Flaw Tolerance in a Viscoelastic Strip

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1 Introduction

It has been well established that micro- and nano-structured materials can exhibit drastically different properties compared to their macroscopic counterparts. For example, while an ideal defect-free solid could, in principle, attain its theoretical strength irrespective of the sample size, the load-carrying capacity of a macroscopic material is always compromised by the inevitable existence of cracklike flaws. As a result, the actual strength of the material is usually size-dependent and can only approach theoretical strength when the sample size is reduced to nanoscale.

The notion that biological systems have achieved flaw tolerance via size reduction and organized structural hierarchy has been extensively discussed in the literature [1–8]. A structure is said to be flaw tolerant if pre-existing flaws do not propagate until it ultimately fails through a uniform rupture near the theoretical strength of the material. This concept can be related to many classical studies in fracture mechanics on notch insensitivity, fracture size effects, large scale yielding, and bridging [9–23] and has been particularly helpful in understanding the hierarchical structures of bone-like materials and biological adhesion systems [1,3–5,7,8,24–32]. It has been shown that biological materials/systems tend to fail by uniform rupture, rather than by crack propagation, due to the selection of characteristic sizes in their hierarchical structures.

Gao and Chen [6] investigated the conditions under which an elastic strip becomes flaw tolerant. The basic length scale of flaw tolerance is defined as $l_{th} = \frac{\Gamma E}{\sigma_f^2}$, where $\Gamma$ is the fracture energy, $E$ is the Young’s modulus, and $\sigma_f$ is the theoretical strength of the material. According to Gao and Chen [6], the Griffith model of crack growth leads to the following critical structural size for flaw tolerance

$$\frac{W_0}{l_{th}} = \min_{0 \leq \beta < 1} \left( \frac{\kappa}{\pi \beta (1 - \beta)^2 F^2(\beta)} \right)$$

where $F(\beta) = (1 - 0.025\beta^2 + 0.06\beta^4)\sqrt{\frac{\pi \beta}{2}}$, $\beta = \frac{a}{W}$

$W$ is the half-width of the strip; $a$ is the half-length of a pre-existing center crack, so that $\beta = a/W$ measures an effective area density of the crack over the cross-section of the strip; here, $\kappa = 1/(1 - \nu^2)$ for the plane strain and $\kappa = 1$ for the plane stress.

In comparison, the corresponding critical size, according to the Dugdale–Barenblatt model, is [6]

$$f(\beta) = \int_0^1 \frac{\sin \pi(\xi + \beta)/2}{\sin \pi(\xi - \beta)/2} d\xi$$

Recently, Kumar et al. [33,34] performed in situ TEM experiments of the tensile fracture in thin aluminum strips with a pre-existing edge notch and observed that the samples sometimes fail away from the notch and there was no measurable stress concentration near the notch tip. Other experiments showed that silk fibril bundles with diameters in the range of 20–150 nm can be scaled up to form macroscopic silk fibers with outstanding mechanical properties in spite of the presence of cavities, tears, and cracks [35].

The mechanical properties of protein-rich biomaterials such as bone are usually time-dependent [36,37] and viscoelastic or poroelastic models are often important for biomedical applications such as developing materials to match the time-dependent behaviors of bone for viscoelastic biocompatibility [38]. In fact, almost all solid biomaterials or tissues exhibit viscoelastic behaviors to some extent. The reader can be referred to Fung’s book [39] for numerous examples.

In the present paper, we investigate the flaw tolerance condition of a center-cracked viscoelastic strip under tension, deriving analytical solutions based on the Griffith and Dugdale–Barenblatt models of crack growth in terms of the loading rate and material constants such as the fracture energy, Young’s modulus, and theoretical strength.

2 Model

The model under consideration involves a viscoelastic strip of width $2W$ containing an interior center crack of length $2a$ subject to a uniaxial, time-dependent tensile loading $\sigma(t)$, as shown in Fig. 1, where $f_c(t)$ is a function representing a time-dependent loading profile.

