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THE MECHANICAL CONDITION OF SHEAR BAND BIFURCATION

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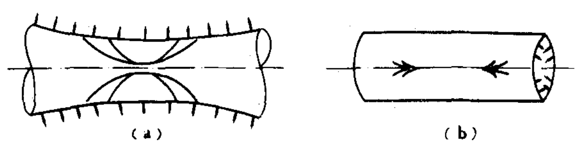
ABSTRACT: Based on the understanding of the role played by the strain-softening .fect in the formation of shear band bifurcation, this paper investigates (a) What is the most favourable condition that stimulates the occurrence of shear band? (b) With what model and characterizing parameters can the curved type of band bifurcation be simulated?

KEYWORDS: shear band, bifurcation, ductile, damage

I. INTRODUCTION

One of the salient features of damage in ductile material is known as the localized band type of bifurcation. The understanding of the influencing factors and the favourable condition that affect the formation of such band is then of practical importance to controlling damage and improving the ductility of material.

The macroscopic form of shear band phenomenon in material sheet testing had been observed by Chakrabarti and Spretnak[1] and they concluded that the necessary condition of this type of localized deformation was for the true stress to attain a maximum. In the tests with hydrostatic pressure, curved shear band was seen near or at the moment of material rupture[2][3], as demonstrated in Fig. 1.



(a) Necking of copper bar under uniaxial tension and hydrostatic compression

(b) Curved cracks in glass tube under hydrostatic pressure

Fig. 1 Examples of curved shear band

Besides those stated above, much smaller localized deformation bands do exist in high strength steel as examined by Anand and Spitzig[4], their width is about 1 μm with the length of the order of 100 μm. Another kind of microstructural band is formed by the linking of tiny voids (~ 1 μm) between larger voids, as shown by Hancock and Cowling[5]. It doesn't matter whether the bands are large or small in size or with whatever microstructure, the different types of bands have a common feature that there is a large deformation localized in the band.

Based on the understanding of the role played by the strain-softening effect in the formation of such shear band bifurcation[6], this paper investigates (a) What is the most favourable condition that stimulates the occurrence of shear band, (b) With what model and characterizing parameters can the curved type of band bifurcation be simulated.

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II. THE PRE-BIFURCATION STRESS/STRAIN STATE AND THE BIFURCATION EQUATION

If the bifurcated shear plane runs through a material point o in a continuum and the normal of this plane is defined as $\boldsymbol{b}$ as shown in Fig. 2, let the principal stress be stated by

$$\sigma_1(=\sigma) \geqslant \sigma_3(=\phi\sigma) \geqslant \sigma_2(=\beta\sigma) \tag{1}$$

and assume the pre-bifurcation stressing to be proportional, we are then able to apply the deformation type of theory in plasticity to the stress-strain relation. So that

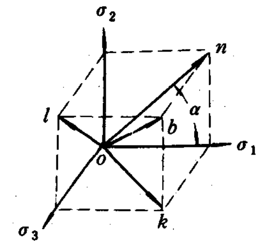


Fig. 2

$$\begin{aligned}
\varepsilon_1 &= \frac{\sigma}{E}\left[1 - \nu(\beta+\phi) + \frac{\psi}{3}(2-\beta-\phi)\right] = \varepsilon \\
\varepsilon_2 &= \frac{\sigma}{E}\left[\beta - \nu(1+\phi) + \frac{\psi}{3}(2\beta-1-\phi)\right] \\
\varepsilon_3 &= \frac{\sigma}{E}\left[\phi - \nu(1+\beta) + \frac{\psi}{3}(2\phi-1-\beta)\right]
\end{aligned} \tag{2}$$

where E, ν are the elastic modulus and the Poisson's ratio, respectively.

