-periodic fiber reinforced composites are calculated, and the first-order approximation formula for the square and hexagonal arrays is presented, which is convenient for engineering application. The numerical results show a good convergency of the presented method, even though the fiber volume fraction is relatively high. Comparisons with the existing analytical and experimental results are made to demonstrate the accuracy and validity of the first-order approximation formula for the hexagonal array.

*Keywords:* effective thermal conductivity; unit cell model; eigenfunction expansion; variational techniques; double period; fiber reinforced materials

## 1. Introduction

Fiber reinforced composites possess some remarkable properties in contrast to the conventional metal materials, thus they are used broadly in aerospace applications. The carbon/epoxy composites have small specific mass, high rupture resistance, very good fatigue strength and good thermal/electrical conductivity. They have been used in the wings, fuselages, horizontal stabilizers, vertical stabilizers, ailerons and so on of airplanes such as Boeing 787 and Airbus A380[1]. Continuous fiber reinforced ceramic composites potentially offer higher specific mechanical properties which can be utilized in a variety of high-temperature aerospace applications[2].

Thermal conductivity is a very important property for the applications of fiber reinforced composites. Lots of researches[3] focused on this topic and can be traced back to L. Rayleigh's research[4]. Rayleigh derived an approximation formula for the effective transverse thermal conductivity for a square array of fibers. Z. Hashin[5] developed a bounded solution for the effective transverse thermal conductivity, in which the lower bound was equivalent to the generalized self-consistent method (GSCM)[3] for circular cross-section fibers. The results of the self-consistent method (SCM)[6] are always lain between the two bounds. G. S. Springer, et al.[7] considered a unit cell for a square array, and derived an equation by using the so-called "series-parallel"model. R.Rolfes, et al.[3] examined some of the existing approximation formulae by comparing with the finite-element calculations and experimental data. They found that these formulae provided rather different results. So it is still necessary to develop a more reliable method for estimating the transverse thermal conductivity. In this article, the two-dimensional steady-state heat conduction of composites with doubly-periodic array of

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circular cross-section fibers is solved with the aid of the presented variational functional and the eigen-function expansion-variational method (EEVM). And then the effective thermal conductivity is calculated and its changing with the fiber volume fraction and arrangement is discussed.

It is similar to the elasticity issue[8-9] that the temperature field is quasi-periodic and the heat flux field is periodic. For a unit cell model these periodic boundary conditions must be prescribed. With the aid of the variational principle which is similar to that for elasticity issue[10], a new variational functional for a unit cell for steady-state heat conduction is constructed which contains the periodic boundary conditions. In contrast to the approaches based on a rectangular unit cell model[4,7], the presented variational functional is based on a general unit cell model. So the composites with a general doubly-periodic array of fibers can be studied with the presented method. The unit cell model can be used to design composites[11], thus the presented method is useful for designing composites.

Numerical results obtained from the presented method are expected to be approaching the exact values with the increasing number of the expansion terms, so the method can serve as a reference for other approximation methods. In addition, a first-order approximation formula for the square and hexagonal arrays is presented for engineering application. Finally, comparisons with the existing analytical and experimental results are carried out.

## 2. Periodicity Conditions and Variational Functional for Unit Cell

Composites with a general doubly-periodic array of circular cross-section fibers are shown in Fig.1 (a), where $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ denote two fundamental periods. For the same periodic microstructure, two different unit cells can be selected as shown in Fig.1(b) and Fig.1(c).

It can be seen that $\partial V_j^+$ and $\partial V_j^-$ are paired. By properly translating $\boldsymbol{p}^j$, the boundary $\partial V_j^-$ will coincide with the boundary $\partial V_j^+$, where $j$=1, 2, 3 for Fig.1(b) and $j$=1, 2 for Fig.1(c). Let the coordinates $\boldsymbol{x}$ and unit normal vectors $\boldsymbol{n}$ on the boundaries $\partial V_j^+$, $\partial V_j^-$ be $\boldsymbol{x}^{j+}$ and $\boldsymbol{n}^{j+}$ and $\boldsymbol{x}^{j-}$ and $\boldsymbol{n}^{j-}$ respectively. The periodicity of the unit cell results in the following relations:

