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THE THREE-DIMENSIONAL UNSTEADY BOUNDARY LAYER OVER AN IMPULSIVELY STARTED ROTATING DISK

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Abstract

The unsteady boundary layer over an impulsively started rotating disk is studied, a complete solution describing the smooth transition from vortex diffusion at $\omega t=0$ *to Kármán's steady solution is obtained by series expansion and its numerical continuation. The angle of body streamlines, together with experimental values, are given as the function of time t as well as the moment coefficient* c_M *and on-coming velocity* $w(\infty)$.

Two- and Three-dimensional inviscid hydrodynamics of turbomachine has been well developed, the studies on different effects, such as unsteadiness or viscosity, are in full swing. One of the basic and simple problems related to rotating bodies is the boundary layer over a steady rotating disk (Kármán[1][2]) . Its generalization is the boundary layer on an impusively started or a periodic perturbed rotating disk, which has both theoretical significance and practical application.

Thiriot[3] first studied the former problem, expanded the solution in series near $t=0$, gave the full first-order terms and the second-order term of circumferential velocity component g_1 which were quoted in several books[4][5]. But his g_1 was in error, nearly one third bigger than the correct value, because his coefficient of the term-$8h_0g_0'$ in the nonhomogeneous equation was wrongly doubled. In addition, the series were valid only for $\tau<1$, hence his attempt to apply his theory to the region $\tau\geqslant1$ to explain the experiment was evidently not reasonable. In the present paper, g_1 is corrected, the full second-order approximations are obtained and taken as the initial values for numerical continuations, the compiete solution describing the transition from the initial state to the final steady state is obtained.

For an impulsively started rotating disk (the angular velocity ω =constant), besides $\zeta=z\sqrt{\omega/\nu}$ introduced by Kármán, similar variables include dimensionless time $\tau=\omega t$, which represents the ratio of two characteristic lengths in z direction: $\sqrt{\tau}=\sqrt{\nu t}/\sqrt{\nu/\omega}$, where $\sqrt{\nu t}$ and $\sqrt{\nu/\omega}$ represent the diffusion length and the viscous affecting distance. Here r, θ , z and u, v, w are radial, circumferential, axial coordinates and velocity components respectively. ν is the kinetic viscosity coefficient.

When $\sqrt{\nu t}\gg\sqrt{\nu/\omega}$, or $\tau\gg1$ the solution should approach the steady one, hence we will express the solution in the following form:

$$u=\omega r f(\zeta,\tau),\quad v=\omega r g(\zeta,\tau),\quad w=\sqrt{\nu\omega}\,h(\zeta,\tau)$$

The governing equations and boundary conditions are as follows:

$$\frac{\partial^2 f}{\partial\zeta^2}-h\frac{\partial f}{\partial\zeta}+g^2-f^2=\frac{\partial f}{\partial\tau},\quad \frac{\partial^2 g}{\partial\zeta^2}-h\frac{\partial g}{\partial\zeta}-2fg=\frac{\partial g}{\partial\tau}\quad \frac{\partial h}{\partial\zeta}+2f=0 \tag{1}$$

$$\left.\begin{aligned}&\zeta=0:\quad f=h=0,\ g=1\\ &\zeta\to\infty:\quad f=g=0\\ &\tau=0:\quad f=g=h=0\end{aligned}\right\} \tag{2}$$

The momentum equation in z direction can be used to find the pressure p (let $p=\rho\nu\omega P$).

$$\frac{\partial P}{\partial\zeta}=\frac{\partial^2 h}{\partial\zeta^2}-h\frac{\partial h}{\partial\zeta}-\frac{\partial h}{\partial\tau}\,.$$

When $\sqrt{\nu t}\ll\sqrt{\nu/\omega}$ (i.e. $\omega t\ll 1$) the process is mainly the diffusion of the vortex line at $z=0$, just as in the case of an impulsively started plate, the appropriately similar variable should be

$$\eta=z/2\sqrt{\nu t}=\zeta/2\sqrt{\tau} \tag{3}$$

The first order approximation is the function of η only, and the effect of τ will occur only in higher order approximations. The solutions are expressed as series of τ :

$$\left.\begin{aligned}&f=\tau f_0(\eta)+\tau^3 f_1(\eta)+\cdots\\ &g=g_0(\eta)+\tau^2 g_1(\eta)+\cdots\\ &h=-4[\tau^{\frac{3}{2}}h_0(\eta)+\tau^{\frac{7}{2}}h_1(\eta)+\cdots]\end{aligned}\right\} \tag{4}$$

The first order approximation is found by substituting (4) into (1) as[3][4]

$$\left.\begin{aligned}&g_0''+2\eta g_0'=0,\quad h_0'=f_0\\ &f_0''+2\eta f_0'-4f_0=-4g_0^2\\ &g_0(0)=1,\quad g_0(\infty)=f_0(0)=f_0(\infty)=h_0(0)=0\end{aligned}\right\} \tag{5}$$

