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A New Limiter for Hybrid Grid

H. W. Zheng<sup>a,\*</sup>

<sup>a</sup>LHD, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China

**Abstract**

For the finite volume method, the reconstruction step is employed to obtain the states for the calculation of numerical fluxes at the faces. To remove the non-physical oscillations, a limiting procedure is required. This limiting procedure is so important that it not only influence the numerical accuracy in the smooth regions but also affect the robustness of the solver. For the unstructured meshes, the design of the limiting procedure is not trivial. The well-known and wide-used limiting procedure is proposed by Barth and Jespersen in 1989 and lately improved by Venkatakrisnan in 1993. However, this method is too dissipative and the overshoot or undershoot phenomenon can still be observed. In this paper, a new limiter for hybrid grid is proposed. It limits the state variables of a face directly from the corresponding variables of this face's neighbor cells. It is so simple that it can be easily adopted in many solvers based on unstructured grid.

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*Keywords:* finite volume, multi-dimensional limiter, hybrid grid.

**Nomenclature**

$u$	velocity (m/s)
$E$	the total energy
$F$	the flux
$F^c$	the convective flux
$F^v$	the viscous flux
$P$	the pressure
$q$	the heat flux
$U$	the state vector of conservative variables, $U = (\rho, \rho u, E)^T$
$W$	the vector of primitive variables, $W = (p, u, T)^T$
$R$	the residual

*Greek symbols*

\* Corresponding author. Tel.: +86-010-82543976; fax: 86-010-82543976.  
E-mail address: [h.zheng@imech.ac.cn](mailto:h.zheng@imech.ac.cn)

$\alpha, \beta$	parameters for the limiter function
$\gamma$	the limiter function
$\Delta$	the difference
$\phi$	the limiter
$\psi$	the limiter function
$\rho$	the density(N/m <sup>3</sup> )
$[\tau]$	the stress tensor

**1. Introduction**

A main computational challenge with nonlinear hyperbolic equations is the resolution of discontinuities. However, any linear scheme higher than first order accuracy cannot generate monotonic solutions. Hence, the non-linear limiting function is introduced to avoid numerical oscillations [1-5].

A good limiting function should be able to remove the non-physical oscillations nearby the shock and can also preserve the numerical accuracy in the smooth regions. Moreover, the limiter function should not affect the convergence to the steady state. Barth and Jespersen (1989) introduced the first limiter for unstructured grids. The Barth and Jespersen limiter is used to enforce a monotone solution. The main idea of their work is to avoid introducing oscillation is that no new local extrema are formed during reconstruction. The scheme consists of finding a value in each control-volume that will limit the gradient in the piecewise linear reconstruction of the solution. However, their method is rather dissipative which leads to smear discontinuities. Furthermore, the limiter may be active in smooth flow regions due to the numerical noise, which causes difficulties for steady state convergence. To improve the convergence, Venkatakrishnan (1993) proposed a smooth differentiable alternative of the minimum function in Barth-Jespersen. However, it does not preserve strict monotonicity, slight oscillations can be observed near shock discontinuities. Moreover, similar to Barth and Jespersen (1989), it is quite dissipative that predicted accuracy also cannot be guaranteed with the fixed stencil when these limiters are used.

In this work, a new multidimensional limiting procedure is proposed to limit the gradient in each direction independently. For each face of one cell, only the gradient along the direction between the two centroids is limited. This will reproduce a limited difference for each face. After having these limited differences, the unlimited differences and the limited differences for each face are limited secondly. This produces a new limited difference which is multidimensional in its very nature. The rest of paper is organized as follows. The numerical method is described in section 2. The numerical experiment is presented in section 3. The final section gives a summary about the main work of the paper.

**2. Numerical Method**

In this paper, the steady Farve-averaged Navier–Stokes equations are considered,

$$\partial_t \int_{\Omega} U dV + \int_S F \cdot \bar{n} dS = 0 \tag{1}$$

Where U is the state vector and F is the flux vector

$$U = \begin{pmatrix} \rho \\ \rho \bar{u} \\ E \end{pmatrix}, F = F^c - F^v, F^c = \begin{pmatrix} \rho \bar{u} \\ \rho \bar{u} \otimes \bar{u} + P[I] \\ (E + P)\bar{u} \end{pmatrix}, F^v = \begin{pmatrix} 0 \\ [\tau] \\ ([\tau] \cdot \bar{u}) + \bar{q} \end{pmatrix} \tag{2}$$

