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Z_n-EQUIVARIANT SINGULARITY THEORY*

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Abstract

The basic concepts, normal forms and universal unfoldings of Z_n-equivariant singularity are investigated in the present paper. As an example, the normal forms and universal unfoldings of Z_3-singularity are formulated. As a matter of fact, the theory provides a useful tool to study the subharmonic resonance bifurcation of the periodic parameter-excited system.

Key words . Z_n-equivariant singularity, resonance bifurcation, periodic parameter-excited system

I. Introduction

Parameter-excited systems are important in vibration engineering. The governing equation (e.g. Mathieu equation) is a periodic non-autonomous system. To investigate their subharmonic resonant bifurcation the Liapunov-Schmidt reduction with time symmetry is necessarily used to obtain the Z_n-equivariant algebraic bifurcation equations, which is Z_n-equivalent to the governing equations. In order to study the Z_n-equivariable bifurcation problem, the Z_n-equivariant singularity must be formulated.

Although the singularity theory was used to investigate the local bifurcation problems a few years ago, the symmetric singularity theory still under developing has just been made use of quite recently. The study of symmetric singularity was initially carried out by Sattinger[2]. The general results have been given by Golubitsky[1], who studied in detail the singularity theory under the action of contact Lie groups $O(2)$, $SO(2)$, $Z_2\oplus Z_2$ and D_n. Buzano and co-workers[3] formulated the normal forms and universal unfoldings of singularity under the action of the dihedral group D_n while studying the buckling behavior of a thin rod. The present paper purports to deal with Z_n-equivariant theory. Followed by a brief introduction, the Z_n-equivariant mapping over complex field is formulated in the second section. The basic concepts and methods of the Z_n-equivariant singularity are discussed in the next section, and as an example we illustrate how normal forms and universal unfoldings are derived by using Z_3-singularity. Finally, some useful conclusions are drawn.

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II. Z_n-Equivariant Mapping

Let $Z_n=\left\{\exp\left[i\frac{2k\pi}{n}\right],\ k=0,\ 1,\ 2,\ \cdots,\ n-1\right\}$ be the cyclic group of order n. It acts on the complex field C in a standard way:

$$\exp\left[i\frac{2\pi}{n}\right]\cdot z=z\exp\left[\frac{2\pi}{n}\right]$$

where $z\in C$, $g(z,\ \lambda)$ is a smooth mapping from $C\times R$ to C such that (1) g is Z_n-equivariant about the variable z. (2) (0, 0) is the singular point of g . The goal of this section is to present the general expression of Z_n-equivariant mappings. The bifurcation parameter λ is temporarily omitted since λ is irrelevant to equivariance.

Theorem 2.1 The set of the complex Z_n-invariant polynomials $e_C(Z_n)=\{f(u,v,w),\ f$ is the complex polynomial about $u,\ v,\ w\}$ is a ring over complex field C, which is generated by

$$u=z\bar{z},\quad v=z^n+\bar{z}^n,\quad w=i(z^n-\bar{z}^n)$$

Proof According to Poenagu[1], we may assume f is a polynomial mapping. Writing $f(z,\ \bar{z})=\sum\limits_{\alpha,\beta}a_{\alpha\beta}z^\alpha\bar{z}^\beta$, thus the invariance yields

$$f\left(\exp\left[i\frac{2\pi}{n}\right]z,\exp\left[-i\frac{2\pi}{n}\right]\bar{z}\right)=f(z,\ \bar{z}),$$

that is

$$\sum_{\alpha,\beta}a_{\alpha\beta}z^\alpha\bar{z}^\beta\left[\exp\left[i\frac{2\pi}{n}(\alpha-\beta)\cdot\right]-1\right]=0$$

Hence $a_{\alpha\beta}\neq 0$ unless $\alpha-\beta=\pm kn$. Then

$$f(z,\ \bar{z})=\sum_{k,\beta}a_{\beta+kn,\beta}(z\bar{z})^\beta z^{kn}+\sum_{\alpha,k}a_{\alpha,\alpha+kn}(z\bar{z})^\alpha(\bar{z})^{kn}=0$$

Noting the identities

$$z^{kn}=(z^n)^k,\quad z^n=\frac{1}{2}(v-iw)$$

we obtain the free generators u, v and w of f over C.

