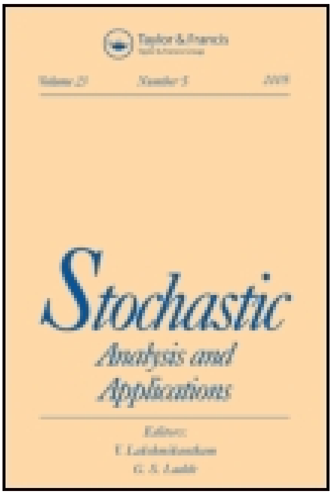

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Ensemble Averaging for Dynamical Systems Under Fast Oscillating Random Boundary Conditions

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Ensemble Averaging for Dynamical Systems Under Fast Oscillating Random Boundary Conditions

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This article is devoted to providing a theoretical underpinning for ensemble forecasting with rapid fluctuations in body forcing and in boundary conditions. Ensemble averaging principles are proved under suitable "mixing" conditions on random boundary conditions and on random body forcing. The ensemble averaged model is a nonlinear stochastic partial differential equation, with the deviation process (i.e., the approximation error process) quantified as the solution of a linear stochastic partial differential equation.

Keywords Multiscale modeling; Ensemble averaging; Random partial differential equations; Stochastic partial differential equations; Random boundary conditions; Martingale.

Mathematics Subject Classifications (2010) 60H15; 60F10; 60G17.

1. Motivation

A complex system often involves with multiple scales, uncertain parameters or coefficients, and fluctuating interactions with its environment. Ensemble forecasting for such a complex system is a prediction method to obtain collective or ensembled view of its dynamical evolution, by generating multiple numerical predictions using different but plausible realizations of a model for the system. The multiple simulations are generated to account

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for errors introduced by sensitive dependence on the initial/boundary conditions and errors introduced due to imperfections in the model [8, 9].

In order to better understand the theoretical foundation of ensemble forecasting, we consider a system modeled by partial differential equations (PDEs) with fast oscillating random forcing in the physical medium or on the physical boundary, and show that ensemble averaged dynamics converges to the original dynamics, as a scale parameter tends to zero.

Some relevant recent works [1, 3, 10, 15] are about averaging or homogenization for random partial differential equations (random PDEs) with fast oscillating coefficients in time or space. Different from the method in the above mentioned works, we present a more direct approach, in order to derive an averaging principle and deviation estimates for PDEs with random oscillating coefficients and random oscillating boundary conditions. We previously studied [13] stochastic partial differential equations (stochastic PDEs) with perturbed white noise dynamical boundary conditions which are measured by a small scale parameter $\epsilon > 0$. In that case, the effectively reduced model does not capture the influence of the random force on boundary.

In this article, we consider the following PDE with a random oscillating body forcing and/or a small random oscillating boundary condition, for a unknown random field $u^\epsilon(x, t, \omega)$

$$u_t^\epsilon = u_{xx}^\epsilon + g(t/\epsilon, u^\epsilon, \omega), \quad u^\epsilon(x, 0) = u_0(x), \tag{1}$$

$$u^\epsilon(0, t) = \sqrt{\epsilon} f(t/\epsilon, \omega), \quad u^\epsilon(l, t) = 0. \tag{2}$$

Here $x \in (0, l), l > 0, t > 0, \epsilon$ is a small positive scale parameter, and ω is in a sample space Ω. A probability $\mathbb{P}$ with a $\sigma-$algebra $\mathcal{F}$ is defined on this sample space. The mathematical expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. We often suppress the $\omega-$dependence for notational clarity.

We prove an ensemble averaging theorem (Theorem 1 in §2) for the random system (1)–(2), that is, we obtain a stochastic model for u^ϵ as $\epsilon \to 0$. It turns out that the random boundary condition appears as a white noise on the dynamical field equation for u, as $\epsilon \to 0$. The ensemble averaged model is a stochastic partial differential equation (stochastic PDE) for u, instead of a random PDE, with a homogenous boundary condition. When the random boundary condition is absent, we further show that the deviation process (i.e., approximation error process), $u^\epsilon - u$, can be quantified as the solution of a linear stochastic PDE (Theorem 2 in §3).

On the technical side, in order to pass the limit $\epsilon \to 0$, we first prove the tightness of the distribution of $\{u^\epsilon\}$, so we just consider $\langle u^\epsilon, \varphi \rangle$ for every bounded continuous function φ with compact support. Then, in §2.2, we construct a process $\mathcal{M}_t^\epsilon$, which is a martingale by Ethier and Kurtz's result [4, Proposition 2.7.6]. This construction is very direct [3]. By passing the limit $\epsilon \to 0$ in $\mathcal{M}_t^\epsilon$, we obtain the stochastic PDE satisfied by the limit u of u^ϵ. This method is also applied to show that the deviation process, $u^\epsilon - u$, is the solution of a linear stochastic PDE; see Section 3.

Note that we take $\sqrt{\epsilon}$ as the intensity scale for the noise boundary condition. This is for simplicity. In fact, our approach can also treat the case $\epsilon^{\frac{\alpha}{2} \wedge 1} f(t/\epsilon^\alpha)$, with $0 < \alpha \leq 2$. A similar case is also discussed in [3]. But the case $\alpha > 2$ is more singular, one should consider the limit of $\epsilon^{\frac{\alpha}{2}-1} u^\epsilon$ as $\epsilon \to 0$.

This article is organized as follows. After recalling some basic background, we prove an ensemble averaging theorem for a random PDE system with a random boundary condition and with a random body forcing, in Section 2, and further characterize the deviation process in Section 3.

2. Ensemble Averaging Under Small Fast Oscillating Random Boundary Conditions

We consider the random PDE system (1)–(2). Consider the Hilbert space $H = L^2(0, l)$ with the usual norm $\|\cdot\|_0$ and inner product $\langle\cdot,\cdot\rangle$. Define $A = \partial_{xx}$ with the zero Dirichlet boundary condition. It defines a compact analytic semigroup $S(t)$, $t \geq 0$, on H. Denote by $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ the eigenvalues of $-A$ with the corresponding eigenfunctions $\{e_k\}_{k=1}^{\infty}$, which forms an orthonormal basis of H. For every $\alpha > 0$, define a new norm $\|u\|_\alpha = \|(-A)^{\alpha/2}u\|_0$, for those $u \in H$ such that this quantity is finite.

Here we make the following assumptions about the mixing properties of the random boundary and body forcing in the random PDE system (1)–(2).

($\mathbf{H}_g$) For every t, $g(t,\cdot)$ is Lipschitz continuous in u with Lipschitz constant L_g and $g(t, 0) = 0$. For every $u \in H$, $g(\cdot, u)$ is an H-valued stationary random process and is strongly mixing with an exponential rate $\gamma > 0$. That is,

$$\sup_{s\geq 0}\ \sup_{U\in\mathcal{G}_0^s,\, V\in\mathcal{G}_{s+t}^\infty} |\mathbb{P}(U\cap V) - \mathbb{P}(U)\mathbb{P}(V)| \leq e^{-\gamma t}, \quad t \geq 0,$$

where $0 \leq s \leq t \leq \infty$, and $\mathcal{G}_s^t = \sigma\{g(\tau, u) : s \leq \tau \leq t\}$ is the σ-algebra generated by $\{g(\tau, u) : s \leq \tau \leq t\}$.