Here,  $\rho$  is the density,  $u$  is the velocity,  $E$  is the total energy,  $P$  is the pressure,  $[I]$  is the identity tensor,  $q$  is the heat flux,  $[\tau]$  is the stress tensor,

$$[\tau] = (\mu_L + \mu_{tur}) \left\{ \nabla \bar{u} + \nabla^T \bar{u} - \frac{2}{3} (\nabla \cdot \bar{u}) [I] \right\} \quad (3)$$

The numerical semi-discretization of Eq. (1) is

$$\frac{\partial}{\partial t} \int_{\Omega} U dV = R \approx \sum_f \left[ F_f^c(U_f^-, U_f^+, \bar{n}_f) - F_f^v(U_f^-, U_f^+, \bar{n}_f) \right] \cdot A_f \quad (4)$$

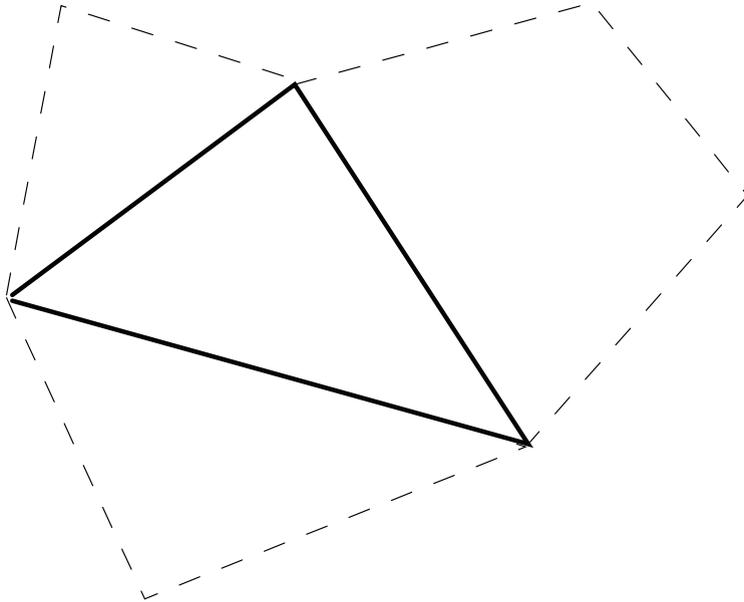


Fig. 1. Illustration of hybrid mesh.

To increase the solution accuracy in space, the left and right states are constructed from extrapolated values from cell centers to cell interfaces (see Fig 1) and then used to construct fluxes. To make the solution be monotonic, the slope limiters are enforced in the extrapolation. For example, Barth-Jespersen's reconstruction and limiter reads

$$W_f^k = W^k + \phi^k \bar{\nabla} W^k \cdot d\vec{r}^k, \quad k=-,+ \quad (5)$$

With

$$\phi^k = \min_{i \in N(k)} \phi^{ki} \quad (6)$$

and

$$\phi^{ki} = \begin{cases} \psi(1.0, \Delta_+ / \Delta_-), \Delta_+ = \max_{i \in N(k)} W^i - W^k & \text{if } \Delta_- > 0 \\ \psi(1.0, \Delta_+ / \Delta_-), \Delta_+ = \min_{i \in N(k)} W^i - W^k & \text{if } \Delta_- < 0 \\ 1.0 & \text{if } \Delta_- = 0 \end{cases} \quad (7)$$

In Barth-Jespersen’s method, they use a non-differential limiter  $\psi(1,y)=\min(1,y)$ . This adversely affects the convergence properties of the solver. For this reason, Venkatakrishnan introduces a smooth alternative of the minimum function in Barth-Jespersen procedure,

$$\psi(1.0, \Delta_+ / \Delta_-) = \frac{(\Delta_+^2 + \varepsilon^2) + 2\Delta_- \Delta_+}{\Delta_+^2 + 2\Delta_-^2 + \Delta_- \Delta_+ + \varepsilon^2} \quad (8)$$

Although the Venkatakrishnan limiter is used to prevent the non-physical oscillations nearby the shock region, the overshoot or undershoot phenomenon can still be observed. Moreover, the numerical accuracy is degraded by using Venkatakrishnan limiter. Besides, it could be easily observed that the gradient in Barth-Jespersen’s and Venkatakrishnan version is limited by multi-plying a scalar limiter  $\phi$ . That is, the limiter is the same for each direction. This shows they may be too dissipative.