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form as relation coelasticity (see Fig. 2), with the following uniaxial constitutive

\[ 
\sigma(t) = E(t)^* d\varepsilon(t) = \varepsilon(0)E(t) + \int_{0^+}^{t} E(t - \tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau 
\]

(5b)

where \( \cdot \) denotes the Stieltjes convolution and \( J(t) \) and \( E(t) \) are
creep compliance and stress relaxation functions

\[ 
J(t) = \frac{1}{q_0} \frac{q_1 - p_1 q_0}{q_1 q_0} e^{-q_0 t/\tau_e} = J_\infty - (J_\infty - J_0) e^{-t/\tau_c} 
\]

(6)

\[ 
E(t) = q_0 + \left( \frac{q_1}{p_1} - q_0 \right) e^{-t/p_1} = E_\infty - (E_\infty - E_0) e^{-t/\tau_a} 
\]

(7)

In the preceding expressions, \( E_0 \) and \( E_\infty \) are the instantaneous and
long-time elastic moduli, \( J_0 \) and \( J_\infty \) are the instantaneous and
long-time compliances, \( \tau_0 \) is the relaxation time, and \( \tau_c \) is the
retardation time. One can easily verify the following relations

\[ 
E_0 = \frac{1}{J_0} = E_1 
\]

(8a)

\[ 
E_\infty = \frac{1}{J_\infty} = E_1 \frac{E_2}{E_1 + E_2} 
\]

(8b)

\[ 
\tau_0 = p_1 = E_\infty \tau_c / E_0 = \eta / (E_1 + E_2) 
\]

(8c)

For simplicity, Poisson’s ratio of the viscoelastic strip is taken here as a time-independent constant.

3 Flaw Tolerance Analysis Based on Griffith Model of Crack Growth

3.1 Auxiliary Elastic Model. We first consider the auxiliary problem of an elastic strip of width \( 2W \) containing a central crack of length \( 2a \), as shown in Fig. 3(a). The corresponding viscoelastic problem is shown in Fig. 3(b). In the auxiliary elastic model, \( \sigma^* \) denotes the applied uniaxial stress and \( \sigma^*_e(x,0), \varepsilon^*_e(x,0), \) and \( u^*_e(x,0) \) denote the normal stress, strain, and displacement along the crack plane, respectively. In the viscoelastic model, \( \sigma^e(t) \) denotes the applied load and \( \sigma^e(x,0,t), \varepsilon^e(x,0,t), \) and \( u^e(x,0,t) \) are the normal stress, strain, and displacement along the crack plane, respectively.

The two models satisfy the following relations

\[ 
\sigma^e(t) = \sigma^* f_e(t) = \sigma^* f_e(t) 
\]

(9)

between the applied loads and we have

\[ 
\sigma^e(x,0,t) = \sigma^*_e(x,0) f_e(t) 
\]

(10a)

\[ 
\varepsilon^e(x,0,t) = \varepsilon^*_e(x,0) f_e(t) 
\]

(10b)

\[ 
\varepsilon^e(x,0,t) = \varepsilon^*_e(x,0) f_e(t) 
\]

(10c)

along the crack plane, where \( f_e(t) \) and \( f_e(t) \) are the time-dependent
parts of the stress and displacement, respectively.

In fracture mechanics, the energy release rates for the elastic and viscoelastic models can be written in the form

\[ 
G^e_\infty = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_{0}^{\Delta a} \sigma^e(x,0,t) u^e(x,0,t) dx = K^e_\infty \sqrt{\pi a} f(\beta) 
\]

(11)

and

\[ 
G^e_\infty(t) = \lim_{\Delta a(t) \to 0} \frac{1}{\Delta a(t)} \int_{0}^{\Delta a(t)} \sigma^e(x,0,t) u^e(x,0,t) dx 
\]

(12)

where \( E^e \) is the Young’s modulus in the auxiliary elastic problem,
which can be chosen arbitrarily; here, \( \kappa = 1/(1 - \nu^2) \) for the
plane strain and \( \kappa = 1 \) for the plane stress and \( K^e_\infty = \sigma_\infty \sqrt{\pi a} f(\beta) \)
denotes the elastic stress intensity factor under the external loading \( \sigma_\infty \), where \( F(\beta) \) is defined in Eq. (1). The energy release rate
associated with crack initiation in the viscoelastic material is calculated by substituting Eqs. (10a)-(10c) into Eq. (12). This leads to [40]

\[
G_I(t) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[ \sigma_y(x,0)f_e(t) \right] dx = G_0 f_e(t) f_a(t) 
\]

where \( G_I(t) \) denotes the energy release rate from the corresponding elastic problem in Eq. (11). The functions \( f_e(t) \) and \( f_a(t) \) will be determined from the correspondence principle between the viscoelastic and elastic boundary value problems [41,42].