So $g_0=1-\mathrm{erf}(\eta)=\mathrm{erfc}(\eta)$ error function. Following[6][7] we introduce the function

$$\varphi_a(\eta)=\frac{2}{\sqrt{\pi}\,\Gamma(2a+1)}\int_\eta^\infty (x-\eta)^{2a}e^{-x^2}dx$$

which satisfies

$$\varphi_a''+2\eta\varphi_a'-4a\varphi_a=0$$

and has the properties

$$\varphi_a(\infty)=0,\quad \varphi_a(0)=1/(2^{2a}\Gamma(a+1)),\quad \varphi_a'(\eta)=-\varphi_{a-\frac{1}{2}}(\eta)$$

and we express the first-order approximation as

$$\left.\begin{aligned}&f_0(\eta)=\frac{8}{\pi}\varphi_1(\eta)-2\,[\varphi_{\frac{1}{2}}(\eta)]^2,\quad f_0'(0)=\frac{4}{\sqrt{\pi}}\left(1-\frac{2}{\pi}\right)\\ &h_0(\eta)=\int_0^\eta f_0(x)dx\end{aligned}\right\} \tag{6}$$

The second-order approximation is in terms of nonhomogeneous linear equations

$$\left.\begin{aligned}&g_1''+2\eta g_1'-8g_1=8f_0g_0-8h_0g_0'\\&f_1''+2\eta f_1'-12f_1=4f_0^2-8h_0f_0'-8g_0g_1\\&h_1'=f_1\\&g_1(0)=g_1(\infty)=f_1(0)=f_1(\infty)=h_1(0)=0\end{aligned}\right\} \tag{7}$$

These two-point boundary problems can be reduced to the initial value problems of the corresponding homogeneous equations

$$\left.\begin{aligned}&\psi_1''+2\eta\psi_1'-8\psi_1=0,\ \ \chi_1''+2\eta\chi_1'-12\chi_1=0\\&\psi_1(0)=\chi_1(0)=0,\ \ \psi_1'(0)=\chi_1'(0)=1\end{aligned}\right\} \tag{8}$$

and the solution G_1, F_1 of the nonhomogeneous equations (7) satisfying zero initial condition $G_1(0)=F_1(0)=G_1'(0)=F_1'(0)=0$. Hence,

$$g_1(\eta)=G_1(\eta)-G_1(\infty)\psi_1(\eta)/\psi_1(\infty),\ \ f_1(\eta)=F_1(\eta)-F_1(\infty)\chi_1(\eta)/\chi_1(\infty) \tag{9}$$

Now turn to the problem of continuously smooth transition from the starting state to the steady state. Firstly, the solution at some time τ_0 (0.1 in our case) can be found from the expansion, e.g.

$$f(\zeta,\tau_0)=\tau_0f_0(\zeta/2\sqrt{\tau_0})+\tau_0^3f_1\ (\zeta/2\sqrt{\tau_0})$$

Using these as the initial value we can numericably integrate the parabolic eq. (1). The central difference for s-derivative and the implicit three-point forward difference formula (which has second-order accuracy) for τ -derivative are adopted.

$$\left.\begin{aligned}&\frac{\partial^2 f}{\partial\zeta^2}=\frac{f_{i+1,j}+f_{i-1,j}-2f_{i,j}}{\Delta\zeta^2}+O(\Delta\zeta^2)\\&\frac{\partial f}{\partial\zeta}=\frac{f_{i+1,j}-f_{i-1,j}}{2\Delta\zeta}+O(\Delta\zeta^2)\\&\frac{\partial f}{\partial\tau}=\frac{3f_{i,j}-4f_{i,j-1}+f_{i,j-2}}{2\Delta\tau}+O(\Delta\tau^2)\end{aligned}\right\} \tag{10}$$

here $f_{i,j}$ stands for $f(i\Delta\zeta,\tau_0+j\Delta\tau)$. The difference equations are in a linear equation system with three-diagonal coefficient matrix; for example

$$a_if_{i-1,j}+b_if_{i,j}+c_if_{i+1,j}=d_i \tag{11}$$

$$\left.\begin{aligned}&a_i=2\Delta\tau/\Delta\zeta^2+4h_{i,j}\Delta\tau/\Delta\zeta,\ c_i=2\Delta\tau/\Delta\zeta^2-4h_{i,j}\Delta\tau/\Delta\zeta\\&b_i=-4\,\Delta\tau/\Delta\zeta^2-3,\ d_i=-4f_{i,j-1}+f_{i,j-2}-2(g_{i,j}^2-f_{i,j}^2)\Delta\tau\end{aligned}\right\} \tag{12}$$