($\mathbf{H}_f$) The process $f(t)$ is a bounded continuous differentiable process, and the time derivative process $f_t(t)$ is a bounded stationary process with $\mathbb{E}f_t = 0$. Furthermore, f_t is strong mixing with exponential rate, that is

$$\sup_{s\geq 0}\ \sup_{U\in\mathcal{F}_0^s,\, V\in\mathcal{F}_{s+t}^\infty} |\mathbb{P}(U\cap V) - \mathbb{P}(U)\mathbb{P}(V)| \leq e^{-\lambda t}, \quad t \geq 0,$$

where $0 \leq s \leq t \leq \infty$, $\lambda > 0$, and $\mathcal{F}_s^t = \sigma\{f_t(\tau) : s \leq \tau \leq t\}$ is the σ-algebra generated by $\{f_t(\tau) : s \leq \tau \leq t\}$.

Remark 1. Taking time derivative on the random boundary condition, we have

$$u_t^\epsilon = u_{xx}^\epsilon + g(t/\epsilon, u^\epsilon), \quad u^\epsilon(x, 0) = u_0$$
$$u_t^\epsilon(0, t) = \frac{1}{\sqrt{\epsilon}} f_t(t/\epsilon), \quad u_t^\epsilon(l, t) = 0,$$

which is a system with a random dynamical boundary condition.

To "homogenize" the inhomogeneous boundary condition in the system (1)–(2), we transform the random boundary condition to the field equation by introducing a new random field $\hat{u}^\epsilon(x, t) = u^\epsilon(x, t) - \sqrt{\epsilon} f(t/\epsilon)(1 - \frac{x}{l})$. Then, $\hat{u}_{xx}^\epsilon = u_{xx}^\epsilon$ and the system (1)–(2) becomes

$$\hat{u}_t^\epsilon(x, t) = \hat{u}_{xx}^\epsilon(x, t) + g(t/\epsilon, u^\epsilon(x, t)) - \frac{1}{\sqrt{\epsilon}} f_t(t/\epsilon)\left(1 - \frac{x}{l}\right), \tag{3}$$

$$\hat{u}^\epsilon(x, 0) = u_0 - \sqrt{\epsilon} f(0)\left(1 - \frac{x}{l}\right), \tag{4}$$

$$\hat{u}^\epsilon(0, t) = 0,\ \ \hat{u}^\epsilon(l, t) = 0, \tag{5}$$

which is a random system with homogeneous boundary conditions. By the assumption $(\mathbf{H}_f), f$ is bounded. Thus,

$$\mathbb{E} \sup_{t_0 \le t \le T} \|\hat{u}^\epsilon(t) - u^\epsilon(t)\|_0 = \sqrt{\epsilon}\mathbb{E}\big[\sup_{t_0 \le t \le T} |f(t/\epsilon)|\big] \|1 - \frac{x}{l}\|_0 \to 0\,, \quad \epsilon \to 0\,. \tag{6}$$

So in the following subsections, we consider $\hat{u}^\epsilon$ in space $C(t_0, T; H)$ for some $T > t_0 > 0$, and derive an ensemble averaged equation to be satisfied by the limit of $\hat{u}^\epsilon$.

2.1. Tightness

In this section, we examine the tightness of the distribution of $\hat{u}^\epsilon$ in space of continuous functions, $C(t_0, T; H)$, for all fixed $T > t_0 > 0$.

In the mild or integral formulation, Equation (3) becomes

$$\hat{u}^\epsilon(t) = S(t)\hat{u}^\epsilon(0) + \int_0^t S(t-s) g\left(\frac{s}{\epsilon}, u^\epsilon(s)\right) ds - \frac{1}{\sqrt{\epsilon}} \int_0^t S(t-s) f_t\left(\frac{s}{\epsilon}\right)\left(1 - \frac{x}{l}\right) ds\,.$$

By the properties of the semigroup $S(t)$, we have

$$\|\hat{u}^\epsilon(t)\|_0 \le \|\hat{u}^\epsilon(0)\|_0 + \int_0^t \|g\left(\frac{s}{\epsilon}, u^\epsilon(s)\right)\|_0\, ds + \frac{1}{\sqrt{\epsilon}} \left\| \int_0^t S(t-s) f_t\left(\frac{s}{\epsilon}\right)\left(1 - \frac{x}{l}\right) ds \right\|_0 ,$$

and by the assumption $(\mathbf{H}_g)$,

$$\|g\left(\frac{s}{\epsilon}, u^\epsilon(s)\right)\|_0 \le L_g \|u^\epsilon(s)\|_0 \le L_g(\|\hat{u}^\epsilon(s)\|_0 + \sqrt{\epsilon} l C_f)\,. \tag{7}$$

Then we have, for every $T > 0$ and $0 < t \le T$,

$$\begin{aligned}\mathbb{E} \sup_{0 \le s \le t} \|\hat{u}^\epsilon(s)\|_0 \le \|\hat{u}^\epsilon(0)\|_0 + L_g \int_0^t \sup_{0 \le r \le s} \|\hat{u}^\epsilon(r)\|_0\, ds \\ + C_{T,1} + \sup_{0 \le s \le t} \|I^\epsilon(s)\|_0,\end{aligned} \tag{8}$$

where $C_{T,1}$ is a positive constant depending only on L_g, l, and C_f, and

$$I^\epsilon(t) := \frac{1}{\sqrt{\epsilon}} \int_0^t S(t-s) f_t\left(\frac{s}{\epsilon}\right)\left(1 - \frac{x}{l}\right) ds\,.$$

Next we treat the singular term $I^\epsilon(t)$. By the factorization method [2], for some $0 < \alpha < 1$,

$$I^\epsilon(t) = \frac{\sin \pi\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1} S(t-s) Y^\epsilon(s)\, ds,$$

with

$$Y^\epsilon(s) = \frac{1}{\sqrt{\epsilon}} \int_0^s (s-r)^{-\alpha} f_t\left(\frac{r}{\epsilon}\right) S(s-r)\left(1 - \frac{x}{l}\right) dr\,. \tag{9}$$

Then, for every $T > 0$, there is a positive constant $C_{T,2}$ such that

$$\mathbb{E} \sup_{0 \le s \le t} \|I^\epsilon(s)\|_0^2 \le C_{T,2} \int_0^t \mathbb{E}\|Y^\epsilon(s)\|_0^2\, ds\,, \quad 0 \le t \le T\,.$$

Notice that

$$\mathbb{E}\|Y^\epsilon(s)\|_0^2 = \frac{1}{\epsilon}\int_0^s\int_0^s (s-r)^{-\alpha}(s-\tau)^{-\alpha}\mathbb{E}\left[f_t\left(\frac{r}{\epsilon}\right)f_t\left(\frac{\tau}{\epsilon}\right)\right] \times S(s-r)\left(1-\frac{x}{l}\right)S(s-\tau)\left(1-\frac{x}{l}\right)drd\tau\,,$$

by the assumption ($\mathbf{H}_f$). For every $T > 0$, there is a positive constant $C_{T,3}$ such that for all $0 \le t \le T$,

$$\mathbb{E}\sup_{0\le s\le t}\|I^\epsilon(s)\|_0 \le C_{T,3}\,. \tag{10}$$

Hence, for every $T > 0$, applying the Gronwall inequality to (8), we obtain

$$\mathbb{E}\sup_{0\le t\le T}\|\hat{u}^\epsilon(t)\|_0 \le C_T(1+\|\hat{u}^\epsilon(0)\|_0), \tag{11}$$

for some constant $C_T > 0$. Furthermore, from the mild form of $\hat{u}^\epsilon$, by the fact that $\|S(t)u\|_1 \le \frac{1}{\sqrt{t}}\|u\|_0$, we have

$$\|\hat{u}^\epsilon(t)\|_1 \le \frac{1}{\sqrt{t}}\|\hat{u}^\epsilon(0)\|_0 + \int_0^t \frac{1}{\sqrt{t-s}}\|g(\frac{s}{\epsilon},u^\epsilon(s))\|_0\,ds + \|I^\epsilon(s)\|_1\,. \tag{12}$$

We now consider the term $\|I^\epsilon(s)\|_1$. Still by the factorization method,

$$\begin{aligned}\|I^\epsilon(t)\|_1 &\le \frac{\sin\alpha}{\pi}\int_0^t (t-s)^{\alpha-1}\|S(t-s)Y^\epsilon(s)\|_1\,ds\\ &\le \frac{\sin\alpha}{\pi}\int_0^t (t-s)^{\alpha-1}\frac{1}{\sqrt{t-s}}\|Y^\epsilon(s)\|_0\,ds,\end{aligned}$$

where $Y^\epsilon(s)$ is defined by (9). Then, choose α with $1/2 < \alpha < 1$, and by the same discussion for (10), we conclude that for every $T > 0$

$$\mathbb{E}\|I^\epsilon(t)\|_1 \le C_{T,5}\,,\quad 0\le t\le T$$

for some constant $C_{T,5} > 0$. Then for $t_0 > 0$, from (7) and (12), and by Gronwall inequality we have

$$\mathbb{E}\|\hat{u}^\epsilon(t)\|_1 \le C_T\,,\quad t_0\le t\le T, \tag{13}$$

for some constant $C_T > 0$.