### 3.2 Energy Release Rate for Crack Initiation in Viscoelastic Material

Applying the Laplace transform to the stress, strain, and displacement functions in Eqs. (10a)-(10c) and the constitutive relation Eq. (3) leads to

\[
\begin{align*}
\bar{\sigma}_y(x,0, s) &= \sigma_y(x,0)f_e(s) \\
\bar{\varepsilon}_y(x,0, s) &= \varepsilon_y(x,0)f_e(s) \\
\bar{u}_y(x,0, s) &= u_y(x,0)f_e(s) \\
\bar{\sigma}(1 + p_1s) &= \bar{\varepsilon}(q_0 + q_1s)
\end{align*}
\]

Here, an overbar on a variable denotes its Laplace transform. Equation (17) corresponds to the transformed uniaxial stress-strain relation.

For a mode I crack, the stress state ahead of the crack tip is equi-biaxial, i.e., \( \sigma_y = \sigma_x \). In this state, the stress-strain relation could be written as

\[
\varepsilon_y = \frac{\rho}{E} \sigma_y 
\]

where \( \rho = (1 - \nu - 2\nu^2) \) under plane strain and \( \rho = (1 - \nu) \) under plane stress.

After the Laplace transform, Eq. (18) becomes

\[
\bar{\varepsilon}_y = \frac{(1 + p_1s)}{(q_0 + q_1s)} \rho \bar{\sigma}_y 
\]

Substituting Eqs. (14) into (19) leads to

\[
\bar{\varepsilon}_y(x,0, s) = \rho \sigma_y(x,0)f_e(s)\left[1 + q_1s\right] \left[q_0 + q_1s\right]^{-1} 
\]

which, after an inverse Laplace transform, results in

\[
\varepsilon_y(x,0, t) = \rho \sigma_y(x,0)L^{-1}\left[1 + q_1s\right] \left[q_0 + q_1s\right]^{-1} 
\]

Here, \( L^{-1}[\cdots] \) denotes the inverse Laplace transform of the bracketed term.

Combining Eqs. (10c), (18), and (21) yields the following relation

\[
f_a(t) = E' L^{-1}\left[1 + q_1s\right] \left[q_0 + q_1s\right]^{-1} 
\]

The time-dependent energy release rate can be obtained from Eqs. (13) and (22) as

\[
G_I(t) = G_0 f_e(t) E' L^{-1}\left[1 + q_1s\right] \left[q_0 + q_1s\right]^{-1} 
\]

Substituting Eqs. (11) into (23), we have

\[
G_I(t) = \frac{(K_I^0)^2 f_e(t)}{\kappa} L^{-1}\left[1 + q_1s\right] \left[q_0 + q_1s\right]^{-1} 
\]

### 3.3 Flaw Tolerant Analysis With Time-Dependent Energy Release Rate

For the center-cracked viscoelastic strip to be flaw tolerant, the energy release rate should not exceed the fracture energy of the material (assumed to be a material constant), i.e.,

\[
G_I(t) \leq \Gamma 
\]

before the strip fails at its theoretical strength \( (1 - \beta)\sigma_s \), regardless of the crack size.

This condition is expressed as

\[
\frac{(1 - \beta)\sigma_s \sqrt{\pi a F(\beta)}}{\kappa} L^{-1}\left[1 + q_1s\right] \left[q_0 + q_1s\right]^{-1} \leq \Gamma 
\]
For a given $\beta$ and loading time $t_0$ at which the external loading reaches the theoretical strength of the strip, Eq. (26) gives rise to a critical strip width $W_{cr}$ from $G_{f}(t) = \Gamma$ as

$$W_{cr} = \frac{\Gamma \kappa}{\pi \beta (1 - \beta)^2 F^2(\beta) \sigma_f^2 f_0(t) L^{-1}\left[\frac{f_0(t)(1 + p_1 \gamma)}{(q_0 + q_1 \gamma)}\right]_{t = t_0}}, \quad (0 \leq \beta < 1)$$

The flaw tolerant width $W_f$ is then defined as

$$W_f = \min_{0 \leq \beta < 1} (W_{cr})$$

Introducing an intrinsic length scale [6]

$$L_{in} = \frac{\Gamma E_{\infty}}{\sigma_f^2}$$

the flaw tolerant size can be normalized as

$$\frac{W_f}{L_{in}} = \min_{0 \leq \beta < 1} \left(\frac{\kappa}{\pi \beta (1 - \beta)^2 F^2(\beta)}\frac{1}{f_0(t)L^{-1}\left[\frac{f_0(t)(1 + p_1 \gamma)}{(1 + q_1 \gamma/q_0)}\right]_{t = t_0}}\right)$$

In the preceding equation, the time independent part within the round bracket on the right side is identical to the solution for the corresponding elastic problem [6]; the remaining part represents the effects of the viscoelastic parameters of the strip and the loading rate.