$$\left.\begin{aligned} \boldsymbol{x}^{j+} &= \boldsymbol{x}^{j-} + \boldsymbol{p}^j \\ \boldsymbol{n}^{j+} &= -\boldsymbol{n}^{j-} \end{aligned}\right\} \tag{1}$$

where $\boldsymbol{p}^1 = \boldsymbol{d}_1 - \boldsymbol{d}_2$, $\boldsymbol{p}^2 = \boldsymbol{d}_1$ and $\boldsymbol{p}^3 = \boldsymbol{d}_2$ for Fig.1(b) and $\boldsymbol{p}^1 = \boldsymbol{d}_1$ and $\boldsymbol{p}^2 = \boldsymbol{d}_2$ for Fig.1(c).

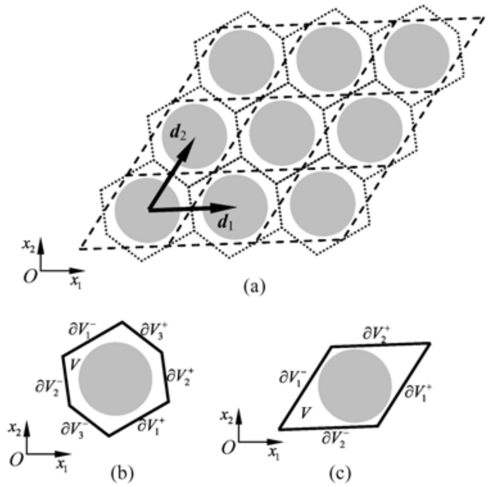


Fig.1 Composites with doubly-periodic microstructure and their two different unit cells.

For the steady-state heat conduction, the temperature field $T(\boldsymbol{x})$ is quasi-periodic and the heat flux field $\boldsymbol{q}(\boldsymbol{x})$ is periodic. Thus the corresponding temperatures $T$ and boundary heat flux $q$ ($q = \boldsymbol{q}\cdot\boldsymbol{n}$) on $\partial V_j^+$ and $\partial V_j^-$ satisfy the following coupling condition and periodicity condition:

$$T(\boldsymbol{x}^{j+}) - T(\boldsymbol{x}^{j-}) = \langle \boldsymbol{H} \rangle \cdot \boldsymbol{p}^j \tag{2}$$

$$q(\boldsymbol{x}^{j+}) + q(\boldsymbol{x}^{j-}) = 0 \tag{3}$$

where $\langle \boldsymbol{H} \rangle$ denotes the average temperature gradient within a unit cell and is the same for all unit cells. The temperature coupling conditions (Eq.(2)) together with the periodicity conditions of boundary heat flux (Eq.(3)) form the periodic boundary conditions and can be written as

$$\begin{cases} T^{j+} - T^{j-} = \langle \boldsymbol{H} \rangle \cdot \boldsymbol{p}^j \\ \quad q^{j+} + q^{j-} = 0 \end{cases} \tag{4}$$

For steady-state conditions with no internal heat source, the heat flux, temperature gradients and temperature fields satisfy the following three equations[12]:

Fourier's law: $\boldsymbol{q} = -\boldsymbol{k} \cdot \boldsymbol{H}$ (5a)

Temperature gradient: $\boldsymbol{H} = \nabla T$ (5b)

Equilibrium equation: $\nabla \cdot \boldsymbol{q} = 0$ (5c)

where $\boldsymbol{k}$ is the thermal conductivity tensor. According to the variational principle which is similar to that for elasticity issue[10] and presupposing that the

Eqs.(5a)-(5b) and fiber-matrix interface conditions are met, a generalized potential energy functional which takes account of periodic microstructure attributes can be defined as

$$\Pi = \int_V \frac{1}{2}\boldsymbol{q}\cdot\boldsymbol{H}\mathrm{d}V - \sum_j \int_{\partial V_j^+} q^{j+}(T^{j+} - T^{j-} - \langle \boldsymbol{H}\rangle \cdot \boldsymbol{p}^j)\mathrm{d}S \tag{6}$$

where the second term is corresponding to the temperature coupling conditions derived from the quasi-periodicity of temperature field. Eq.(6) is suitable for the general unit cell of any periodic array, where the symmetry or antisymmetry of the unit cell may not exist. For the present study, the fiber-matrix interface conditions are that the heat transfer rate and temperature are continuous along the fiber boundary.

