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Stochastic analysis of localised damage for short fatigue cracks

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Abstract

The initiation and the growth of short fatigue cracks present two typical characteristics: collective evolution and stochastic response. In this paper, the method taking into account the balance of crack number density is adopted to investigate the stochastic behaviour of short-crack collective evolution. The results obtained from the analysis of the stochastic equation for Local-Crack-Number-Density (LCND) illustrate the stochastic tendency of crack development. Moreover, the evolution equation of Overall-Crack-Number-Density (OCND) is derived by the analytic procedure of LCND. The result indicates that the stochastic behaviour of OCND is dominated by the fluctuation of local crack-growth-rate.

1 Introduction

It has been observed that, the evolution process of short fatigue cracks in some metallic materials presents collective damage characteristics. The cumulation of the damage is produced by a number of localized short fatigue cracks[1-6]. The extent of damage is not dependent on a single crack, but on the whole response of total short cracks. For this situation, we adopted the method of the balance of crack-number-density to describe such an evolutionary process[7, 8]. The basic consideration of the model is

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that, at a certain time duration, the number of cracks with a given length is due to two aspects: (a) crack nucleation, and (b) crack growth. The evolution equation of short cracks is derived and its non-dimensional form is[7]:

$$\frac{\partial}{\partial t}\bar{n}'(c,t)+\frac{\partial}{\partial c}\left[\bar{A}'(c)\cdot\bar{n}'(c,t)\right]=N_{\mathrm{g}}\cdot\bar{n}'_{\mathrm{N}}(c) \ , \quad (1)$$

where $\bar{A}'$ is crack growth rate, $\bar{n}'_{\mathrm{N}}$ is crack nucleation rate, and $\bar{n}'$ is crack number density with $\bar{n}'\mathrm{d}c$ being the number of cracks with the length between c and $c+\mathrm{d}c$ at time t. Also in eqn. (1), the non-dimensional parameter $N_{\mathrm{g}}=(n_{\mathrm{N}}^{*}\cdot d)/(n^{*}\cdot A^{*})$, with n_{N}^{*} being the characteristic crack nucleation rate, A^{*} the characteristic crack growth rate, n^{*} the characteristic crack number density, and d the intrinsic dimension of the material concerned (e.g. grain diameter). After setting the expressions for $\bar{A}'$ and $\bar{n}'_{\mathrm{N}}$, we are able to show the evolution characteristics of crack number density[7, 8].

In the above analysis, the evolution of short cracks is considered uniform in the gauge area of a specimen. However, during fatigue damage process, the initiation and the growth of short cracks are always randomly distributed. In some local areas, short cracks may densely appear; simultaneously, there may exist some other areas even without any short crack damage[1-4]. This suggests that the collective damage of short fatigue cracks is of stochastic feature, which calls for further analyses for a better understanding for such a process. In this paper, a stochastic differential equation is established to analyze the evolution of crack number density with the consideration of the effect of localized crack damage. Consequently, the evolution equation of Overall-Crack-Number-Density (OCND) is derived based on the Local-Crack-Number-Density (LCND) analysis. The analytic results show the difference between the stochastic model and the mean-field theory, which have then been discussed.

2 Stochastic model and its equations

Denote $n(c,t,\vec{x})$ as LCND to describe the fatigue damage, which represents the crack-number-density of a local area. Such an area is small enough comparing with the macro-scale of material, and in turn, it must contains enough number of short cracks to fulfil the requirement by the LCND analysis method. The evolution equation is:

$$\frac{\partial}{\partial t}n(c,t,\vec{x})+\frac{\partial}{\partial c}\left[A(c,t,\vec{x})\cdot n(c,t,\vec{x})\right]=N_{\mathrm{g}}\cdot n_{\mathrm{N}}(c,t,\vec{x}) \ , \quad (2)$$