To show the tightness of the distributions of $\hat{u}^\epsilon$, we need a Hölder estimate in time. For every $0 \le s < t \le T$,

$$\begin{aligned}\|\hat{u}^\epsilon(t)-\hat{u}^\epsilon(s)\|_0 \le \|(S(t)-S(s))\hat{u}^\epsilon(0)\|_0 + \left\|\int_s^t S(t-\sigma)g\left(\frac{\sigma}{\epsilon},u^\epsilon(\sigma)\right)d\sigma\right\|_0\\ + \frac{1}{\sqrt{\epsilon}}\left\|\int_s^t S(t-\sigma)f_t\left(\frac{\sigma}{\epsilon}\right)\left(1-\frac{x}{l}\right)d\sigma\right\|_0\\ + \left\|\int_0^s [S(t-\sigma)-S(s-\sigma)]g\left(\frac{\sigma}{\epsilon},u^\epsilon(\sigma)\right)d\sigma\right\|_0\end{aligned}$$

$$+\frac{1}{\sqrt{\epsilon}}\left\|\int_0^s [S(t-\sigma)-S(s-\sigma)]\, f_t\left(\frac{\sigma}{\epsilon}\right)\left(1-\frac{x}{l}\right) d\sigma\right\|_0 .$$

By the estimate on $\|\hat{u}^\epsilon(t)\|_0$ and (7), we have for some constant $C_T > 0$,

$$\mathbb{E}\left\|\int_s^t S(t-\sigma) g\left(\frac{\sigma}{\epsilon}, u^\epsilon(\sigma)\right) d\sigma\right\|_0^2 \le C_T|t-s|.$$

Moreover, by the strong continuity of the semigroup $S(t)$, we also have

$$\mathbb{E}\left\|\int_0^s [S(t-\sigma)-S(s-\sigma)]\, g\left(\frac{\sigma}{\epsilon}, u^\epsilon(\sigma)\right) d\sigma\right\|_0^2 \le C_T|t-s| .$$

Now consider for the singular terms. First, notice that $(1-x/l)$ is smooth. We have

$$S(t-\sigma)\left(1-\frac{x}{l}\right) \in L^\infty(0,l) .$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\frac{1}{\epsilon}\left\|\int_s^t \int_s^t S(t-\sigma) f_t\left(\frac{\sigma}{\epsilon}\right)\left(1-\frac{x}{l}\right) d\sigma\right\|_0^2 \\
&= \int_s^t \int_s^t \frac{1}{\epsilon}\mathbb{E}\left[f_t\left(\frac{\sigma}{\epsilon}\right) f_t\left(\frac{\tau}{\epsilon}\right)\right] \int_0^l S(t-\sigma)\left(1-\frac{x}{l}\right) S(t-\tau)\left(1-\frac{x}{l}\right) dx\, d\sigma d\tau \\
&\le C_{l,T}\int_s^t \int_s^t \frac{1}{\epsilon}\mathbb{E}\left[f_t\left(\frac{\sigma}{\epsilon}\right) f_t\left(\frac{\tau}{\epsilon}\right)\right] d\sigma d\tau,
\end{aligned}$$

for a positive constant $C_{l,T}$ depending on T and l. Now by $(\mathbf{H}_f)$, we have

$$\mathbb{E}\frac{1}{\epsilon}\left\|\int_s^t S(t-\sigma) f_t\left(\frac{\sigma}{\epsilon}\right)\left(1-\frac{x}{l}\right) d\sigma\right\|_0^2 \le C_{l,T}|t-s| .$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}\frac{1}{\epsilon}\left\|\int_0^s [S(t-\sigma)-S(s-\sigma)] f_t\left(\frac{\sigma}{\epsilon}\right)\left(1-\frac{x}{l}\right) d\sigma\right\|_0^2 \\
&= \frac{1}{\epsilon}\sum_k l_k \int_0^s \int_0^s \mathbb{E} f_t\left(\frac{\sigma}{\epsilon}\right) f_t\left(\frac{\tau}{\epsilon}\right)\left[e^{-\lambda_k(t-\sigma)} - e^{-\lambda_k(s-\sigma)}\right] \\
&\times \left[e^{-\lambda_k(t-\tau)} - e^{-\lambda_k(s-\tau)}\right] d\sigma\, d\tau,
\end{aligned}$$

where

$$l_k = \int_0^l (1-\frac{x}{l}) e_k(x)\, dx .$$

Then still by $(\mathbf{H}_f)$, we have for some constant $C_T > 0$

$$\mathbb{E}\frac{1}{\epsilon}\left\|\int_0^s [S(t-\sigma)-S(s-\sigma)]\, f_t\left(\frac{\sigma}{\epsilon}\right)\left(1-\frac{x}{l}\right) d\sigma\right\|_0^2 \le C_T|t-s| .$$

Combining all the above estimates we have for some constant $C_T > 0$ such that for $0 \leq s < t \leq T$

$$\mathbb{E}\|\hat{u}^\epsilon(t) - \hat{u}^\epsilon(s)\|_0^2 \leq C_T|t-s|\,. \tag{14}$$

Now we need the following lemma [7]. Suppose $\mathcal{X}_1$ and $\mathcal{X}_2$ are two Banach spaces. Let $T > 0$, $1 \leq p \leq \infty$, and $\mathcal{B}$ be a compact operator from $\mathcal{X}_1$ to $\mathcal{X}_2$. That is, $\mathcal{B}$ maps bounded subsets of $\mathcal{X}_1$ to relatively compact subsets of $\mathcal{X}_2$.

Lemma 1 ([7]). *Let $\mathcal{H}$ be a bounded subset of $L^1(0,T;\mathcal{X}_1)$ such that $G = \mathcal{B}\mathcal{H}$ is a subset of $L^p(0,T;\mathcal{X}_2)$ bounded in $L^r(0,T;\mathcal{X}_2)$ with $r > 1$. If*

$$\lim_{\sigma\to 0}\|u(\cdot+\sigma) - u(\cdot)\|_{L^p(0,T;\mathcal{X}_2)} = 0 \quad \textit{uniformly for } u \in G\,,$$

then G *is relatively compact in $L^p(0,T;\mathcal{X}_2)$ (and in $C(0,T;\mathcal{X}_2)$ if $p = +\infty$).*

Then by the above lemma we have the following tightness result.