If the heat flux fields satisfy the equilibrium equations, only the periodic boundary conditions of the unit cell are to be satisfied, then the stationary condition can be written as

$$\sum_j \int_{\partial V_j^+} \delta q^{j+}(T^{j+} - T^{j-})\mathrm{d}S - \sum_j \int_{\partial V_j^+} (q^{j+} + q^{j-})\delta T^{j-}\mathrm{d}S = \sum_j \int_{\partial V_j^+} \delta q^{j+}\langle \boldsymbol{H}\rangle \cdot \boldsymbol{p}^j \mathrm{d}S \tag{7}$$

where δ (·) denotes the variational calculus.

In the following sections, the variational functional Eq.(7) will be used to develop an eigenfunction expansion-variational method based on a unit cell to solve the problem of steady-state heat conduction for a generally doubly-periodic array of circular cross-section fibers.

## 3. Eigenfunction Expansion of Complex Potentials in a Unit Cell with a Fiber

For the (transversely) isotropic material under two-dimensional, steady-state conditions with no internal heat source, the temperature field satisfies Laplace's equation[12]

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} = 0 \tag{8}$$

So the temperature $T$, heat flux $[q_1 \;\; q_2]$ and heat transfer rate $\Phi$ can be formulated by a complex potential $\omega(z)$:

$$q_1 - \mathrm{i}q_2 = -k\omega'(z) \tag{9a}$$

$$T = \frac{1}{2}[\omega(z) + \overline{\omega(z)}] \tag{9b}$$

$$\Phi = \frac{-k}{2\mathrm{i}}[\omega(z) - \overline{\omega(z)}]_\mathrm{A}^\mathrm{B} \tag{9c}$$

where $\mathrm{i} = \sqrt{-1}$, $z=x_1+\mathrm{i}x_2$ is complex variable, the upper bar denotes the conjugation (below is the same) , the prime denotes the derivative with respect to $z$, $[\cdot]_A^B$ denotes the difference of the taken values of points $B$ and $A$ within the bracket, and $k$ is the thermal conductivity.

Referring to Figs.1(b)-1(c), assume that the unit cell and the circular fiber share the thermal common center. Then the complex potential $\omega(z)$ satisfies

$$\omega(z) = -\omega(-z) \tag{10}$$

The complex potential $\omega_\mathrm{f}(z)$ is an analytical function within the fiber domain, and can be expanded as a Taylor series:

$$\omega_\mathrm{f}(z) = \sum_{n=1}^{\infty} E_n z^{2n-1} \tag{11}$$

The complex potential $\omega_\mathrm{m}(z)$ within the matrix domain can be expanded as a Laurent series:

$$\omega_\mathrm{m}(z) = \sum_{n=1}^{\infty} G_n z^{-(2n-1)} + \sum_{n=1}^{\infty} F_n z^{2n-1} \tag{12}$$

where $E_n$, $G_n$ and $F_n$ are complex coefficients.

The heat transfer rate and temperature are continuous along the fiber boundary, that is

$$\Phi_\mathrm{f} = \Phi_\mathrm{m}, \quad T_\mathrm{f} = T_\mathrm{m} \quad \text{at} \quad |z| = R \tag{13}$$

where $R$ denotes the radius of the fiber cross-section. Substituting Eqs.(11)-(12) into Eq.(9) and then into Eq.(13) yields

$$G_n = \eta R^{4n-2}\overline{F}_n \tag{14}$$

where $\eta = (k_\mathrm{m} - k_\mathrm{f})/(k_\mathrm{m} + k_\mathrm{f})$, $k_\mathrm{m}$ and $k_\mathrm{f}$ are the thermal conductivities of the matrix and fiber respectively. Hence the eigenfunction expansion of the complex potential in the matrix domain is

$$\omega_\mathrm{m}(z) = \sum_{n=1}^{\infty} \eta R^{4n-2}\overline{F}_n z^{-(2n-1)} + \sum_{n=1}^{\infty} F_n z^{2n-1} \tag{15}$$