$$\psi = \frac{3}{2}\frac{E}{E_{se}} - (1+\nu) \tag{3}$$

E_{se} is the secant modulus on the equivalent stress-strain curve $(\sigma_e - \varepsilon_e)$. For the exponential-hardening material it follows that

$$\frac{E_{se}}{E} = \left(\frac{\sigma_e/\sigma_y}{\varepsilon_e/\varepsilon_y}\right) = \left(\frac{\varepsilon_e}{\varepsilon_y}\right)^{n-1} \tag{4}$$

where σ_y, ε_y are the yield stress and the yield strain respectively and n is the exponent parameter of the material concerned. According to the definition of the equivalent stress, we have

$$\sigma = \sigma_e/(1-\beta-\phi-\beta\phi+\beta^2+\phi^2)^{1/2} \tag{5}$$

In the following examples we shall mainly analyse the cases where σ_1 and σ_2 constitute the maximum shear stress, that is $\phi \geqslant \beta$. At the moment of the occurrence of any arbitrarily curved shear band (including straight ones), let the velocity variation within the band differ from that outside the band by

$$\delta V_i = v_i(\boldsymbol{n}, \boldsymbol{t}) \tag{6}$$

in which, $\boldsymbol{n}$ is the unit normal of the band plane,

$$\boldsymbol{n} = n_i \boldsymbol{x}_i \tag{7}$$

$\boldsymbol{t}$ is the unit vector along the tangential direction, and

$$\boldsymbol{t} = t_i \boldsymbol{x}_i \qquad (i = 1, 2, 3) \tag{8}$$

From the requirement of equilibrium, the variation of the nominal stress rate in the band should satisfy[6]

$$\delta \dot{T}^{ij}{}_{i} = \left[\frac{D\delta\tau^{ij}}{Dt} - \sigma^{ik}\delta D_k^j - \sigma^{jk}\delta D_k^i + \sigma^{ik}\delta V^j|_k \right]\Bigg|_i = 0 \tag{9}$$

where $|_i$ is the symbol for covariant derivative, σ^{ik} is the true stress tensor, $\delta D_k^i = \delta D_{jk} g^{ij} = \frac{1}{2}(\delta V_j|_k + \delta V_k|_j)g^{ij}$, g^{ij} is the metric tensor and $D\delta\tau^{ij}/Dt$ denotes the Jaumann rate of the variation of Kirchhoff stress. Based on the plastic dilatational theory[7] for the constitutive description of damaged material, we have

$$\frac{D\delta\tau^{ij}}{Dt} = L^{ijkl*}\delta D_{kl} \tag{10}$$

where
$$L^{ijkl*} = \frac{E}{1+\nu}\left[\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) + g^{ij}g^{kl}\frac{-E/3E_{tm}^{(p)}}{1-2\nu+E/E_{tm}^{(p)}} - \frac{3}{2\sigma_e^2}\frac{S^{ij}S^{kl}}{1+2(1+\nu)E_{te}^{(p)}/3E}\right]$$

S^{ij} is the deviatoric stress tensor, $E_{te}^{(p)}$, $E_{tm}^{(p)}$ are the plastic tangent moduli on the equivalent stress-strain curve $(\sigma_e - \varepsilon_e)$ and the mean stress-strain curve $(\sigma_m - \varepsilon_m)$, respectively. During the strain-hardening stage $E_{te}^{(p)}$ and $E_{tm}^{(p)}$ are positive, but when strain-softening effect takes control they become negative. The asterisk attached to the stiffness L^{ijkl} represents the possible bifurcation of material behaviour which is simultaneous with the occurrence of velocity variation within certain field of the material. The method for the solution of a shear band bifurcation, with the equations in (6)—(10) under uniform stressing as described by (1)—(5), is similar to that stated previously[6], will not be recited here for brevity.

III. PLANE-STRAIN CONDITION

On the assumption that the velocity disturbance and the material behaviour at bifurcation vary only along the normal direction while keeping constant along the tangential side, from Eq. (6) in Cartesian coordinates we have

$$\delta V_{i,j} = v_i' n_j, \; n_1 = \cos\alpha, \; n_2 = \sin\alpha, \; n_3 = 0 \tag{11}$$

and
$$m = n_2/n_1 \tag{12}$$

where α is the angle between the normal vector $\boldsymbol{n}$ and the coordinate axis x_1, the preceding comma before a suffix j denotes the partial derivative with respect to the coordinate axis x_j, the prime denotes derivative with respect to the normal $\boldsymbol{n}$. The proportional stressing condition before bifurcation is now only approximately satisfied. It is easy to derive from Eq. (2) that under plane-strain condition, $\varepsilon_3 = 0$,

$$\phi = (1+\beta)(\nu + \psi/3)/(1+2\psi/3) \approx (1+\beta)/2$$

since in plastic case $\psi \gg 1$. When β is fixed, ϕ is only approximately constant.