Lemma 2. ***(Tightness)*** *Assume that both $(\mathbf{H}_g)$ and $(\mathbf{H}_f)$ hold. For every $0 < t_0 < T$, the distribution of $\{\hat{u}^\epsilon\}_{0<\epsilon\leq 1}$ is tight in space $C(t_0,T;H)$.*

Proof. Let $\mathcal{X}_1 = H_0^1(0,l)$, $\mathcal{X}_2 = L^2(0,l)$ and $\mathcal{B}$ being the embedding from $\mathcal{X}_1$ to $\mathcal{X}_2$. By estimate (13) for some constant $C_T > 0$

$$\mathbb{E}\|\hat{u}^\epsilon\|_{L^1(t_0,T;\mathcal{X}_1)} < C_T\,,$$

then by the Markov inequality for each $\kappa > 0$, there is $K_1 > 0$ such that

$$\mathbb{P}\{\|\hat{u}^\epsilon\|_{L^1(t_0,T;\mathcal{X}_1)} \leq K_1\} \geq 1 - \frac{C_T}{K_1} \geq 1 - \frac{\kappa}{2}. \tag{15}$$

Further, noticing estimate (14), by the Garcia–Rademich–Rumsey theorem [5], for some constant $C_T > 0$

$$\mathbb{E}\|\hat{u}^\epsilon\|_{C^{1/2}(0,T;L^2(0,l))} \leq C_T\,.$$

Then still by Markov inequality for each $\kappa > 0$ there is $K_2 > 0$ such that

$$\mathbb{P}\left\{\sup_{0\leq t\leq T, 0<\sigma<T-t}\frac{\|\hat{u}^\epsilon(t+\sigma) - \hat{u}^\epsilon(t)\|_0}{\sqrt{\sigma}} \leq K_2\right\} \geq 1 - \frac{C_T}{K_2} \geq 1 - \frac{\kappa}{2}. \tag{16}$$

Now define sets

$$S_1 = \{u \in L^1(t_0,T;\mathcal{X}_1) : \|u\|_{L^1(t_0,T;\mathcal{X}_1)} \leq K_1\}$$

and

$$S_2 = \left\{u \in S_1 : \frac{\|u(t+\sigma) - u(t)\|_0}{\sqrt{\sigma}} \leq K_2\,, \quad t_0 < t < T\,, \quad 0 < \sigma < T - t\right\}.$$

Then by the estimates (15) and (16)

$$\mathbb{P}\{\hat{u}^\epsilon \in S_2\} > 1 - \kappa .$$

By the definition of S_2, for each $u \in S_2$

$$\lim_{\sigma \to 0} \sup_{t_0 < t < T} \|u(t+\sigma) - u(t)\|_0 = 0.$$

Then by Lemma 1, set S_2 is compact in space $C(t_0, T; \mathcal{X}_2)$, that is the distributions of $\{\hat{u}^\epsilon\}_{0<\epsilon\leq 1}$ is tight in space $C(t_0, T; H)$. The proof is complete. □

2.2. Ensemble Averaging

Next we use the weak convergence method [6] to pass the limit $\epsilon \to 0$. In this approach we construct a martingale which has the following form

$$\Phi(t) - \int_0^t A^\epsilon \Phi(s)\, ds,$$

for some $\mathcal{F}_0^t$-process $\Phi(t)$ defined by $z_1^\epsilon(t)$ and A^ϵ, which is a pseudo differential operator to be introduced later.

Because of the tightness of $\hat{u}^\epsilon$ in space $C(t_0, T; H)$ for every fixed $t_0 > 0$, in order to determine the limit equation of $\hat{u}^\epsilon$ in space $C(t_0, T; H)$, we consider the limit of $\Phi(\langle \hat{u}^\epsilon(t), \varphi\rangle)$, for every bounded second-order differentiable function $\Phi : \mathbb{R} \to \mathbb{R}$ and for every compactly supported smooth function $\varphi \in C_b^\infty(0, l)$.

First, we have

$$\begin{aligned}
\Phi(\langle \hat{u}^\epsilon(t), \varphi\rangle) - \Phi(\langle u_0, \varphi\rangle) &= \int_0^t \Phi'(\langle \hat{u}^\epsilon(s), \varphi\rangle)\langle \hat{u}^\epsilon(s), \varphi_{xx}\rangle\, ds \\
&\quad + \int_0^t \Phi'(\langle \hat{u}^\epsilon(s), \varphi\rangle)\left\langle g\left(\frac{s}{\epsilon}, \hat{u}^\epsilon(s)\right), \varphi\right\rangle ds \\
&\quad - \frac{1}{\sqrt{\epsilon}} \int_0^t \Phi'(\langle \hat{u}^\epsilon(s), \varphi\rangle)\left\langle f_t\left(\frac{s}{\epsilon}\right)\left(1 - \frac{x}{l}\right), \varphi\right\rangle ds\,. \quad (17)
\end{aligned}$$

To treat the singular term in (17), we apply a perturbation method in [6, Chapter 7]. To this end, we define the following two processes

$$F_0^\epsilon(t) := \frac{1}{\sqrt{\epsilon}} \int_t^\infty \mathbb{E}\left[f_t^\epsilon\left(\frac{s}{\epsilon}\right) |\mathcal{F}_0^{t/\epsilon}\right] ds \quad (18)$$

and

$$\begin{aligned}
F_1^\epsilon(t) &:= \frac{1}{\sqrt{\epsilon}} \mathbb{E}\left[\int_t^\infty \Phi'(\langle \hat{u}^\epsilon(t), \varphi\rangle)\left\langle f_t\left(\frac{s}{\epsilon}\right)\left(1 - \frac{x}{l}\right), \varphi\right\rangle ds \middle| \mathcal{F}_0^{t/\epsilon}\right] \\
&= \Phi'(\langle \hat{u}^\epsilon(t), \varphi\rangle)\left\langle 1 - \frac{x}{l}, \varphi\right\rangle F_0^\epsilon(t). \quad (19)
\end{aligned}$$

Then we have the following lemma.

Lemma 3. *Assume that* $(\mathbf{H}_f)$ *holds. Then*

$$\mathbb{E}|F_1^\epsilon(t)| \le C\sqrt{\epsilon},$$

for some constant $C > 0$*and*

$$\mathbb{E} \sup_{0\le t\le T} |F_1^\epsilon(t)| \to 0\,, \quad \epsilon \to 0\,,$$

for every $T > 0$.

Proof. By the boundedness of $f_t$and the strong mixing property, we have

$$\mathbb{E}|F_0^\epsilon(t)| \le C\sqrt{\epsilon} \tag{20}$$

for some constant $C > 0$. Then by the choice of Φ, the proof is complete. □

Now we apply a diffusion approximation to derive the limit of $\hat{u}^\epsilon$ in the sense of distribution. For this we introduce the following operator

$$A^\epsilon \Phi(t) = \mathbb{P}\text{-}\lim_{\delta\to 0} \frac{1}{\delta}\mathbb{E}\left[\Phi(t+\delta) - \Phi(t)|\mathcal{F}_0^{t/\epsilon}\right] \tag{21}$$

for $\mathcal{F}_0^{t/\epsilon}$ measurable function $\Phi(t)$ with $\sup_t \mathbb{E}|\Phi(t)| < \infty$. Using Ethier and Kurtz's result [4, Proposition 2.7.6], we know that

$$\Phi(t) - \int_0^t A^\epsilon \Phi(s)\, ds$$

is a martingale with respect to $\mathcal{F}_0^{t/\epsilon}$. Define processes Y^ϵ and Z^ϵ as follows

$$Y^\epsilon(t) = \Phi(\langle \hat{u}^\epsilon(t), \varphi\rangle) - F_1^\epsilon(t)\,, \quad Z^\epsilon(t) = A^\epsilon Y^\epsilon(t)\,.$$