If the unit cell is symmetrical additionally, such as the hexagonal and square unit cells shown in Fig.2, and the boundary conditions are also symmetrical, then the coefficients also satisfy

$$\overline{G}_n = -G_n, \quad \overline{F}_n = -F_n \tag{16}$$

i.e. $G_n$ and $F_n$ are imaginary constants. Hence Eq.(15) reduces to

$$\omega_\mathrm{m}(z) = \sum_{n=1}^{\infty} F_n[-\eta R^{4n-2} z^{-(2n-1)} + z^{2n-1}] \tag{17}$$

The eigenfunction expansions expressed as Eq.(15) and Eq.(17) satisfy the fiber-matrix interface conditions of Eq.(13) and Eqs.(5a)-(5c). The remaining work is the determination of unknown coefficients $G_n$ and $F_n$ which can be completed by using the stationary conditions (Eq.(7)) .

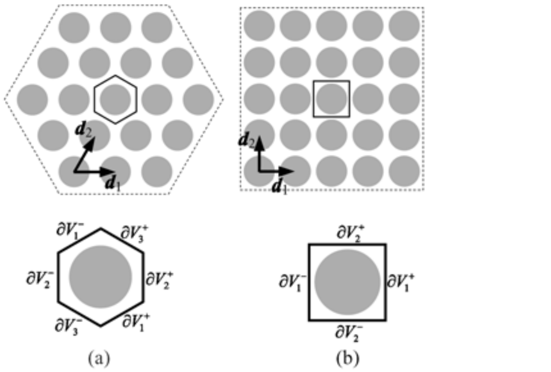


Fig.2 Hexagonal and square arrays of fibers and their symmetrical unit cells.

## 4. Determination of Unknown Coefficients in Eigenfunction Expansions

In this section, only the detailed solving procedure of the unknown coefficients for a nonsymmetrical unit cell is given, because the solving procedure for a symmetrical unit cell is simpler.

By substituting Eq.(15) into Eq.(9), and taking appropriate truncation of the expansions, the heat flux, temperature and heat transfer rate can be expressed as follows[13]:

$$\left.\begin{aligned} q_i &= \sum_{n=1}^{2N} B_n q_i^{(n)} \\ T &= \sum_{n=1}^{2N} B_n T^{(n)} \\ \varPhi &= \sum_{n=1}^{2N} B_n \varPhi^{(n)} \\ &(i=1,2) \end{aligned}\right\} \tag{18}$$

where

$$B_n = \begin{cases} \eta R^{4n-2}\overline{F}_n & 1 \le n \le N \\ F_{n-N} & N+1 \le n \le 2N \end{cases} \tag{19}$$

$$q_1^{(n)} = \begin{cases} -\frac{1}{2}k_{\mathrm{m}}[(1-2n)z^{-2n} + (2n-1)\eta^{-1}R^{2-4n}\overline{z}^{2n-2}] \\ \quad 1 \le n \le N \\ -\frac{1}{2}k_{\mathrm{m}}(2n-2N-1)[z^{2(n-N)-2} - \eta R^{4(n-N)-2}\overline{z}^{-2(n-N)}] \\ \quad N+1 \le n \le 2N \end{cases} \tag{20a}$$

$$q_2^{(n)} = \begin{cases} \dfrac{1}{2\mathrm{i}}k_{\mathrm{m}}[(1-2n)z^{-2n} - (2n-1)\eta^{-1}R^{2-4n}\overline{z}^{2n-2}] \\ \quad 1 \le n \le N \\ \dfrac{1}{2\mathrm{i}}k_{\mathrm{m}}(2n-2N-1)[z^{2(n-N)-2} + \eta R^{4(n-N)-2}\overline{z}^{-2(n-N)}] \\ \quad N+1 \le n \le 2N \end{cases} \tag{20b}$$

$$T^{(n)} = \begin{cases} \dfrac{1}{2}[z^{1-2n} + \eta^{-1}R^{2-4n}\overline{z}^{2n-1}] \\ \quad 1 \le n \le N \\ \dfrac{1}{2}[z^{2(n-N)-1} + \eta R^{4(n-N)-2}\overline{z}^{1-2(n-N)}] \\ \quad N+1 \le n \le 2N \end{cases} \tag{20c}$$