It follows that the flaw tolerant size of a viscoelastic strip is

$$W_f = \chi \frac{\Gamma E_{\infty}}{\sigma_f^2} \frac{1}{f_0(t)L^{-1}\left[\frac{f_0(t)(1 + p_1 \gamma)}{(1 + q_1 \gamma/q_0)}\right]_{t = t_0}}$$

where $\chi = \min_{0 \leq \beta < 1} (\kappa/\pi \beta (1 - \beta)^2 F^2(\beta))$. Taking $\nu = 0.3$ yields $\chi = 2.1$ for the plane strain and $\chi = 1.91$ for the plane stress.

### 3.4 A Simple Example Case.

As a simple example, consider a bi-linear loading profile to the theoretical strength $\sigma_\infty = (1 - \beta)\sigma_s$, as shown in Fig. 4. The applied loading function

$$\sigma = \sigma_0 f_0(t) = \sigma_\infty \left[\frac{t - (t_0 - t)}{t_0}\right], \quad 0 \leq t < \infty$$

leads to the following displacement function

$$f_0(t) = E' t + (t_0 - t) H(t - t_0) + [H(t - t_0) + 1] (q_1 - p_1 q_0) [1 + e^{-q_0 (t - t_0)/q_1}]$$

where $H(t)$ denotes the Heaviside function: $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geq 0$.

For viscoelastic materials such as butyl rubber, we take the material constants as $E_1 = 115$ MPa, $E_2 = 130$ MPa, $\eta = 0.49$ MPa s, $\Gamma = 0.01$ J/m$^2$, and $\sigma_s = 0.61$ MPa [43], which results in $\tau_0 = 2$ ms. The stress-strain relations for the butyl rubber under different loading rates are shown in Fig. 5, from which one can see that the stiffness of the viscoelastic material increases with the loading rate $\tau_0/t_0$. At very low loading rates, the viscoelastic rubber behaves like an elastic material with a long-time modulus $E_{\infty}$, while it assumes the instantaneous modulus $E_0$ at very high loading rates. Figure 6 shows the relationship between the normalized critical strip width for crack growth versus the relative crack size under different loading rates and plane stress conditions. The flaw tolerance width, corresponding to the minimum strip width below which cracks of all sizes are tolerated, is seen to increase with the loading rate. This is consistent with the fact that a higher loading rate corresponds to a larger Young’s modulus; hence, a larger flaw tolerant size.

![Fig. 4 A bi-linear loading profile in the viscoelastic problem](image)

### 4 Flaw Tolerant Analysis Based on Dugdale Model

#### 4.1 Elastic Solution.

Consider the elastic model shown in Fig. 7, where an elastic strip of half-width $W$ containing a center crack of length $2a$ is subject to an applied uniaxial stress $\sigma_\infty$. The cohesive zone ahead of the crack is $l$, leading to an effective crack length of $2c$ with a crack tip opening displacement $\delta$ and

$$c = a + l$$

Within the cohesive zone, the normal traction $\sigma(\delta)$ is related to the effective range of cohesive interaction $\delta$, as follows

![Fig. 5 The stress-strain relationship of a viscoelastic butyl rubber under different loading rates $\tau_0/t_0$](image)
\[ \sigma(\delta) = \begin{cases} \sigma_s, & 0 < \delta \leq \delta_c \\ 0, & \delta > \delta_c \end{cases} \] (35)

The flaw tolerance solution corresponds to a uniform distribution of the normal stress \(\sigma_s\) outside the crack region. To find the crack tip opening displacement, the original model is divided into three subproblems, as shown in Fig. 8: (1) a perfect crack-free strip subjected to uniform stress \(\sigma_s\), as shown in Fig. 8(a), (2) a center-cracked strip with crack length \(2c\) subject to a uniform normal compressive stress \(\sigma_s\) on the crack face, as shown in Fig. 8(b), and (3) a uniform normal tensile stress \(\sigma_s\) in the cohesive zone of length \(l\), as shown in Fig. 8(c). Since there is no contribution to the crack opening displacement from the first subproblem, the total crack opening displacement from the other two subproblems can be found from the classical distributed dislocation density method as (see the Appendix)

\[ \delta = \frac{8\sigma_s W}{\pi K E^r} I(x) \] (36)

where

\[ I(x) = \frac{c'}{\pi} \cos X \int_0^{\pi/2} \frac{c' \cos X}{\pi \sqrt{1 - c'^2 \sin^2 X}} \ln \left( \frac{\sin \theta - \sin X}{1 - \cos \theta + X} \right) dX \]