A direct calculation yields

$$\begin{aligned}
Z^\epsilon(t) \;=\;& \Phi'(\langle \hat{u}^\epsilon(t), \varphi\rangle)\big[\langle \hat{u}^\epsilon(t), \varphi_{xx}\rangle + \langle g(t/\epsilon, u^\epsilon(t)), \varphi\rangle\big] \\
& - \Phi''(\langle \hat{u}^\epsilon(t), \varphi\rangle)\left\langle\left(1 - \frac{x}{l}\right), \varphi\right\rangle^2 \frac{1}{\sqrt{\epsilon}} f_t\left(\frac{t}{\epsilon}\right) F_0^\epsilon(t) \\
& + \Phi''(\langle \hat{u}^\epsilon(t), \varphi\rangle)\left[\langle \hat{u}^\epsilon(t), \varphi_{xx}\rangle + \left\langle g\left(\frac{t}{\epsilon}, u^\epsilon(t)\right), \varphi\right\rangle\right]\left\langle 1 - \frac{x}{l}, \varphi\right\rangle F_0^\epsilon(t) \\
:=\;& Z_1^\epsilon(t) + Z_2^\epsilon(t) + Z_3^\epsilon(t)\,.
\end{aligned} \tag{22}$$

Next we pass the limit $\epsilon \to 0$ for $\hat{u}^\epsilon(t)$ in space $C(t_0, T; H)$. By the convergence result of Walsh [12, Theorem 6.15], we only need to consider finite dimensional distributions of $\{\langle \hat{u}^\epsilon(t), \varphi_1\rangle, \ldots, \langle \hat{u}^\epsilon(t), \varphi_n\rangle\}$ for every $\varphi_1, \ldots, \varphi_n \in C_b^\infty(0, l)$. That is, we pass limit $\epsilon \to 0$ in

$$\mathbb{E}\left\{\left[Y^\epsilon(t) - Y^\epsilon(s) - \int_s^t Z^\epsilon(r)\, dr\right] h(\langle \hat{u}^\epsilon(r_1), \varphi_1\rangle, \ldots, \langle \hat{u}^\epsilon(r_n), \varphi_n\rangle)\right\} = 0$$

for every bounded continuous function h and $0 < r_1 < \cdots < r_n < T$ with $T > 0$. Denote by $\hat{u}$ one limit point in the sense of distribution of $\hat{u}^\epsilon$ as $\epsilon \to 0$ in space $C(t_0, T; H)$. For simplicity we assume $\hat{u}^\epsilon$ converges in distribution to $\hat{u}$ as $\epsilon \to 0$. Then by the estimates in Lemma 3 we have

$$Y^\epsilon(t) - Y^\epsilon(s) \to \Phi(\langle \hat{u}(t), \varphi \rangle) - \Phi(\langle \hat{u}(s), \varphi \rangle) \tag{23}$$

in distribution.

Consider the integral term in (17). First, we need the following lemma whose proof is given in Appendix A.

Lemma 4. *The following convergence in probability holds:*

$$\int_0^t \left[g(r/\epsilon, u^\epsilon(r)) - \bar{g}(u^\epsilon(r)) \right] dr \to 0, \quad \epsilon \to 0.$$

Then by this lemma, we have

$$\int_s^t Z_1^\epsilon(r)\, dr \to \int_s^t \Phi(\langle \hat{u}(r), \varphi \rangle) \left[\langle \hat{u}(r), \varphi_{xx} \rangle + \langle \bar{g}(\hat{u}(r)), \varphi \rangle \right] dr \tag{24}$$

in distribution as $\epsilon \to 0$. By the the estimate (20), we have

$$\mathbb{E} \int_s^t |Z_3^\epsilon(r)|\, dr \to 0. \tag{25}$$

Now we consider $Z_2^\epsilon(t)$. Define a bilinear operator

$$\langle \Sigma\varphi, \varphi \rangle = b \int_0^l \int_0^l \left(1 - \frac{x}{l}\right) \varphi(x) \left(1 - \frac{y}{l}\right) \varphi(y)\, dx\, dy, \tag{26}$$

where b is the variance of f_t, which is constant defined as

$$b := 2\mathbb{E} f_t(t) f_t(t) > 0. \tag{27}$$

We again apply a perturbation method. Set

$$F_2^\epsilon(t) := -\Phi''(\langle \hat{u}^\epsilon(t), \varphi \rangle) \left\langle \left(1 - \frac{x}{l}\right), \varphi \right\rangle^2 \frac{1}{\sqrt{\epsilon}} \int_t^\infty \mathbb{E}\left[f_t\left(\frac{s}{\epsilon}\right) F_0^\epsilon(s) - \frac{1}{2} b \middle| \mathcal{F}_0^{t/\epsilon} \right] ds. \tag{28}$$

By the properties of conditional expectation, the definition of F_0^ϵ and the fact of $\mathcal{F}_0^t \subset \mathcal{F}_0^s$ for $s \geq t$,

$$\begin{aligned} &\frac{1}{\sqrt{\epsilon}} \int_t^\infty \mathbb{E}\left[f_t\left(\frac{s}{\epsilon}\right) F_0^\epsilon(s) - \frac{1}{2} b \middle| \mathcal{F}_0^{t/\epsilon} \right] ds \\ &\quad = \frac{1}{\epsilon} \int_t^\infty \int_s^\infty \mathbb{E}\left[f_t\left(\frac{s}{\epsilon}\right) f_t\left(\frac{\tau}{\epsilon}\right) - \frac{1}{2} b | \mathcal{F}_0^{t/\epsilon} \right] d\tau ds. \end{aligned}$$

Then, by the strong mixing properties of f_t in the assumption $(\mathbf{H}_f)$, we have as $\epsilon \to 0$

$$\sup_{t \geq 0} \mathbb{E} F_2^\epsilon(t) = \mathcal{O}(\epsilon).$$

Furthermore, by the same calculation as for $Z^\epsilon(t)$, we have the following lemma.

Lemma 5. *The following result holds:*

$$A^\epsilon F_2^\epsilon(t) = -\Phi''(\langle \hat{u}^\epsilon(t), \varphi\rangle)\left\langle\left(1-\frac{x}{l}\right), \varphi\right\rangle^2 \frac{1}{\sqrt{\epsilon}}\left[f_t\left(\frac{t}{\epsilon}\right)F_0^\epsilon(t) - \frac{1}{2}b\right] + R_1^\epsilon(t) \tag{29}$$

with $\mathbb{E}|R_1^\epsilon(t)| = \mathcal{O}(\epsilon)$ *as* $\epsilon \to 0$.

Now we have the following $\mathcal{F}_0^{t/\epsilon}$-martingale

$$\begin{aligned}\mathcal{M}_t^\epsilon := \Phi(\langle \hat{u}^\epsilon(t), \varphi\rangle) - F_1^\epsilon(t) - F_2^\epsilon(t) - \int_0^t \Phi'(\langle \hat{u}^\epsilon(s), \varphi\rangle)\big[\langle \hat{u}^\epsilon(s), \varphi_{xx}\rangle \\ + \langle \bar{g}(\hat{u}^\epsilon(s)), \varphi\rangle\big]\,ds + \frac{1}{2}\int_0^t \Phi''(\langle \hat{u}^\epsilon(s), \varphi\rangle)\langle \Sigma\varphi, \varphi\rangle\,ds + R^\epsilon(t)\end{aligned}$$

with $\mathbb{E}|R^\epsilon(t)| = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. Then by passing the limit $\epsilon \to 0$, the distribution of the limit u of $\hat{u}^\epsilon$ solves the following martingale problem

$$\begin{aligned}\mathcal{M}(\tau) = \Phi(\langle u(t), \varphi\rangle) - \int_0^t \Phi'(\langle u(s), \varphi\rangle)\big[\langle u(s), \varphi_{xx}\rangle + \langle \bar{g}(u(s)), \varphi\rangle\big]\,ds \\ + \frac{1}{2}\int_0^t \Phi''(\langle u(s), \varphi\rangle)\langle \Sigma\varphi, \varphi\rangle\,ds,\end{aligned} \tag{30}$$

which is equivalent to the fact that u is the martingale solution of the following stochastic PDE:

$$du = [u_{xx} + \bar{g}(u)]\,dt - \sqrt{b}\left(1 - \frac{x}{l}\right)dB(t), \tag{31}$$

where B is a usual scalar Brownian motion, and b is the variance of f_t as defined in (27).