$$\varPhi^{(n)} = \begin{cases} \dfrac{-1}{2\mathrm{i}}k_{\mathrm{m}}[z^{1-2n} - \eta^{-1}R^{2-4n}\overline{z}^{2n-1}]_A^B \\ \quad 1 \le n \le N \\ \dfrac{-1}{2\mathrm{i}}k_{\mathrm{m}}[z^{2(n-N)-1} - \eta R^{4(n-N)-2}\overline{z}^{1-2(n-N)}]_A^B \\ \quad N+1 \le n \le 2N \end{cases} \tag{20d}$$

Substituting Eq.(18) and Eq.(20) into the stationary conditions (Eq.(7)) yields the following linear algebraic equation:

$$\sum_{n=1}^{2N} D_{mn}B_n = C_m \qquad m = 1,2,\cdots,2N \tag{21a}$$

where

$$D_{mn} = \sum_j \int_{\partial V_j^+} \boldsymbol{n}^{j+} \cdot \boldsymbol{q}_{(m)}^{j+}(T_{(n)}^{j+} - T_{(n)}^{j-})\mathrm{d}S - \sum_j \int_{\partial V_j^+} \boldsymbol{n}^{j-} \cdot (\boldsymbol{q}_{(n)}^{j-} - \boldsymbol{q}_{(n)}^{j+})T_{(m)}^{j-}\mathrm{d}S \tag{21b}$$

$$C_m = \sum_j \int_{\partial V_j^+} (\boldsymbol{n}^{j+} \cdot \boldsymbol{q}_{(m)}^{j+})(\langle \boldsymbol{H} \rangle \cdot \boldsymbol{p}^j) dS = \sum_j [\varPhi_{(m)}^{j+} \langle \boldsymbol{H} \rangle \cdot \boldsymbol{p}^j] \tag{21c}$$

where $\boldsymbol{q}_{(m)}^{j+}$, $T_{(n)}^{j+}$ and $\varPhi_{(m)}^{j+}$ denote taking the values of $q_i^{(m)}$, $T^{(n)}$ and $\varPhi^{(m)}$ from $\partial V_j^+$, and the quantities with superscript "$j$-" are corresponding to taking values from $\partial V_j^-$. Once the unknown coefficients are determined by Eq.(21) , the heat flux and temperature fields can be obtained by Eq.(18) and Eq.(20).

## 5. Average Fields and Effective Thermal Conductivities

The effective thermal conductivities are determined with the aid of the average field theory. It is worth noting that the period boundary conditions are prescribed for the unit cell by setting the average temperature gradient, and then the average heat flux are solved for calculating the effective thermal conductivities. The solving procedure for the effective thermal conductivity is shown in Table 1.

The average heat flux in Table 1 are calculated by the following formula

$$\langle q_i \rangle = \frac{1}{V}\int_V q_i \mathrm{d}V = \frac{1}{V}\sum_j \varPhi^j p_i^j \tag{22}$$

where $\Phi^j$ is the heat transfer rate through the boundary $\partial V_j^+$.

For composites with a general doubly-periodic array of fibers as shown in Fig.1, the in-plane effective conductivity is anisotropic and two sets of boundary conditions (see Table 1) need to be prescribed to calculate all its components. Due to the symmetry of the conductivity tensor, there are three independent components. For overall orthotropic composites, $k_{c,12}=k_{c,21}=0$. Moreover, for composites with microstructure as shown in Fig.2, the in-plane effective conductivity is isotropic, that is $k_{c,11}=k_{c,22}=k_c$.