where $A(c,t,\vec{x})$ and $n_{\mathrm{N}}(c,t,\vec{x})$ denote respectively the crack growth rate and the crack nucleation rate in the corresponding local area. The research by Hong & Qiao[9] indicated that the damage extent due to collective short cracks dominated by the 0-th order of damage moment $D_0(t,\vec{x})$, i.e. the number of total cracks. If only the correlation between the nearest neighbouring cracks is taken into account, one may write

$$A(c,t,\vec{x}) = A\left[c,\vec{x},D_0(t,\vec{x}),\frac{\partial D_0(t,\vec{x})}{\partial \vec{x}}\right], \tag{3}$$

and

$$n_N(c,t,\vec{x}) = n_N\left[c,\vec{x},D_0(t,\vec{x}),\frac{\partial D_0(t,\vec{x})}{\partial \vec{x}}\right]. \tag{4}$$

Consider that the influence caused by damage cumulation and crack length in above equations can be treated separately and assume that the number of total cracks can be expressed as an exponential type. Such that,

$$A(c,t,\vec{x}) = \left[A_0(c) + L(c)\cdot W_1(\vec{x})\right]\cdot\left[\eta\cdot D_0^{\xi}(t,\vec{x})\right], \tag{5}$$

and

$$n_N(c,t,\vec{x}) = \left[n_{N0}(c) + L^*(c)\cdot W_2(\vec{x})\right]\cdot\left[p\cdot D_0^{q}(t,\vec{x})\right], \tag{6}$$

where $A_0(c)$ and $n_{N0}(c)$ are the average part of the crack growth rate and the crack nucleation rate respectively; $W_1(\vec{x})$ and $W_2(\vec{x})$ are the white noise representing the two independent stochastic processes; $L(c)$ and $L^*(c)$ are variables governing the stochastic process; and η, ξ, p and q are material constants. Combining eqns. (5), (6) and (2), one may derive the following stochastic equation for the concerned problem:

$$\frac{\partial}{\partial t}n(c,t,\vec{x}) + \eta_1\left\{t\left[a + bW_2(\vec{x})\right]\right\}^{\frac{\xi}{1-q}}\cdot\frac{\partial}{\partial c}\left\{n(c,t,\vec{x})\left[A_0(c) + L(c)W_1(\vec{x})\right]\right\}$$

$$= N_g\cdot p_1\left[n_{N0}(c) + L^*(c)\cdot W_2(\vec{x})\right]\left\{t\left[a + b\cdot W_2(\vec{x})\right]\right\}^{\frac{q}{1-q}}, \tag{7}$$

where $\eta_1 = \eta(1-q)^{\frac{\xi}{1-q}}$, $p_1 = p(1-q)^{\frac{\xi}{1-q}}$, $a = \int_0^{\infty} p\cdot n_{N0}(c)\,\mathrm{d}c$, and $b = \int_0^{\infty} p\cdot L^*(c)\,\mathrm{d}c$.

3 Numerical results and discussion

In the calculation of eqn. (7), we set $\xi = -q = 0.5$, $\eta = p = 1.0$ and $A_0(c)$ and $n_{N0}(c)$ are assumed to be the following forms:

$$A_0(c) = \begin{cases} 1-(1-A_\mathrm{d})c & (c \le 1) \\ \bar{d} \cdot c & (c > 1) \end{cases} \tag{8}$$

$$n_{\mathrm{N}0}(c) = \begin{cases} 1-\dfrac{c}{2} & (c \le 2) \\ 0 & (c > 2) \end{cases} \tag{9}$$

where A_d is the crack growth rate at $c=1$ and $\bar{d}$ is the normalized average grain size. $L(c)$ and $L^*(c)$ are assumed with the same form as eqns. (8) and (9). For the convenience in the following discussion, we denote $\widetilde{D}_0(t)$ as the number of total cracks in the whole area of observation and $\widetilde{c}_{\max}(t)$ as the largest crack length in the whole area of observation.