Here, \(\sin \theta = a'/c', \ a' = \sin(\pi \beta/2), \) and \(c' = \sin(\pi x/2).\) The normalized effective crack length \(\alpha = c/W\) is deduced in the Appendix as

\[ \alpha = \frac{c}{W} = 2 \arcsin \left( \frac{\sin(\pi/2)}{\cos(\pi \beta/2)} \right) \] (37)

Using the preceding crack opening displacement, the flaw tolerant condition for the elastic strip requires

\[ \sigma_{sc} = (1 - \beta)\sigma_s, \quad x = 1 \] (38)

and

\[ \delta = \frac{8\sigma_s W}{\pi K E^r} I(x) \bigg|_{x=1} \leq \delta_c = \frac{\Gamma}{\sigma_s} \] (39)

The flaw tolerant width is thus
Here, \( \beta = \frac{a}{W} \) is the normalized flaw tolerant width below which, cracks of all sizes are tolerated by the strip. The normalized flaw tolerant width \( W_\text{ft} \) is defined as

\[
W_\text{ft} = \min_{0 \leq \beta < 1} \left( \frac{\delta \pi E \sigma_x}{8 \sigma_x J(1)} \right)
\]

below which, cracks of all sizes are tolerated by the strip. The normalized flaw tolerant width \( W_\text{ft} \) is defined as

\[
W_\text{ft} = \min_{0 \leq \beta < 1} \left( \frac{\pi E \sigma_x}{8 \sigma_x J(1)} \right)
\]

where \( E \) is the tensile modulus, \( \sigma_x \) is the tensile stress, \( J(1) \) is the effective interaction range, \( \delta \) is the effective interaction range, \( \sigma_x \) is the tensile stress, \( E \) is the tensile modulus, \( J(1) \) is the effective interaction range, \( \beta \) is the normalized flaw tolerant width, and \( \pi \) is the mathematical constant.

\[
W_\text{ft} \approx \min_{0 \leq \beta < 1} \left( \frac{\pi E \sigma_x}{8 \sigma_x J(1)} \right)
\]

4.2 Viscoelastic Solution. Graham’s extended correspondence principle \([44,45]\) can be used to determine the viscoelastic solution of the crack opening displacement based on the Dugdale model. This principle, when applied to a moving crack, states that the stress distribution in quasi-static plane problems is the same for elastic and viscoelastic solids under the conditions that: (1) the crack size does not decrease \((\alpha c / \alpha t \geq 0)\), (2) the elastic stress normal to the plane of crack propagation is independent of elastic constants, and (3) any dependence of the normal displacement along the crack face on elastic constants can be written in the separation form \( u_t = f_1(E, \nu) J_2(x) \).

Consider a center-cracked viscoelastic strip subject to uniaxial tensile loading

\[
\sigma(x, t) = \sigma_c f_2(t)
\]

where \( f_2(t) \) is a monotonically increasing function and \( 0 \leq f_2(t) \leq 1 \). It can be shown that all three conditions of Graham’s extended correspondence principle are satisfied in the present model.

Replacing the external loading \( \sigma_c \) in the elastic solution to the effective crack length \( \alpha = c / W \) by the time-dependent loading in the viscoelastic model, we obtain the time-dependent effective crack length directly from Eq. (37)

\[
\alpha(t) = c(t) = \frac{2}{\pi} \sin \frac{\pi \beta^2}{2 \sigma_0 f_2(t)}
\]

Applying the extended correspondence principle to Eq. (36) yields the time-dependent crack opening displacement in the viscoelastic model as

\[
\delta(t) = \frac{8 \pi E \sigma_x}{\pi \kappa} \left[ J(t) I(c(0)) + \int_{0^-}^{t} J'(t - \tau) \frac{\partial I(c(\tau))}{\partial \tau} d\tau \right]
\]

where

\[
\frac{W_\text{ef}}{W_\text{f}} = \min_{0 \leq \beta < 1} \left( \frac{W_\text{ef}}{W_\text{f}} \right)
\]