Hence, in this paper, a new multidimensional limiting procedure is proposed to limit the gradient in each direction independently. The main idea is to limit the gradient normal to the face direction for each face. For each face of one cell, only the difference along the direction between the two centroids is limited,

$$\Delta^{k,\text{lim}} = \chi^k L(\Delta^{k,-}, \Delta^{k,+}) \quad (9)$$

with

$$\begin{aligned} \Delta^{k,-} &= 2\vec{\nabla} W^k \cdot (\vec{r}^i - \vec{r}^k) - (W^i - W^k) \\ \Delta^{k,+} &= W^i - W^k \end{aligned} \quad (10)$$

This will reproduce a limited difference for each face. After having these limited differences, the unlimited differences and the limited differences for each face are limited secondly

$$\phi^k = \Upsilon(\Delta^{k,\text{lim}}, \Delta^{k,f}) \quad (11)$$

With the unlimited difference is calculated by

$$\Delta^{k,f} = \vec{\nabla} W^k \cdot (\vec{r}^f - \vec{r}^k) \quad (12)$$

In order to have a better convergence, the differential limiter function reads

$$\Upsilon(a,b) = \beta \frac{a^2 + \alpha ab + \varepsilon^2}{a^2 + \alpha b^2 + ab + \beta \varepsilon^2}, \beta = \frac{\alpha + 2}{\alpha + 1} \quad (13)$$

It is clearly it recovers the solution without limiter at the extrema and in the near const regions where  $\varepsilon^2$  dominates a and b.

### 3. Results and Discussions

In order to validate the proposed limiter, the case of turbulent boundary layer flows over a flat plate is chosen as the first validation case. It has been widely adopted as the standard test case for turbulence models. The mesh used in this study is shown in Fig. 2. It consists of 5500 brick (hex) cells and 1860 prim cells.

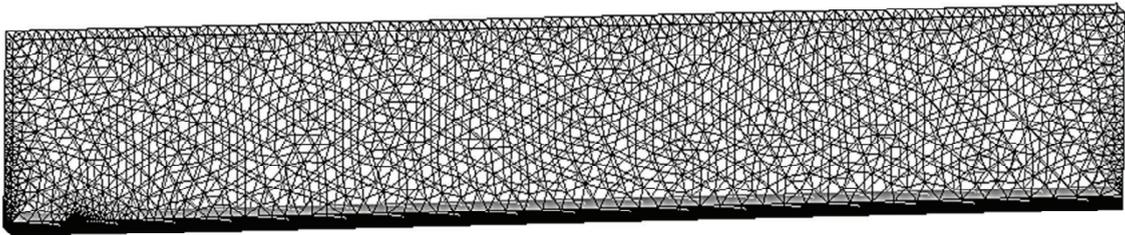


Fig. 2. The hybrid mesh for flows over flat plate.

The upstream, downstream and upper boundaries are specified as the far-field, and the two side boundaries are treated as symmetry planes or in-viscid walls. The different far field conditions ( $Ma=0.2, 1, 2$ ) are used in this study. For this low Mach (incompressible) flow, the velocity profile in the boundary layer is known to satisfy the famous wall law. That is, within the viscous sub-layer, there is a linear relation between the normalized stream-wise velocity  $u^+$  and the distance from the wall  $y^+$ . Besides, in the logarithmic layer, the velocity varies in a logarithmic relation with the distance. In fact, there is unified formula (Spalding, 1961) to describe the velocity profile in regions which include both the sub-layer and log-layer. It reads,

$$y^+ = u^+ + e^{-\kappa B} \left[ e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6} \right] \quad (14)$$

The velocity and skin friction profile for Ma=0.2 are plotted in Fig. 3.