The critical strain ε, maximum shear strain $\gamma_m = \varepsilon_1 - \varepsilon_2$, critical stress $\sigma_1 = \sigma$, maximum shear stress $\tau_m = (1-\beta)\sigma/2$ and the angle α of the shear band are related to the proportional stressing parameter β and ϕ under plane-strain condition at bifurcation with $E/E_{te}^{(p)} = -100$. Fig. 3 shows the results.

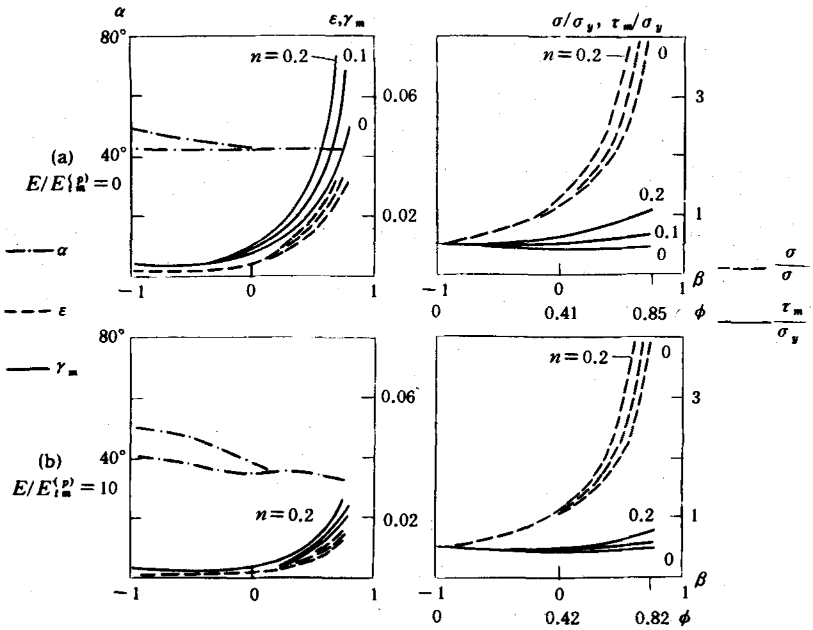


Fig. 3 The relations of the critical strain ε, maximum shear strain γ_m, critical stress σ, maximum shear stress τ_m and the angle α to the parameters β and ϕ under plane-strain condition with $E/E_{te}^{(p)} = -100$.

The results of material bifurcation without plastic dilatation are depicted in the case (a) of Fig. 3, whilst those with plastic dilatation are shown in (b). All the listed results are restricted to $\phi \geqslant \beta$. Calculations with the other parameters of $E/E_{te}^{(p)} = -1000, -10$ have similar trends as shown in Fig. 3. We can then conclude:

(a) The maximum shear stress τ_m at bifurcation is rather insensitive to the biaxial loading condition, characterized by the parameter β.

(b) m may have two real roots in the compression loading case, hence two values of α can be seen in this part.

(c) Plastic dilatation can further reduce the critical values of stress/strain.

IV. AXISYMMETRIC CONDITION

Let the axial and radial variations of velocities in this case be

$$\delta\dot{w} = v_z(\boldsymbol{n}) \qquad \delta\dot{u} = v_r(\boldsymbol{n}) \tag{13}$$

respectively, and

$$n_z = \cos\alpha \qquad n_r = \sin\alpha \qquad m = n_r/n_z \tag{14}$$

Owing to the fact that drastic variation of velocity mainly occurs along the normal to the band, when

$r \gg 0$ we find

$$v_i'' \gg v_i'/r \gg v_i/r^2 \qquad (i = z, r)$$

In search of an asymtotic solution, it is possible to neglect all the lower order terms in the governing equation of bifurcation. The final results have an equation form similar to that in the previous case of plane-strain condition. Physically, this means that for a very thin layer of shear band ring, if the radius of that ring is large enough (i.e. $r \gg 0$) then the influence of the curvature and the circumferential strain rate can be neglected.