**Table 1 Solving procedure for the effective thermal conductivity**

| | | Loading case 1 | Loading case 1 |
|---|---|---|---|
| Input | Average temperature gradient | $\langle H_1\rangle \neq 0$ , $\langle H_2\rangle = 0$ | $\langle H_1\rangle = 0$ , $\langle H_2\rangle \neq 0$ |
| | Periodic boundary conditions | $T^{j+} - T^{j-} = \langle H_1\rangle p_1^j$ | $T^{j+} - T^{j-} = \langle H_2\rangle p_2^j$ |
| Output | Average heat flux | $\langle q_i\rangle^{(1)}$ ($i=1,2$) | $\langle q_i\rangle^{(2)}$ ($i=1,2$) |
| | Effective thermal conductivity | $k_{c,i1} = \frac{-\langle q_i\rangle^{(1)}}{\langle H_1\rangle}$ ($i=1,2$) | $k_{c,i2} = \frac{-\langle q_i\rangle^{(2)}}{\langle H_2\rangle}$ ($i=1,2$) |

For the square array and hexagonal array shown in Fig.2, if $N$=1, the first-order approximation formula of effective conductivity is obtained as

$$\frac{k_c}{k_m} = \frac{(\pi - \alpha\eta\lambda)^2}{2\pi^2(1+\eta\lambda) - (\pi + \alpha\eta\lambda)^2} \tag{23a}$$

where $\lambda$ is the fiber volume fraction, and

$$\alpha = \begin{cases} 2 & \text{Square array} \\ 3\sqrt{3}/2 & \text{Hexagonal array} \end{cases} \tag{23b}$$

This formula is useful for engineering application and its accuracy will be verified in the next section.

## 6. Numerical Examples and Discussions

### 6.1. *Convergency analysis*

To examine the convergency of the presented method, the composites with square and hexagonal array of fibers as shown in Fig.2 are considered, in which the fibers are transversely isotropic and the fiber-to-matrix conductivity ratio is taken as $k_f/k_m$=50.

The normalized effective thermal conductivities are listed in Table 2, from which a rapid convergence is observed with increasing the number $N$ of expanded terms, even though the fiber volume fraction $\lambda$ is relatively high (the maximum volume fraction is 0.79 for square array, and 0.91 for hexagonal array). The smaller the value of $\lambda$ is, the more rapidly the results converge.

**Table 2 Variation of the normalized effective thermal conductivity with the term number $N$ of the eigenfunction expansion ($k_f/k_m$=50).**

| $N$ | $k_c/k_m$ | | | | | |
|---|---|---|---|---|---|---|
| | Square array | | | Hexagonal array | | |
| | $\lambda$=0.3 | $\lambda$=0.5 | $\lambda$=0.7 | $\lambda$=0.4 | $\lambda$=0.6 | $\lambda$=0.8 |
| 1 | 1.851 | 3.060 | 6.220 | 2.267 | 3.804 | 8.107 |
| 3 | 1.815 | 2.924 | 6.267 | 2.249 | 3.741 | 8.035 |
| 5 | 1.813 | 2.915 | 6.319 | 2.249 | 3.743 | 8.247 |
| 7 | 1.813 | 2.915 | 6.333 | 2.249 | 3.743 | 8.260 |
| 9 | 1.813 | 2.915 | 6.335 | 2.249 | 3.743 | 8.260 |
| 11 | 1.813 | 2.915 | 6.336 | 2.249 | 3.743 | 8.260 |

### 6.2. *Comparison with existing analytical results*

From Table 2, it can be seen that the numerical results of the presented eigenfunction expansion-variational method converge rapidly and the accuracy is expected to be higher enough when $N$=11. So they can serve as a reference for other approximate methods.

In Figs.3-4, the present numerical results (EEVM, $N$=11) and the first-order approximate results (EEVM, $N$=1) for the square array (Square, SQU) and hexagonal array (Hexagonal, HEX) are compared with the existing results obtained from Rayleigh's approach[4], Hashin's upper and lower bound method[5], SCM[6] and Springer/Tsai's approach[7].

It can be seen that all the results are within Hashin's bounds except Springer/Tsai's results. For the square array, Rayleigh's results are closest to the present numerical results. Whereas, for the hexagonal array, the present first-order approximate results are most close to the present numerical results. The conductivity of square array is higher than that of hexagonal array, for as much as 6.5% for $\lambda$=0.6 and $k_f/k_m$=50. It is worth noting that the hexagonal array can serve as a model of random array, thus the first-order approximation formula for the hexagonal array is useful in engineering application.