Figure 1 is the result of a simulation, showing the stochastic evolution of $c_{\max}$ for different local areas at different time stages. It is seen that the four local areas labelled A, B, C and D are four typical evolution sites. Area A—the value of $c_{\max}$ is very small for the whole process; area B—$c_{\max}$ is small at the beginning of fatigue damage but it grows up rapidly and becomes a main crack site in the whole field; area C—the crack at this site is always one of the largest cracks in the whole field; and area D—the crack at this site is one of the largest at the beginning of fatigue process but its growth rate slows down and eventually it cannot act as one of the largest cracks for the whole field.

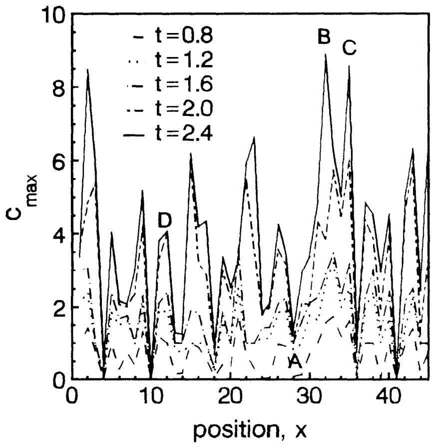

Figure 1: Variation of $c_{\max}$ with position x and time t.

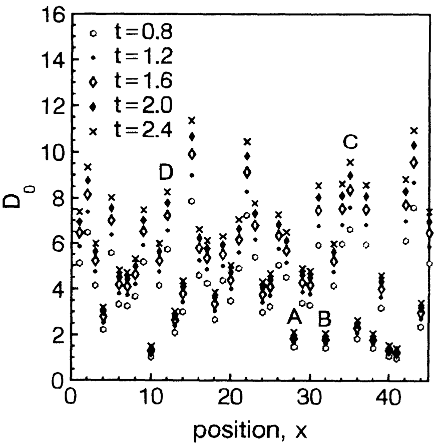

Figure 2: Variation of D_0 with position x and time t.

Figure 2 illustrates the distribution of D_0 at different local areas for the same simulation as Fig. 1. A small value of D_0 for areas A and B is observed from beginning to later stages, and a comparative large value of D_0 is seen at areas C and D. The results of Figs. 1 and 2 illustrate 4 distinct situations: area A—unfavourable both for crack initiation and growth; area B—unfavourable for crack initiation but favourable for crack growth; area C—favourable both for crack initiation and growth; and area D—favourable for crack initiation but unfavourable for crack growth. The results demonstrate the stochastic distribution of short crack development caused by the independent influence of crack initiation and crack growth, which imply that fast damage development area in the beginning stage may not be the location of a main crack, whereas the local area with slow speed of damage development at the beginning may form a main crack leading to final fracture.

Figure 3 shows the results of multiple simulations. The solid curve represents the result of the mean-field theory, i.e. setting $L(c) \equiv 0$ and $L^*(c) \equiv 0$ in eqns. (5) and (6) and the dash curve represents the best-fit result from the datum points of stochastic simulations. It is seen from Fig. 3 that, with the progression of fatigue process, the difference between the two cases becomes evident, showing the value of $\tilde{c}_{max}$ obtained by the mean-field theory larger than that obtained by the stochastic analysis. In addition, Fig. 4 demonstrates the correlation between $\tilde{c}_{max}$ and $\tilde{D}_0$, in which the solid curve represents the result from the mean-field theory and the datum points denote the results from the stochastic simulations. It is seen that the results from the two methods have the same trend.

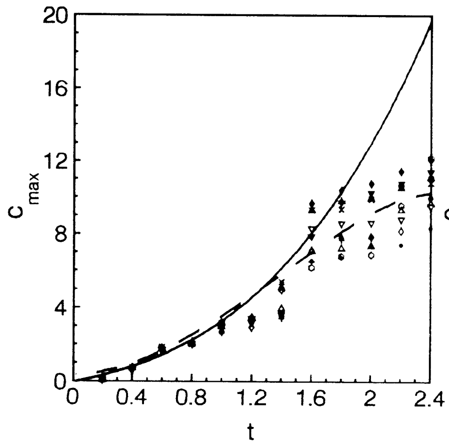

Figure 3: Results of $\tilde{c}_{max}$ versus t.