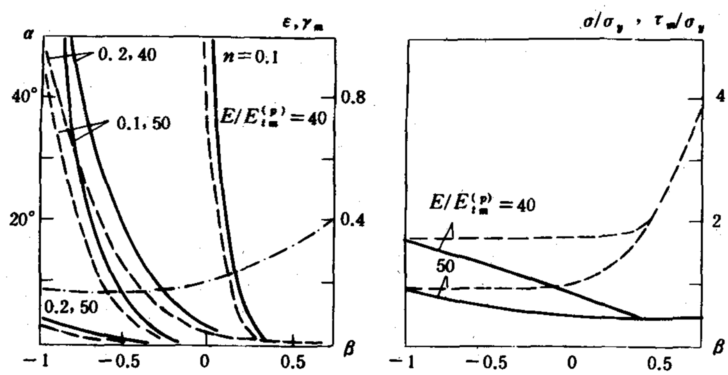


Fig. 4 The relations of the critical strain ε, maximum shear strain γ_m, critical stress σ, maximum shear stress τ_m and the angle α to the parameter β in axisymmetric case with $E/E_{te}^{(p)} = -100$.

Fig. 4 shows the variations of the critical stress / strain and the shear band angle with respect to the radial stressing parameter β. The calculations had actually covered a wide range of softening parameters $(E/E_{te}^{(p)} = -1000, -100, -10, -4)$, only the typical case of $E/E_{te}^{(p)} = -100$ is given here. The general trend in the axisymmetric case has some striking contrast to that of the previous one in plane-strain condition, that is,

(a) If the whole range of softening is classified as normally damaged $(E/E_{te}^{(p)} = -1000 \sim -10)$ and seriously damaged $(E/E_{te}^{(p)} = -10 \sim -1)$, we can say that the axisymmetric loading is able to trigger shear band bifurcation only when the normally damaged case is accompanied by large plastic dilatation, unless if the latter damage condition becomes dominating, and in that case plastic dilatancy would not be so indispensable. There is no such limitation in the plane-strain loading condition.

(b) Only one root value for m is found in the cases shown in Fig. 4. So there is only one curve for α.

V. CURVED SHEAR BAND

It will be exemplified in this section that instead of straight band, curved band-type bifurcation can also occasionally seen and that may be possibly explained by using the notion that the material behaviour at bifurcation varies along the tangential side of the band. To make the solution feasible, we focussed on asymtotic analysis. Therefore the following assumptions can be taken as acceptable: (a) Only the highest order of derivative of the velocity variation with respect to the normal $\boldsymbol{n}$ needs to be retained. (b) The stiffness parameter in the band varies smoother than that of the velocity variation field at bifurcation, then

$$\left(L^{ijkl*}\,\delta D_{kl}\right)_{,i} = \left(L^{ijkl*}\right)_{,i}\delta D_{kl} + L^{ijkl*}\left(\delta D_{kl}\right)_{,i}$$
$$\approx L^{ijkl*}\left(\delta D_{kl}\right)_{,i}$$

With these treatments the asymtotic equation for curved shear band analysis results in the same form as that applied to the previous cases, except the bifurcated stiffness parameter in the band is not constant but varies along the tangential side.