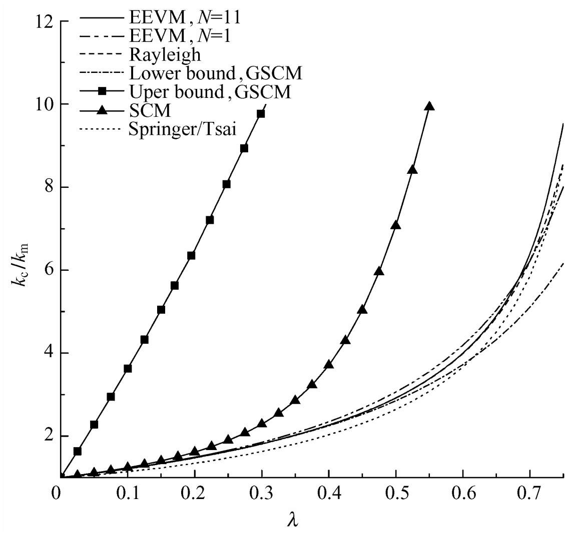


Fig.3 Comparison of present results with existing approximate results (square array, $k_f/k_m$=50).

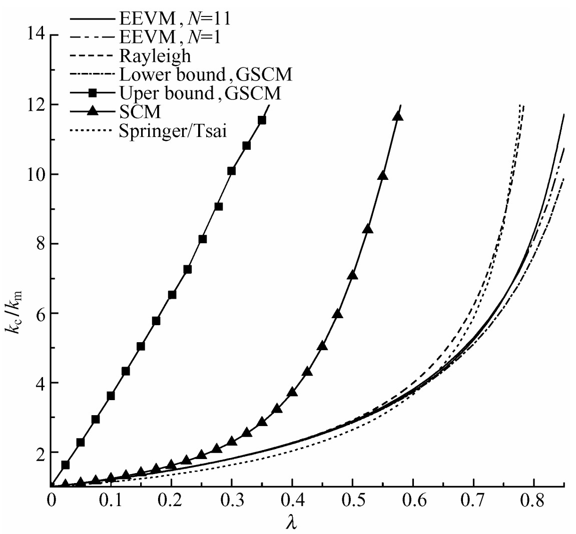


Fig.4 Comparison of present results with existing approximate results (hexagonal array, $k_f/k_m$=50).

### 6. 3. Comparison with experimental data

To our knowledge, the experimental data are very limited. M. W. Pilling, et al.[14] reported the experimental data of the thermal conductivity of carbon/epoxy composites. The comparison of the present first-order approximate results for the hexagonal array with the experimental data is depicted in Fig.5, which demonstrates the validity of the first-order approximation formula ($k_f/k_m$=23).

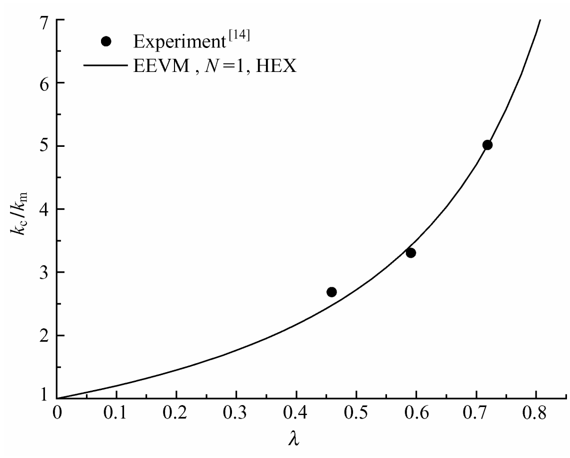


Fig.5 Comparison of present first-order approximate results ($k_f/k_m$=23) for hexagonal array with experimental data[14].

## 7. Conclusions

The two-dimensional steady-state heat conduction of composites with a doubly-periodic array of circular cross-section fibers is studied. A new variational functional for a unit cell is constructed and an eigenfunction expansion-variational method is developed for calculating the effective transverse thermal conductivity. The numerical analysis shows a good convergency of the presented method, even though the fiber volume fraction is relatively high. A first-order approximation formula for the square and hexagonal arrays is presented for engineering application. Comparisons with the existing analytical and experimental results demonstrate the accuracy and validity of the first-order approximation formula for the hexagonal array.

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