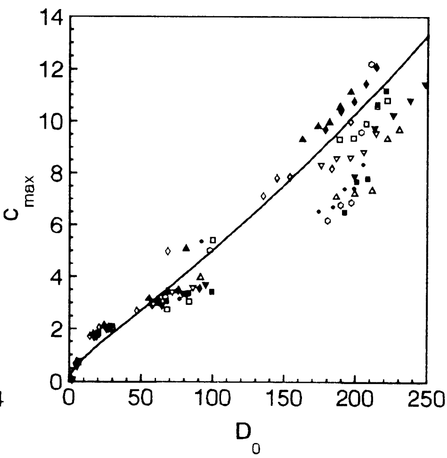

Figure 4: Results of $\tilde{c}_{max}$ versus $\tilde{D}_0$.

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4 Evolution equation of OCND

Define that $\bar{n}(c,t)$ is the average crack number density of unit surface area S corresponding to the region of Ω, i.e.

$$\bar{n}(c,t)=\frac{1}{S}\int_{\Omega} n(c,t,\vec{x})\,\mathrm{d}x \ . \tag{10}$$

Define also that $\tilde{n}(c,t,\vec{x})$ is the fluctuation of crack number density within local area $\vec{x}$. Thus we have

$$n(c,t,\vec{x})=\bar{n}(c,t)+\tilde{n}(c,t,\vec{x}) \ . \tag{11}$$

Similarly, for crack growth rate and crack nucleation rate, we may write

$$A(c,t,\vec{x})=\bar{A}(c,t)+\tilde{A}(c,t,\vec{x}) \ , \tag{12}$$

and

$$n_{\mathrm{N}}(c,t,\vec{x})=\bar{n}_{\mathrm{N}}(c,t)+\tilde{n}_{\mathrm{N}}(c,t,\vec{x}) \ , \tag{13}$$

where $\bar{A}(c,t)$ and $\bar{n}_{\mathrm{N}}(c,t)$ are the average part of crack growth rate and crack nucleation rate respectively; and $\tilde{A}(c,t,\vec{x})$ and $\tilde{n}_{\mathrm{N}}(c,t,\vec{x})$ are the fluctuation of crack growth rate and crack nucleation rate respectively. Substituting eqns. (11)—(13) into (2), one derives

$$\frac{\partial \bar{n}(c,t)}{\partial t}+\frac{\partial\left[\bar{A}(c,t)\bar{n}(c,t)\right]}{\partial c}=N_{\mathrm{g}}\,\bar{n}_{\mathrm{N}}(c,t)-\overline{\left\{\frac{\partial\left[\tilde{A}(c,t,\vec{x})\tilde{n}(c,t,\vec{x})\right]}{\partial c}\right\}} \ . \tag{14}$$

Eqn. (14) is the evolution equation of OCND, which describes the balance of crack number density in the whole field and takes into account the influence of fluctuation during the stochastic evolution of localized short crack damage. Note that there is one extra item in the right hand side of eqn. (14) upon comparing with eqn. (1). We denote this as fluctuation influence term (FIT), i.e.

$$\overline{\left\{\frac{\partial\left[\tilde{A}(c,t,\vec{x})\cdot\tilde{n}(c,t,\vec{x})\right]}{\partial c}\right\}}=\frac{1}{S}\int_{\Omega}\left\{\frac{\partial\left[\tilde{A}(c,t,\vec{x})\cdot\tilde{n}(c,t,\vec{x})\right]}{\partial c}\right\}\mathrm{d}\vec{x}$$

$$=\frac{\partial}{\partial c}\left\{\frac{1}{S}\int_{\Omega}\left[\tilde{A}(c,t,\vec{x})\cdot\tilde{n}(c,t,\vec{x})\right]\mathrm{d}\vec{x}\right\}=\frac{\partial \mu(c,t)}{\partial c} \ , \tag{15}$$

where $\mu(c,t)$ is the covariance of $\tilde{A}(c,t,\vec{x})$ and $\tilde{n}(c,t,\vec{x})$. It is reasonable to anticipate that the difference between the results of the mean-field theory and the stochastic analysis shown in Fig. 3 is associated with FIT.