Finally, by the uniqueness of the solution to equation (31), we have the following main result on ensemble averaging under a random boundary condition.

Theorem 1. *(Ensemble averaging under a random boundary condition)*

For every $t_0 > 0$ and $T > t_0$, the solution u^ϵ, of the random PDE system (1), converges in distribution to u *in space $C(t_0, T; H)$, with* u *solving the limit equation (31).*

3. Ensemble Averaging Under Fast Oscillating Random Body Forcing

In this section, we consider the special case when the random boundary condition is absent. The approach to derive ensemble averaged model in the last section is applicable in this case. But our goal here is to further show that the deviation process, $u^\epsilon - u$, can be quantified as the solution of a linear stochastic partial differential equation.

We consider the following PDE with random oscillating body forcing on a bounded interval $(0, l)$

$$u_t^\epsilon = u_{xx}^\epsilon + g(t/\epsilon, u^\epsilon), \quad u^\epsilon(x, 0) = u_0, \;\; u^\epsilon(0, t) = 0, \;\; u^\epsilon(l, t) = 0. \tag{32}$$

Here, we still make the assumption $(\mathbf{H}_g)$ on the random body forcing g.

We introduce the notation $\varphi(t)$ to quantify the mixing as follows

$$\varphi(t) \triangleq \sup_{s\geq 0} \sup_{U\in\mathcal{G}_0^s, V\in\mathcal{G}_{s+t}^\infty} |\mathbb{P}(U\cap V) - \mathbb{P}(U)\mathbb{P}(V)| .$$

By the above assumption, for each $\alpha > 0$

$$\int_0^\infty \varphi^\alpha(t)\, dt < \infty .$$

For the random oscillating PDE (32) we have an averaging principle as above. Introduce the following averaged equation

$$u_t = u_{xx} + \bar{g}(u), \quad u(0) = u_0, \tag{33}$$

where $\bar{g}(u) = \mathbb{E}g(t,u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(s/\epsilon, u)\, ds$. Define the deviation process

$$z^\epsilon(t) := \frac{1}{\sqrt{\epsilon}}(u^\epsilon(t) - u(t)) . \tag{34}$$

Then the following averaging principle will be established.

Theorem 2. *(Ensemble averaging under random body forcing)*
Assume that (**H**) *holds. Then, given a $T > 0$, for every $u_0 \in H$, the solution $u^\epsilon(t, u_0)$ of (32) converges in probability to the solution* u *of (33) in $C(0, T; H)$. Moreover, the rate of convergence is $\sqrt{\epsilon}$, that is, for each $\kappa > 0$ there is $C_T^\kappa > 0$ such that*

$$\mathbb{P}\left\{\sup_{0\leq t\leq T} \|u^\epsilon(t) - u(t)\|_0 \geq C_T^\kappa\sqrt{\epsilon}\right\} \leq \kappa . \tag{35}$$

Furthermore, the deviation process z^ϵ converges in distribution in the space $C(0, T; H)$ to z, *which solves the following linear stochastic PDE*

$$dz(t) = [z_{xx}(t) + \overline{g'}(u(t))z(t)]\, dt + d\widetilde{W}, \quad z(0) = z(l) = 0, \tag{36}$$

where

$$\overline{g'}(u) = \mathbb{E}g'_u(t,u)$$

and $\widetilde{W}(t)$ is an H*-valued Wiener process defined on a new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with the covariance operator*

$$\tilde{B}(u) = 2\int_0^\infty \mathbb{E}\left[(g(t,u) - \bar{g}(u)) \otimes (g(0,u) - \bar{g}(u))\right] dt .$$

Remark 2. This deviation result is similar to the averaging results for random PDEs in [3, 11].

Proof. First by the assumption of Lipschitz property on g in (**H**), and noticing that there is no singular term here, standard energy estimates yield that for every $T > 0$

$$\sup_{0\leq t\leq T} \|u^\epsilon(t)\|_1^2 \leq C_T, \tag{37}$$

and

$$\|u^\epsilon(t) - u^\epsilon(s)\|_0 \leq C_T|t-s|, \quad 0 \leq s \leq t \leq T, \tag{38}$$

with some positive constant C_T. Then we have the tightness of the distributions of u^ϵ in space $C(0,T;H)$ for every $T > 0$.

Notice also that $u^\epsilon(t)$ satisfies

$$u^\epsilon(t) = S(t)u_0 + \int_0^t S(t-s)g(s/\epsilon, u^\epsilon(s))\, ds\,,$$

and for every $\varphi \in H$

$$\langle u^\epsilon(t), \varphi\rangle = \langle S(t)u_0, \varphi\rangle + \int_0^t \langle S(t-s)g(s/\epsilon, u^\epsilon(s)), \varphi\rangle\, ds\,.$$

By passing the limit $\epsilon \to 0$, we can just consider the integral term in the above equation. By the tightness of the distributions of u^ϵ, we can follow the same discussion as in Appendix A which yields the averaged equation (33) and the esitmate (35).

We next consider the deviation process z^ϵ. By the definition of z^ϵ,

$$\dot{z}^\epsilon = z^\epsilon_{xx} + \frac{1}{\sqrt{\epsilon}}[g(t/\epsilon, u^\epsilon) - \bar{g}(u)]\,, \quad z^\epsilon(0) = 0,$$

with the zero Dirichlet boundary condition. For every $\alpha > 0$,

$$\begin{aligned}
\|A^\alpha z^\epsilon(t)\|_0 &= \left\| \frac{1}{\sqrt{\epsilon}} \int_0^t A^\alpha e^{A(t-s)} \left[g\left(\frac{s}{\epsilon}, u^\epsilon(s)\right) - \bar{g}(u(s))\right] ds \right\|_0 \\
&\leq \left\| \frac{1}{\sqrt{\epsilon}} \int_0^t A^\alpha e^{A(t-s)} \left[g\left(\frac{s}{\epsilon}, u^\epsilon(s)\right) - g\left(\frac{s}{\epsilon}, u(s)\right)\right] ds \right\|_0 \\
&\quad + \left\| \frac{1}{\sqrt{\epsilon}} \int_0^t A^\alpha e^{A(t-s)} \left[g\left(\frac{s}{\epsilon}, u(s)\right) - \bar{g}(u(s))\right] ds \right\|_0 \\
&:= I_1(t) + I_2(t)\,.
\end{aligned}$$

Notice that for $0 < \alpha < 1/2$,

$$\frac{1}{\sqrt{\epsilon}} \|A^\alpha e^{A(t-s)} \left[g\left(\frac{s}{\epsilon}, u^\epsilon(s)\right) - g\left(\frac{s}{\epsilon}, u(s)\right)\right] \|_0 \leq C\left(1 + \frac{1}{\sqrt{s}}\right) L_g \|z^\epsilon\|_0,$$

for some constant $C > 0$. Then

$$\mathbb{E} \sup_{0\leq t\leq T} I_1(t) \leq C_T,$$

for some constant $C_T > 0$. For I_2, by the factorization method again, we have

$$I_3 = \frac{\sin \pi\theta}{\theta} \int_0^t (t-s)^{\theta-1} e^A(t-s) A^\alpha Y^\epsilon(s)\, ds,$$

where Y^ϵ is defined as

$$Y^\epsilon(s) = \frac{1}{\sqrt{\epsilon}} \int_0^s (s-r)^\theta e^{A(s-r)} \left[g\left(\frac{r}{\epsilon}, u(r)\right) - \bar{g}(u(r))\right] dr.$$