4.3 Flaw Tolerance Analysis Based on the Dugdale Model. Since the ultimate load carrying capacity of the viscoelastic strip is \( (1 - \beta) \sigma_x \), the load on the remote boundary is taken to be

\[
\sigma = \sigma_c f_2(t) = (1 - \beta) \sigma_c f_2(t)
\]

In the state of flaw tolerance, the crack tip opening displacement remains below the effective interaction range \( \delta_0 \), i.e.,

\[
\delta(t) = \delta_0 \frac{8 \pi E \sigma_x}{\pi \kappa} \left[ J(t) I(c(0)) + \int_{0^-}^{t} J'(t - \tau) \frac{\partial I(c(t))}{\partial \tau} d\tau \right] \leq \delta_0
\]

Here, \( \delta_0 \) is assumed to be a material constant.

For a given \( \beta \) and loading time \( t_0 \), there is a critical strip width \( W_\text{cr} \) for the crack growth, i.e., \( \delta(t) = \delta_0 \).

\[
W_\text{ft} = \min_{0 \leq \beta < 1} \left( \frac{W_\text{ef}}{W_\text{f}} \right)
\]

where \( \delta_0 = \delta E / \sigma_x \) is the intrinsic length \([6]\) and \( J \) is the normalized creep compliance defined as

\[
J = E \chi J
\]
4.4 A Simple Example Case. Consider the same bilinear loading profile as in Eq. (32). The normalized effective crack length is shown to increase with the loading time in Fig. 10. The theoretical strength is reached as the effective crack length reaches the strip width. Figure 11 shows the normalized crack opening displacement versus the normalized loading time for the case of $t_0/\bar{t}_0 = 0.5$, $\beta = 0.33$, and $W/l_0 = 1.83$. The crack opening displacement is found to increase monotonously and nonlinearly with the loading time. It is seen that even when the external loading reaches the theoretical strength, the crack opening displacement is still smaller than the effective interaction distance $\delta_c$, indicating the strip is flaw tolerant under this loading rate.

Figure 12 plots the normalized critical strip width for the crack growth $W_{cr}/l_0$ as a function of the normalized crack size $b$ under different loading rates. For a given loading rate, there always exists a minimum width indicated by the horizontal dashed line labeled as $W_{ft}$, which is the flaw tolerant width below which cracks of all sizes are tolerated. The finding that the flaw-tolerance length scale increases with the loading rate suggests an additional protection mechanism of biological materials against impact loading: a viscoelastic material may be more robust against catastrophic failure at rapid loading rates. This is also consistent with the hypothesis that viscoelasticity could protect bone from dynamic loads (impact and vibration) by dissipating the parts of mechanical energy that could otherwise contribute to structural damage processes [38]. It can also be seen from Fig. 13 that, for very high or low loading rates, the flaw tolerant width approaches the corresponding elastic solution with the instantaneous or long-time Young’s modulus, respectively. The result is consistent with the numerical analysis of flawtolerance between a viscoelastic cylinder and a rigid substrate [31].

5 Discussion

The most elementary level of hierarchical structures of load-bearing biological materials such as bone exhibits a generic structure on the nanometer length scale. At the lowest level of bone organization, type-I collagen ($\sim 250$ nm in diameter) is assembled into fibrils (up to 15 $\mu$m in length and 50–70 nm in diameter), and apatite crystals, $50 \times 25$ nm in length and width and 2–3 nm in thickness, are nucleated at specific regions on or within the collagen fibrils [38]. Previous studies have been devoted to investigating why the elementary structure of biomaterials [1,26,27,46] along with a superhard nanocrystalline coating [47] is generally organized at the nanometer length scale. A central hypothesis adopted in these studies is that the load-bearing biological
materials have been evolved to tolerate cracklike flaws at multiple size scales. The optimal state of a material which induces the maximum strength corresponds to a uniform distribution of stress at failure, even in the presence of cracklike flaws. In this state, the material fails by uniform rupture, rather than by crack propagation. This optimal state can be achieved simply by a size reduction [1,6,8]. Most of the previous studies have been restricted to purely elastic structures. However, as emphasized by Gao [8], the mechanical properties of bone and bonelike biomaterials should be time-dependent and Lakes and Katz [48] have analyzed different physical processes that are responsible for the viscoelasticity of bone. Sasaki et al. [49] showed that the viscoelasticity of bone can be attributed to its protein constituents. Inside the polymeric network of many types of protein, the statistical binding and rupture of cross links result in a small long-time modulus but a relatively stiff instantaneous response [26].