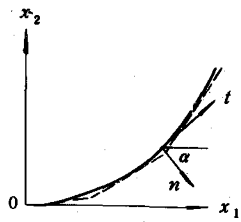


Fig. 5 Curved band

Furthermore, we can approximately substitute any curve by a series of segmented lines, as shown by the straight, broken lines in Fig. 5. Within each part of the segmented line, the bifurcated material behaviour is taken to be the same, however it may change from part to part along the

Table 1

Shear band angle α under axisymmetric loading

$(E/E_{im}^{(p)} = 0,\ \beta = -1 \sim 0.75)$

$-E/E_{te}^{(p)}$	0.958	1.25	1.33	1.43	1.67	2.50	3.33	4.00
α	90	72	70	68 5	64 14	55 26	49 33	42

Table 2

Shear band angle α under plane-strain loading

$(E/E_{te}^{(p)} = 0,\ \phi > \beta)$

β \ $-E/E_{te}^{(p)}$	1	1.11	1.25	1.67	2.50	4.00	6.67	10.0
0.75						43	43	43*
0.50						48	45	43*
0		87	79	71 14	64 22	59 28	55 32	52 38
−0.50			83	73 15	66 22	61 28	57 32	54 35
−1.0			85	74 16	67 23	62 28	58 32	55 35

tangential direction. Therefore, if it is able to find out a continuous variation of the shear band angle α with respect to some corresponding material softening conditions, then it may indicate that there is velocity bifurcation located within the appointed curve band that suffers different levels of damaged condition along its tangential side. As only plane-strain mode of bifurcation is concerned so the curved band plane is perpendicular to the plane composed of the axes x_1 and x_2.

In Table 1 and Table 2, the results of the plane-strain mode of curved band bifurcation under axisymmetric and plane-strain pre-bifurcation loading are listed, respectively.

In the case of axisymmetric loading, the value of the angle α (in degrees) is insensitive to the variation of β, so only one line of α values is listed in Table 1. It is obvious that there are two groups of solution for α in each case, one is from 90° to 45° and the other is from 0° to 45°. This indicates that if there is a variation of bifurcated material behaviour, following the data given in the top line in each of the two Tables, as its consequence, the band-type bifurcation is likely to occur in that curved plane. Except the data attached with asterisk, most of the critical strains listed in Tables 1 and 2 are approximately near to the yield strain ε_y. This means that once the material is yielded the curved band bifurcation becomes possible if the material is actually and simultaneously suffering the corresponding damage localized in that band. This can be true for brittle materials. However, for ductile materials the situation is that in order to accumulate enough damage which may yield the corresponding softening tangent modulus listed in Tables 1 and 2 it may take a long period of straining. In that case, this kind of band localization can only be seen when the material is near to rupture.

VI. THREE DIMENSIONAL SOLUTION

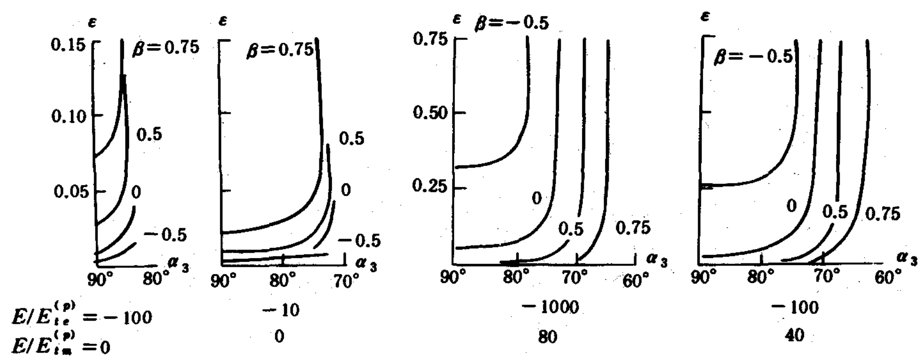
All the previous results are based on the assumption of the plane-strain mode in bifurcation, i.e. $n_3 = \delta V_3 = 0$. We shall abandon this restriction and study the influence of n_3 on shear band bifurcation. Furthermore, the range of the stressing condition before bifurcation will also be extended to include other cases than just the axisymmetric or plane-strain loading already studied. Generally speaking, three dimensional analysis will be considered in this section.

If bifurcation occurs in a plane with its normal as $\boldsymbol{n}$, then according to Eq. (7)

$$n_1 = \cos\alpha_1 \qquad n_2 = \cos\alpha_2 \qquad n_3 = \cos\alpha_3$$

and

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{15}$$



(a) Plane-strain loading (b) Axisymmetric loading

Fig. 6 Critical strain ε versus shear band component α_3 (strain-hardening exponent $n = 0.2$)

where $\alpha_1, \alpha_2, \alpha_3$ are respectively the angles between the normal and the axes x_1, x_2 and x_3.