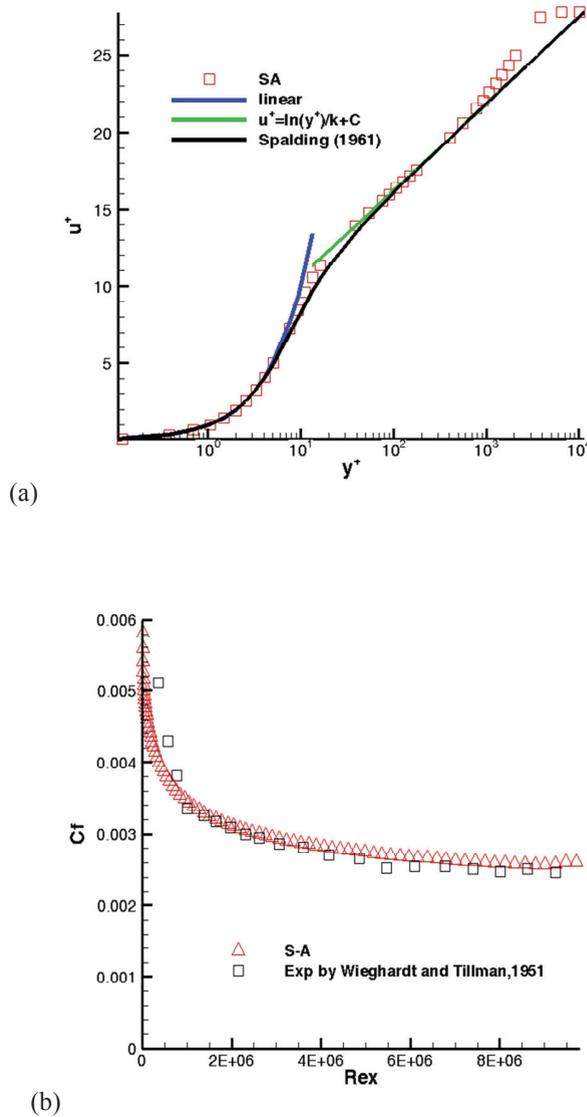


Fig. 3. The profiles of (a) velocity and (b) skin friction for Ma=0.2.

It is clear that the numerical velocity profile agrees quite well with the wall law and Spalding's formula in both the viscous sub-layer and log-layer. What is more, the skin friction also matches the experimental data by Wieghardt et al. (1951).

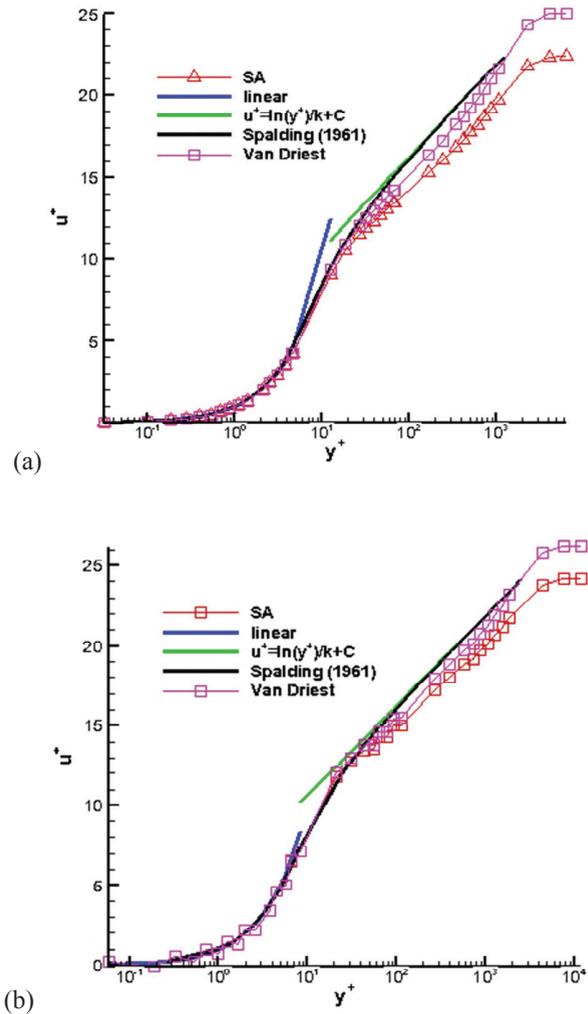


Fig. 4. The velocity profiles for (a)  $Ma=1$  and (b)  $Ma=2$ .

For the high Mach (compressible) flow, the velocity profile in the boundary layer is usually transformed by Van Driest Transformation. The results for  $Ma=1$  and  $Ma=2$  are plotted in Fig. 4. It could be easily observed that the numerical results match the Wall law very well.

#### 4. Conclusion

In this paper, a new limiter for hybrid grid is proposed. It is so simple that it can be easily adopted in many solvers based on hybrid grid. The numerical results agree well with the theory and the experiment results. For example, the velocity profile agrees quite well with the wall law and Spalding's formula in both the viscous sub-layer and log-layer. What is more, the skin friction also matches the experimental data by Wieghardt et al. (1951).

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