On the other hand, one may define the effective crack growth rate as

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$$\hat{A}(c,t) = \frac{\overline{A(c,t,\vec{x}) \cdot n(c,t,\vec{x})}}{\overline{n}(c,t)} = \frac{\overline{A}(c,t) \cdot \overline{n}(c,t) + \overline{\widetilde{A}(c,t,\vec{x}) \cdot \widetilde{n}(c,t,\vec{x})}}{\overline{n}(c,t)} \quad (16)$$

Substituting eqn. (16) into (14), one may write the equation for OCND as

$$\frac{\partial \overline{n}(c,t)}{\partial t} + \frac{\partial\left[\hat{A}(c,t) \cdot \overline{n}(c,t)\right]}{\partial c} = N_{\mathrm{g}} \cdot \overline{n}_{\mathrm{N}}(c,t) \quad (17)$$

This is another type of OCND evolution equation. Comparing eqn. (17) with (1), one sees that the effective crack growth rate replaces the average crack growth rate when the fluctuation effect is taken into consideration.

5 Discussion on difference of results from mean-field theory and stochastic analysis

Experimental observations have reported that short crack growth rate predominantly depends on the crack length and it is less sensitive to the fatigue cycles that the specimen experienced. Thus we assume that the crack growth rate is independent of t. Then eqn. (14) becomes:

$$\frac{\partial \overline{n}(c,t)}{\partial t} + \frac{\partial\left[\overline{A}(c)\overline{n}(c,t)\right]}{\partial c} = N_{\mathrm{g}}\, \overline{n}_{\mathrm{N}}(c,t) - \overline{\left\{\frac{\partial\left[\widetilde{A}(c,\vec{x})\widetilde{n}(c,t,\vec{x})\right]}{\partial c}\right\}} \quad (18)$$

The FIT in the above equation can be expressed as:

$$\frac{\partial \mu}{\partial c} = n_{\mathrm{N}}(c)\left[1 - \overline{A}(c)\overline{\left(\frac{1}{A(c,\vec{x})}\right)}\right] - \frac{\partial}{\partial c}\left[\overline{A}(c)\overline{\left(\frac{1}{A(c,\vec{x})}\right)}\right]\int_0^c n_{\mathrm{N}}(c')\,\mathrm{d}c' \quad (19)$$

Assume that the distribution of local crack growth rate is a logarithm-normal function, then the probability for $A(c,\vec{x}) = v$ is

$$f(v) = \begin{cases} \dfrac{1}{\sqrt{2\pi} \cdot \sigma'(c) \cdot v} \exp\left\{-\dfrac{\left[\ln v - \mu'(c)\right]^2}{2\sigma'^2(c)}\right\} & (v \geq 0) \\ 0 & (v < 0) \end{cases} \quad (20)$$

The mean value of $f(v)$ is

$$E(c) = \overline{A}(c) = \exp\left[\mu'(c) + \frac{\sigma'^2(c)}{2}\right], \quad (21)$$

and the variance of $f(v)$ is

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$$D(c) = \exp\left[2\mu'(c) + \sigma'^2(c)\right]\left\{\exp\left[\sigma'^2(c)\right] - 1\right\} . \quad (22)$$

Substituting eqns. (21) and (22) into (19), one derives

$$\frac{\partial \mu}{\partial c} = \left\{2\sigma'(c)\cdot e^{\sigma'^2(c)}\int_0^c n_N(c')\mathrm{d}c'\right\}\frac{\partial \sigma'(c)}{\partial c} - n_N(c)\left[e^{\sigma'^2(c)} - 1\right] . \quad (23)$$

Hence, the problem can be calculated by the substitution of eqn. (23) into (14). In the calculation, the distribution parameter $\sigma'(c)$ is set as

$$\sigma'(c) = \alpha \cdot \exp(\beta \cdot c) , \quad (24)$$

where α and β are constants and $n_N(c)$ is of the form of eqn. (9).