Bone, as an organic-inorganic composite, is generally able to maintain the stiffness of mineral platelets despite its soft viscoelastic protein constituents. This stiffness paradox has been explained by a tension-shear chain model of a mineral-protein composite in which the mineral platelets sustain the tensile load while protein transfers the load between the platelets via shear deformation [26]. The strength of the biocomposite thus hinges upon the strength of the mineral platelets; the latter was shown, using the virtual internal bond method [50], to be maximized and insensitive to flaws by their nanoscale size dimension [1,26]. Protein molecules can undergo large deformations as their domains unfold [1,6,8]. Most of the previous studies have been restricted to purely elastic protein constituents. This stiffness paradox has been explained by a tension-shear chain model of a mineral-protein composite in which the mineral platelets sustain the tensile load but a relatively stiff instantaneous response [26].

The optimal state of a material which induces the maximum strength corresponds to a uniform distribution of stress over all size scales. The optimal state of a material which induces the maximum strength corresponds to a uniform distribution of stress over all size scales. The flaw tolerant width, defined as the minimum critical strip width below which cracks of all sizes are tolerated, is found to increase with the applied loading rate.

The flaw tolerant width, defined as the minimum critical strip width below which cracks of all sizes are tolerated, is found to increase with the loading rate and, in the cases of very high or low loading rates, coincides with the corresponding elastic solutions with an instantaneous or a long-time elastic modulus. This agrees with the previous numerical analysis by Chen et al. [31] on the flaw tolerance of an adhesive interface between a viscoelastic cylinder and a rigid substrate.

The flaw tolerant size of an elastic structure is usually on the nanometer length scale. The question of how this concept can be extended to macroscopic scales through structural hierarchy has been discussed by Gao [8]. The present analysis suggests that the loading rate could also be utilized to switch the material between flaw tolerance and flaw sensitive states. It should be interesting and challenging to further study the behaviors of hierarchical structures of rate-sensitive materials such as bone.

Furthermore, the present analysis should be useful for the design of new implant materials. It is well known that biomaterials need to fulfill a series of requirements. Besides interfacial and biological compatibility, the mechanical properties of an implant should match, as closely as possible, those of the tissue in contact. Besides the quasi-static mechanical properties, the need for an implant material to have similar viscoelastic properties to the tissue is, obviously, important. This means, for example, that an implant in contact should follow the same time-dependent behavior as the tissue when subjected to the same stress/strain history. This is a major reason why the viscoelastic properties of new materials or devices for medical applications should be carefully evaluated [38]. Since bone is a typically hierarchical and viscoelastic material, it remains a challenge to develop robust implants with similar mechanical properties as the natural bone.

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Appendix: Approximate Solutions to Crack Opening Displacement and Cohesive Zone Length in a Cracked Elastic Strip

For a finite Dugdale crack in an infinite space shown in Fig. 14, the tractions on the crack face are divided into two regions

$$\sigma(x) = \begin{cases}
\sigma_\infty, & |x| < a \\
\sigma_i - \sigma_\infty, & a < |x| < c
\end{cases}$$

(A1)

In order to find the crack opening displacement, the classical distributed dislocation density method [54] can be used. For a dislocation density $f(x)$ over $-c < x < c$, we have the relation

$$A \int_{-c}^{c} f(x) dx = \sigma(x), \quad |x| < c$$

(A2)

where $A = b x E / 4 \pi$ and $b$ is the Burgers vector. Equation (A2) can be inverted as

$$f(x) = -\frac{1}{\pi^2 A} \int_{-c}^{c} \frac{\sqrt{x^2 - c^2} \sigma(\xi) d\xi}{\xi}, \quad |x| < c$$

(A3)

which can be further rewritten as

$$f(x) = -\frac{\sqrt{x^2 - c^2}}{\pi^2 A} \int_{-c}^{c} \frac{\sigma(\xi) d\xi}{(x - \xi) \sqrt{\xi^2 - c^2}} + \frac{1}{\pi^2 A} \int_{-c}^{c} \frac{\xi \sigma(\xi) d\xi}{\sqrt{\xi^2 - c^2}}$$

(A4)

$$+ \frac{1}{\pi^2 A} \int_{-c}^{c} \frac{\sigma(\xi) d\xi}{\sqrt{\xi^2 - c^2}}$$

Fig. 14 Dugdale model of a finite crack in an infinite space.