When a certain bifurcated material behaviour is determined, then with each appointed group of (n_1, n_2, n_3) the corresponding critical stress / strain can be obtained by solving the bifurcation Eq. (9). Among these, the lowest critical value will be taken as the actual one. Fig. 6 shows the dependence of the critical strain ε on α_3 (when $\alpha_3 \leqslant 90°$ then $n_3 \geqslant 0$). Two cases are considered for pre-bifurcation loading, namely, plane-strain and axisymmetric conditions. The results indicate that the mode of $n_3 = 0$ yields the lowest critical value.

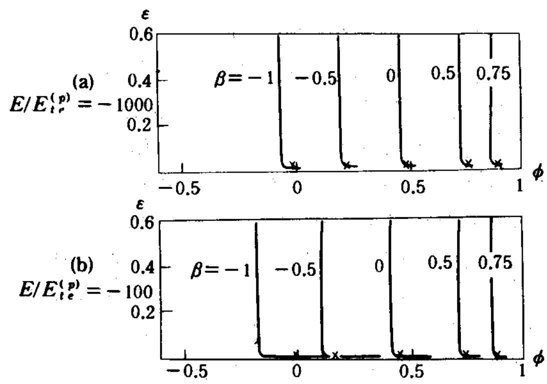


Fig. 7 Critical strain ε versus pre-bifurcation loading parameter (plane-strain loading marked with x, $n = 0.2$, $E/E_{tm}^{(p)} = 0$)

The next part is focussed on studying the consequences of different pre-bifurcation loadings. Here the bifurcation mode is fixed to $n_3 = 0$. The variation of the critical strain ε with respect to the pre-bifurcation loading parameter ϕ and β is shown in Fig. 7. For each value of β there is a small range of ϕ that the critical value grows sharply. The value of ϕ that corresponds to plane-strain loading lies always in the lowest and smooth part of the curve. This smooth part can be widened when strain-softening effect at bifurcation is intensified (compare the (a) and (b) of Fig. 7).

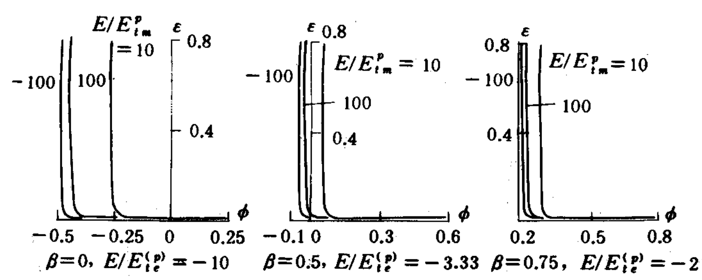


Fig. 8 The enlargement of the range of bifurcation by plastic dilatancy ($E/E_{tm}^{(p)} \neq 0$, $n = 0.2$)

It is interesting to see that from either the pre-bifurcation loading condition or the shear band angle that yields the lowest critical strain, plane-strain condition (including loading and bifurcation mode) is the most favourable case for stimulating shear band. Finally we can see from Fig. 8 that, besides the intensification of strain-softening effect that can enlarge the range of ϕ where bifurcation exists, plastic dilatancy also favours this trend.

VII. CONCLUSIONS

The significance of present study is to show that:

1. Plane-strain loading and plane strain mode of velocity variation are the most favourable conditions for shear band bifurcation. Therefore, if the loading is axisymmetric then shear band bifurcation can only be seen when material is extremely damaged.
2. Curved shear band can be simulated by considering the variation of the bifurcated material behaviour along the tangential direction of the band.
3. The results of this paper has the prerequisite that the bifurcated material behaviour (strain-softening effect and plastic dilatation) has already been determined. However, this point itself actually needs to be answered from the microstructural study of material, which belongs to a field beyond the scope of this paper. It can be expected that only when the material property factors are also included, this study of the mechanical condition of shear band bifurcation will be complete.

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