Figure 5 is the calculation result showing the variation of FIT with c and β. It is seen that the values of FIT are slightly above zero at the negative part of β axis. When $c > 1$ and $\beta > 0$, the values of FIT turn to below zero and sharply decline to form a negative peak.

From eqns. (21) and (22), the following equation can be derived

$$\sigma'(c) = \sqrt{\ln\left[1 + \frac{D(c)}{\bar{A}^2(c)}\right]} . \quad (25)$$

For the further discussion, we define

$$Q = \frac{1}{2}\cdot\frac{\partial \ln\left[D(c)\right]/\partial c}{\partial \ln\left[\bar{A}(c)\right]/\partial c} . \quad (26)$$

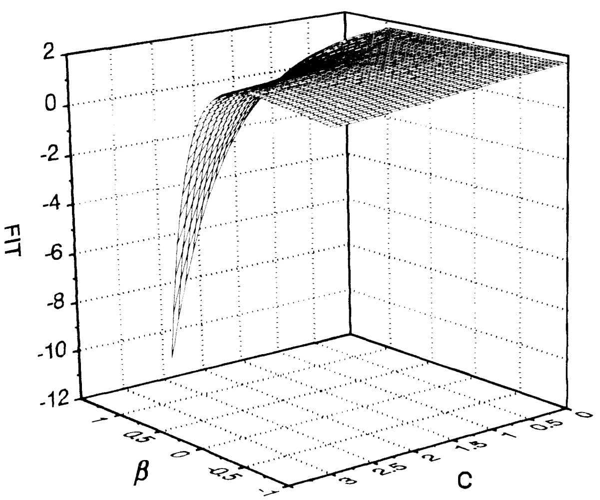

Figure 5: Values of FIT as a function of c and β.

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From eqns. (26) and (25), we see that if $Q > 1$, i.e. the variation speed of $D(c)$ with respect to c is faster than that of $\bar{A}^2(c)$, the variation tendency of $\sigma'(c)$ is of the same trend with that of $\bar{A}(c)$. Because of the deceleration-acceleration pattern for short fatigue cracks (e.g. Lankford[10]), in short-crack regime, there exists $[\partial\sigma'(c)/\partial c] < 0$ with $\beta < 0$. However, in long-crack regime, crack growth rate is proportional to crack length, so that $[\partial\sigma'(c)/\partial c] > 0$ with $\beta > 0$. Referring to Fig. 5, we note that the value of β varies from -1 to $+1$ corresponds to the process that fatigue damage progresses from short-crack regime into long-crack regime. In other words, the collective damage transfers to more localized cracking process. In the primary stage of fatigue damage, the value of β is below zero, and the corresponding values of FIT are around zero or the stochastic fluctuation is less sensitive in this stage. Thus, the results of the maximum crack length obtained by the mean-field theory and the stochastic analysis are almost consistent with each other. With the progress of fatigue damage onto the stage that a few main cracks form, such a process corresponds to the value of β gradually increasing to above zero. At this stage, the values of FIT sharply tern town to produce a negative peak as shown in Fig. 5. This implies that the crack growth is suppressed by the stochastic fluctuation. Therefore, the maximum crack length obtained by the stochastic analysis is evidently smaller than that obtained by the mean-field theory. The above discussion explains the results shown in Fig. 3.

6 Conclusions

The following conclusions are drawn for this research:

(1) Collective fatigue damage presents remarkably stochastic characteristics. The simulation results obtained by the stochastic evolution equation illustrate the random distribution of short crack population.

(2) The stochastic behaviour of short crack damage is dominated by the fluctuation of crack growth rate. The difference in the results of the maximum crack length obtained by the stochastic analysis and the mean-field theory is associated with the propensity of crack growth rate.

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