Then

$$\mathbb{E} \sup_{0 \le t \le T} I_2(t) \le C_T \int_0^T \mathbb{E} \|A^\alpha Y^\epsilon(s)\|_0 \, ds,$$

for some $C_T > 0$. Notice that

$$\|A^\alpha Y^\epsilon(s)\|_0^2 = \frac{1}{\epsilon} \int_0^l \int_0^s \int_0^s (s-r)^\theta (s-\tau)^\theta A^\alpha e^{A(s-r)} \left[g\left(\frac{r}{\epsilon}, u(r,x)\right) - \bar{g}(u(r,x)) \right] \\ \times A^\alpha e^{A(s-\tau)} \left[g\left(\frac{\tau}{\epsilon}, u(\tau,x)\right) - \bar{g}(u(\tau,x)) \right] dr d\tau dx \,. \tag{39}$$

A standard discussion for the averaged equation yields that

$$\sup_{0 \le t \le T} \|u(t)\|_1^2 \le C_T,$$

for some constant $C_T > 0$. Then $A^\alpha e^{A(s-r)}[g(\frac{r}{\epsilon}, u(r,x)) - \bar{g}(u(r,x))] \in \mathcal{G}_0^r$ and $A^\alpha e^{A(s-\tau)}[g(\frac{\tau}{\epsilon}, u(\tau,x)) - \bar{g}(u(\tau,x))] \in \mathcal{G}_\tau^\infty$ and they are bounded real-valued random variables for fixed $x \in (0,l)$. Applying a mixing property [4, Proposition 7.2.2] and choosing positive parameters α and θ so that $\alpha + \theta < 1/2$, we have

$$\mathbb{E}\|A^\alpha Y^\epsilon(s)\|_0^2 \le C_T \,, \quad 0 \le s \le T,$$

and then

$$\mathbb{E} \sup_{0 \le t \le T} I_2(t) \le C_T,$$

for some constant $C_T > 0$. So for some $\alpha > 0$,

$$\mathbb{E}\|z^\epsilon\|_{C(0,T;H^{\alpha/2})} \le C_T \,.$$

Furthermore, for s, t with $0 \le s < t \le T$,

$$\|z^\epsilon(t) - z^\epsilon(s)\|_0^2 = \frac{2}{\epsilon} \left\| \int_s^t e^{A(t-r)} [g(r/\epsilon, u^\epsilon(r)) - \bar{g}(u(r))] \, dr \right\|_0^2 \\ + \frac{2}{\epsilon} \left\| (I - e^{A(t-s)}) \int_0^s e^{A(s-r)} [g(r/\epsilon, u^\epsilon(r)) - \bar{g}(u(r))] dr \right\|_0^2 .$$

Then via a similar discussion as that for (39), we conclude that for some $0 < \gamma < 1$,

$$\mathbb{E}\|z^\epsilon(t) - z^\epsilon(s)\|_0^2 \le C_T |t-s|^\gamma,$$

which yields the tightness of the distributions of z^ϵ in $C(0,T;H)$.

We decompose $z^\epsilon = z_1^\epsilon + z_2^\epsilon$ so that

$$\dot{z}_1^\epsilon = A z_1^\epsilon + \frac{1}{\sqrt{\epsilon}} [g(t/\epsilon, u) - \bar{g}(u)] \,, \quad z_1^\epsilon(0) = 0,$$

and

$$\dot{z}_2^\epsilon = A z_2^\epsilon + \frac{1}{\sqrt{\epsilon}} [g(t/\epsilon, u^\epsilon) - g(t/\epsilon, u)] \,, \quad z_2^\epsilon(0) = 0 \,.$$

For $\varphi \in C_b^\infty(0,l)$, we also consider the limit $\Phi(\langle z_1^\epsilon(t),\varphi\rangle)$ for every bounded second order differentiable function $\Phi:\mathbb{R}\to\mathbb{R}$ in the weak convergence method. Notice that

$$\Phi(\langle z_1^\epsilon(t),\varphi\rangle)-\Phi(\langle 0,\varphi\rangle)=\int_0^t \Phi'(\langle z_1^\epsilon(s),\varphi\rangle)\langle z_1^\epsilon(s),\varphi_{xx}\rangle\,ds$$
$$+\frac{1}{\sqrt{\epsilon}}\int_0^t \Phi'(\langle z_1^\epsilon(s),\varphi\rangle)\langle g(\frac{s}{\epsilon},u(s))-\bar g(u(s)),\varphi\rangle\,ds\,.$$

Define the following process

$$G_1^\epsilon(t):=\frac{1}{\sqrt{\epsilon}}\mathbb{E}\left[\int_t^\infty \Phi'\left(\langle z_1^\epsilon(t),\varphi\rangle\right)\left\langle g\left(\frac{s}{\epsilon},u(t)\right)-\bar g(u(t)),\varphi\right\rangle ds\Big|\mathcal{G}_0^{t/\epsilon}\right]. \tag{40}$$

A direct calculation yields that

$$G^\epsilon(t):=A^\epsilon\Phi(\langle z_1^\epsilon(t),\varphi\rangle)-A^\epsilon G_1^\epsilon(t)=\Phi'(\langle z_1^\epsilon(t),\varphi\rangle)\langle z_1^\epsilon(t),A\varphi\rangle+\Phi''(\langle z_1^\epsilon(t),\varphi\rangle)$$
$$\times\frac{1}{\epsilon}\int_t^\infty \mathbb{E}\left[\left\langle g\left(\frac{t}{\epsilon},u(t)\right)-\bar g(u(t)),\varphi\right\rangle\left\langle g\left(\frac{s}{\epsilon},u(t)\right)-\bar g(u(t)),\varphi\right\rangle|\mathcal{G}_0^{t/\epsilon}\right]ds$$
$$+\Phi''(\langle z_1^\epsilon(t),\varphi\rangle)\langle z_1^\epsilon(t),A\varphi\rangle\frac{1}{\sqrt{\epsilon}}\int_t^\infty \mathbb{E}\left[\left\langle g\left(\frac{s}{\epsilon},u(t)\right)-\bar g(u(t)),\varphi\right\rangle|\mathcal{G}_0^{t/\epsilon}\right]ds\,.$$

Define two bilinear operators

$$B^\epsilon(u,s,t):=2\left[g\left(\frac{t}{\epsilon},u\right)-\bar g(u)\right]\otimes\left[g\left(\frac{s}{\epsilon},u\right)-\bar g(u)\right],$$

and

$$\tilde B(u):=2\int_0^\infty \mathbb{E}\left[(g(t,u)-\bar g(u))\otimes(g(0,u)-\bar g(u))\right]dt\,.$$