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When \( \sigma(\xi) \) is an even function of \( \xi \) and \( f(x)|_{x=-\xi} = 0 \), the second and third terms in the preceding equation vanish so that
\[
f(x) = -\frac{\sqrt{x^2 - c^2}}{\pi^2 A} \int_{-\infty}^{\infty} \frac{\sigma(\xi)d\xi}{(x-\xi)\sqrt{\xi^2 - c^2}} \quad (A5)
\]
The crack opening displacement is found by integrating the dislocation density from \( c \) to \( a \)
\[
\delta = bN = \int_{c}^{a} f(x)dx = \frac{2b\sigma_\infty a}{\pi^2 A} \ln \frac{c}{a} + \frac{8\sigma_\infty}{\pi \sqrt{E} I} \ln \frac{c}{a} \quad (A6)
\]
Meanwhile, the requirement that the third term on the right side of Eq. (A4) should vanish leads to
\[
\int_{-\infty}^{\infty} \frac{\sigma(\xi)d\xi}{\sqrt{\xi^2 - c^2}} = 0 \quad (A7)
\]
from which the length of the cohesive zone ahead of the crack tip can be deduced as
\[
-\frac{\pi}{2} \sigma_\infty + \sigma_\infty \arccos \frac{a}{a+l} = 0
\]
For a finite-width strip model with a center crack, the solution to a periodic array of cracks can be adopted to approximate the finite width problem, as shown in Fig. 15. In this case, we have
\[
A \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(\lambda)d\lambda}{x-\lambda-2mW} = \sigma(x) - c + 2mW < x < c + 2mW \quad (A9)
\]
Using the following equation
\[
\sum_{n=-\infty}^{\infty} \frac{1}{x - \lambda - 2mW} = \cot \frac{\pi(x - \lambda)}{2W} \quad (A10)
\]
Eq. (A9) can be simplified as
\[
\frac{\pi}{2W} \int_{-\infty}^{\infty} \frac{\cos \frac{\pi x}{2W} f(\lambda)d\lambda}{\sin \frac{\pi x}{2W} - \sin \frac{\pi \lambda}{2W}} = \frac{\sigma(x)}{A} \quad (A11)
\]
Using the following variable transformations
\[
\lambda' = \sin \frac{\pi \lambda}{2W}, \quad x' = \sin \frac{\pi x}{2W} \quad (A12)
\]
Eq. (A11) becomes
\[
\int_{-\infty}^{\infty} \frac{M(\lambda')d\lambda'}{x' - \lambda'} = \frac{\sigma(x')}{A} \quad (A13)
\]
where \( M(\lambda') = f(\lambda) \) and \( \sigma(x') = \sigma(x) \).
Since Eq. (A13) has the same form as Eq. (A1), the solution can be written as
\[
M(x') = -\frac{1}{\pi^2 A \sqrt{x'^2 - c'} + 1} \int_{-\infty}^{\infty} \sqrt{x'^2 - c'^2} a'(\lambda')d\lambda'
\]
\[
- c + 2mH < |x'| < c + 2mH
\]
where \( c' = \sin (\pi c/2W) \).
In a manner similar to the single crack case, the crack opening displacement is
\[
\delta = \frac{8\sigma_\infty W}{\pi \sqrt{E} I} I(c) \quad (A15)
\]
where
\[
I(c) = \int_{0}^{\pi/2} \frac{c' \cos X}{\sin \theta - \sin X} \ln \left( \frac{\sin \theta - \sin X}{1 - \cos(\theta + X)} \right) \frac{dX}{\sin \theta + \sin X} \left( 1 - \cos(\theta - X) \right)
\]
Here, \( \sin X = x'/c' \), \( \sin \theta = d'/c' \), and \( d' = \sin (\pi a/2W) \).
The length of the cohesive zone in the finite-width strip model is
\[
-\frac{\pi}{2} \sigma_\infty + \sigma_\infty \arccos \frac{d'}{c} = 0 \quad (A16)
\]
which can be rewritten in the form
\[
\arcsin \frac{\pi b}{2c_{\infty}} = \frac{\pi}{2} \frac{\arcsin \frac{d}{c}}{\cos \frac{\pi b}{2c_{\infty}}} \quad (A17)
\]
References