The equation system for g is the same except that d_i is replaced by e_i

$$e_i=-4g_{i,j-1}+g_{i,j-2}+4g_{i,j}f_{i,j}\Delta\tau \tag{13}$$

These two systems of equations were solved by Thomas algorithm. Iteration is neccessary due to nonlinearity. The first estimation for $f_{i,j}$, $g_{i,j}$ in the coefficients a_i, b_i, etc. comes from extrapolation, and the corrections are obtained from the solution of the difference equations. The procedure is repeated until the relative error is less than one thousandth. Initial values at $\tau=\tau_0$ and

$\tau=\tau_0-\Delta\tau$ are required, because three-point difference formulae are adopted for $\partial f/\partial\tau$.

$\Delta\zeta=0.05$ and $\Delta\tau=0.05$ are used. It is found that the unsteady solution is already within one percent of the steady solution when $\tau\geqslant 5$. The profiles of $g(\zeta,\tau)$ and $f(\zeta,\tau)$ at three typical values of τ are shown in Fig. 1 and Fig. 2. The variations of moment coefficients c_M and $w(\infty)$ with time are given in Fig. 3, in which

$$M=2\pi\mu\int_0^R r^2\frac{\partial v}{\partial z}\,dr\quad,\quad c_M=2M\Big/\left(\frac{1}{2}\rho\omega^2R^5\right)$$

The outward diffusion of the vortex line, the decrease of c_M and the increase of $w(\infty)$ are clearly displayed. v approaches the steady value faster than u and w. The inclination angle of the body streamlines (logarithmic spirals)

$$m=\frac{1}{r}\,\frac{dr}{d\theta}=[f/(g-1)]_{\zeta=0}$$

is shown in Fig. 4, and found in fair agreement with the experiment.[3][4]

The values of g_l and f_l are given in the following Table.

Table 1

η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$-g_1\times10^4$	181	327	424	474	484	466	428	379	326	272	222	177	138	106	79	58
$-f_1\times10^4$	49	91	120	136	140	133	120	103	85	68	52	39	29	20	14	10

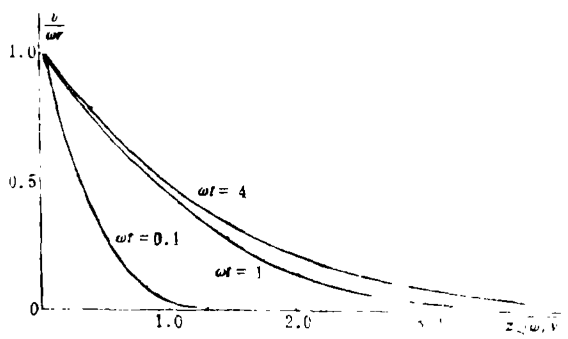

Fig. 1

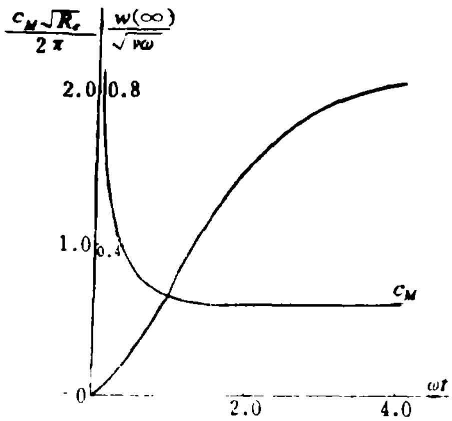

Fig. 2

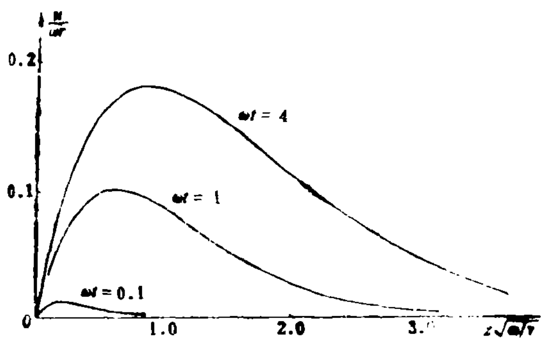

Fig. 3

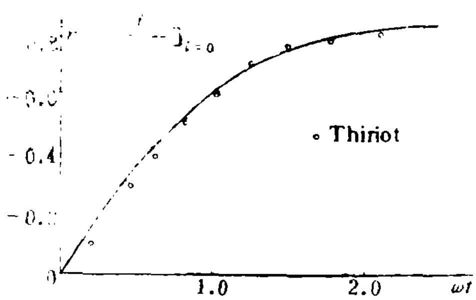

Fig. 4

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