Then by a mixing property [4, Proposition 7.2.2], we have

$$\mathbb{E}|G_1^\epsilon(t)|\to 0\,,$$

$$\mathbb{E}\left|\Phi(\langle z_1^\epsilon(t),\varphi\rangle)\left[\frac{1}{\epsilon}\int_t^\infty \mathbb{E}\left[\frac{1}{2}\langle B^\epsilon(u(t),s,t)\varphi,\varphi\rangle\,|\mathcal{F}_0^{t/\epsilon}\right]ds-\frac{1}{2}\left\langle\tilde B(u(t))\varphi,\varphi\right\rangle\right]\right|\to 0,$$

and

$$\mathbb{E}\left|\Phi''(\langle z_1^\epsilon(t),\varphi\rangle)\langle z_1^\epsilon(t),A\varphi\rangle\frac{1}{\sqrt{\epsilon}}\int_t^\infty \mathbb{E}\left[\left\langle g\left(\frac{s}{\epsilon},u(t)\right)-\bar g(u(t)),\varphi\right\rangle|\mathcal{G}_0^{t/\epsilon}\right]ds\right|\to 0,$$

as $\epsilon\to 0$. Then we also have a martingale

$$\mathcal{M}_t^\epsilon:=\Phi(\langle z_1^\epsilon(t),\varphi\rangle)-\int_0^t \Phi'(\langle z_1^\epsilon(s),\varphi\rangle)\langle z_1^\epsilon(s),\varphi_{xx}\rangle\,ds$$
$$-\frac{1}{2}\int_0^t \Phi''(\langle z_1^\epsilon(s),\varphi\rangle)\langle\tilde B(u)\varphi,\varphi\rangle\,ds+\mathcal{O}(\epsilon)\,.$$

By passing the limit $\epsilon \to 0$ and by the same discussion as in Section 2, we see that z_1^ϵ converges in distribution to z_1, which solves

$$dz_1 = Az_1 + \sqrt{\tilde{B}(u)}d\widetilde{W}\,, \quad z_1(0) = 0, \tag{41}$$

where $\widetilde{W}$ is a cylindrical Wiener process defined on a new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with covariance operator Id_H. Furthermore, z_2^ϵ converges in distribution to z_2, which solves

$$\dot{z}_2 = Az_2 + \overline{g'}(u)z\,, \quad z_2(0) = 0\,.$$

Then z^ϵ converges in distribution to z with z solving (36). The proof is complete. □

Remark 3. The assumption on the strong mixing property in (H) can be weakened as

$$\int_0^\infty \varphi^\alpha(t)\,dt < \infty,$$

for some $\alpha > 0$. In this case, we also have Theorem 2. See [15, 11] for more details.

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Appendix

Proof of Lemma 4

A similar result has been given in [11, Proposition 7]. Here we present another proof which gives a stronger convergence, together with the convergence rate in probability.

First, under the assumption ($\mathbf{H}_g$), we show that for almost all $\omega \in \Omega$,

$$\left\| \int_0^t \left[g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q) \right] dr \right\|_0 = \mathcal{O}(\sqrt{\epsilon}), \quad \epsilon \to 0, \tag{A.1}$$

for every $q \in H$.

Noticing

$$\int_0^t [g(r/\epsilon, q) - \bar{g}(q)]\, dr \in H\,,$$

we get

$$\left\| \int_0^t \left[g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q) \right] dr \right\|_0 = \sup_{\varphi \in H} \frac{1}{\|\varphi\|_0} \left| \left\langle \int_0^t \left[g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q) \right] dr, \varphi \right\rangle \right|.$$

Consider

$$\begin{aligned} &\mathbb{E}\left\langle \int_0^t \left[g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q) \right] dr, \varphi \right\rangle^2 \\ &\quad = \int_0^t \int_0^t \mathbb{E}\left\langle g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q), \varphi \right\rangle \left\langle g\left(\frac{s}{\epsilon}, q\right) - \bar{g}(q), \varphi \right\rangle drds\,. \end{aligned}$$

By a mixing property [4, Proposition 7.2.2], we have

$$\mathbb{E}\left\langle \int_0^t \left[g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q) \right] dr, \varphi \right\rangle^2 = \mathcal{O}(\epsilon)\|\varphi\|_0. \tag{A.2}$$

Noticing that

$$\left\| \int_0^t \left[g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q) \right] dr \right\|_0 = \sup_{\varphi \in H} \frac{\left\langle \int_0^t \left[g\left(\frac{r}{\epsilon}, q\right) - \bar{g}(q) \right] dr, \varphi \right\rangle}{\|\varphi\|_0},$$

by estimate (43) we have (42).

By the estimate in Section 2.1, for every $\kappa > 0$, there is $C_T^\kappa > 0$, which is independent of ϵ, such that

$$\mathbb{P}\left\{\|\hat{u}^\epsilon(t) - \hat{u}^\epsilon(s)\|_0 \le C_T^\kappa\sqrt{t-s}\right\} \ge 1-\kappa, \tag{A.3}$$

for every $t \ge s \ge 0$. Furthermore, by the tightness of the distributions of $\{u^\epsilon\}$ in space $C(0,T;H)$, for every $\kappa > 0$, there is a compact set $K_\kappa \subset C(0,T;H)$ such that

$$\mathbb{P}\{u^\epsilon \in K_\kappa\} \ge 1-\kappa\,. \tag{A.4}$$

So we define

$$\Omega_\kappa = \{\omega \in \Omega : \text{events in (42), (44) and (45) hold}\}\,.$$

Due to the compactness of K_κ, for every $\varepsilon > 0$, we only need to consider a finite ε-net $\{q_1, q_2, \dots, q_N\}$ in $C(0,T;H)$, which covers $\{u^\epsilon\}$. Without loss of generality, we assume that q_j, $j = 1, 2, \dots, N$, are simple functions [11].

Now we consider all $\omega \in \Omega_\kappa$. By the construction of $\hat{u}^\epsilon$ and boundedness of f, we have for $\omega \in \Omega_\kappa$

$$\|u^\epsilon(t) - u^\epsilon(s)\|_0 \le C_T^\kappa\sqrt{t-s} + \sqrt{\epsilon}C,$$

for some constant $C > 0$.

For every $\delta > 0$, we partition the interval $[0,T]$ into subintervals of length of δ. Then for $t \in [k\delta, (k+1)\delta)$, $0 \le k \le [\frac{T}{\delta}]$,

$$\begin{aligned}
\left\|\int_{k\delta}^t \left[g\left(\frac{r}{\epsilon}, u^\epsilon(r)\right) - \bar{g}(u^\epsilon(r))\right] dr\right\|_0 &\le \left\|\int_{k\delta}^t \left[g\left(\frac{r}{\epsilon}, u^\epsilon(r)\right) - g(\frac{r}{\epsilon}, u^\epsilon(k\delta))\right] dr\right\|_0 \\
&+ \left\|\int_{k\delta}^t \left[g\left(\frac{r}{\epsilon}, u^\epsilon(k\delta)\right) - g(\frac{r}{\epsilon}, q_j(k\delta))\right] dr\right\|_0 + \left\|\int_{k\delta}^t \left[g\left(\frac{r}{\epsilon}, q_j(k\delta)\right) - \bar{g}(q_j(k\delta))\right] dr\right\|_0 \\
&+ \left\|\int_{k\delta}^t \left[\bar{g}(q_j(k\delta)) - \bar{g}(u^\epsilon(k\delta))\right] dr\right\|_0 + \left\|\int_{k\delta}^t \left[\bar{g}(u^\epsilon(k\delta)) - \bar{g}(u^\epsilon(r))\right] dr\right\|_0,
\end{aligned}$$

for some q_j. Notice that, by the assumption ($\mathbf{H}_g$) and the definition of $\bar{g}$, $\bar{g}$ is also Lipschitz continuous in u with the same Lipschitz constant L_g. Then by the assumption ($\mathbf{H}_g$) and the definition of Ω_κ,

$$\begin{aligned}
&\left\|\int_0^t \left[g(\frac{r}{\epsilon}, u^\epsilon(r)) - \bar{g}(u^\epsilon(r))\right] dr\right\|_0 \\
&\le T[L_g C_T^\kappa\delta + \sqrt{\epsilon}C + L_g\varepsilon + \mathcal{O}(\sqrt{\epsilon}) + L_g\varepsilon + L_g C_T^\kappa\delta + \sqrt{\epsilon}C]\,.
\end{aligned}$$

Due to the arbitrary choice of δ, ε, and κ, and notice a similar discussion as that in [11], we thus complete the proof. □