The result found in [1] is now provided in the following:

Lemma 2.1 Let compact Lie group T act on C in the standard way. If N_1, N_2,.., N_k are the generators of the complex T-invariant polynomial ring over C, Re (N_1)..., Re (N_k) and Im (N_1), ..., Im (N_k) are the generators over real field R.

According to Lemma 2.1 we can immediately obtain the following result:

Corollary 2.1 The set of real-valued Z_n-invariant polynomials over complex field $e_R=\{\ f\ (u,\ v,\ w),\ f$ is a polynomial with the real coefficient about $u,\ v,\ w\ \}$ is a ring over R, which can be generated by $u,\ v$ and w .

Now we give the general form of Z_n-equivariant mapping:

Lemma 2.2 The set $E(Z_n)$ of the Z_n-equivariant mapping is a module generated by z

and $\bar{z}^{n-1}$ over the ring $e_O(Z_n)$, that is, there exist p and q in $e_O(Z_n)$ such that for all g belonging to $E(Z_n)$

$$g(z, \bar{z}) = p(u, v, w)z + q(u,v,w)\bar{z}^{n-1}$$

Proof Let $g(z, \bar{z}) = \sum_{\alpha, \beta} a_{\alpha\beta} z^\alpha \bar{z}^\beta$. The equivariance leads to the equation:

$$\sum_{\alpha, \beta} a_{\alpha\beta} \bar{z}^\alpha \bar{z}^\beta \left(\exp\left[i\frac{2\pi}{n}(\alpha-\beta-1) \right] - 1 \right) = 0.$$

Hence $a_{\alpha\beta} \neq 0$, unless $\alpha-\beta-1=0$ (module n). Thus

$$g(z, \bar{z}) = \left\{ \sum_{\beta, l} a_{\beta+1+ln} (z\bar{z})^\beta (z^n)^l \right\} z$$
$$+ \left\{ \sum_{\alpha, l} a_{\alpha, \alpha+ln+1} (z\bar{z})^\alpha (\bar{z}^n)^l \right\} \bar{z}^{n-1}$$

Noting the coefficient polynomials of z and $\bar{z}^{n-1}$, denoted by p and q, are Z_n-invariant, p and q belong to $e_O(Z_n)$. the result is true.

Lemma 2.2 shows $E(Z_n) = \{p(u, v, w)z + q(u, v, w)\bar{z}^{n-1},\ p, q \in e_O(Z_n)\}$. Although it is impossible to reduce generators of $E(Z_n)$ the form of g can be simplified.

Theorem 2.2 $E(Z_n) = \{ p(u, v)\ z + q(u, v)\bar{z}^{n-1},\ \ p,\ q \in e\ (Z_n)\}$ in which $e(Z_n) = \{f(u, v);\ f$ is a polynomial with the complex coefficients about $u, v\}$ is a closed subring of $e_O(Z_n)$.

In order to prove the theorem, we show at first the following result.

Lemma 2.3 The mapping $w^r z$ and $w^r \bar{z}^{n-1}$ are Z_n-equivariant. Thus there exist Z_n-invariant polynomials $f_i(u, v) \in e(Z_n)$, $i = 1, 2, 3, 4$, such that

$$w^r z = f_1(u, v)z + f_2(u, v)\bar{z}^{n-1}$$
$$w^r \bar{z}^{n-1} = f_3(u, v)z + f_4(u, v)\bar{z}^{n-1}$$

where r is the nonnegative integral number.

Proof When $r=0$, the result is readily verified.

When $r=1$, $\quad wz = (iv)z - (2iu)\bar{z}^{n-1}$
$$w\bar{z}^{n-1} = (2iu^{n-1})z - (iv)\bar{z}^{n-1}.$$

The result for any integral number r is readily verified in the mathematical induction.

We easily obtain Theorem 2.2 from Lemmas 2.2 and 2.3. And now, several important statements used in the following sections are presented.

Corollary 2.2 Let

$$p(u,v) = p_1(u,v) + ip_2(u, v)$$
$$q(u,v) = q_1(u,v) + iq_2(u,v)$$

where p_1, p_2, q_1, q_2 are the Z_n-invariant polynomials with real coefficients. Then

$$g(z, \bar{z}) = [p_1(u,v)z + q_1(u, v)\bar{z}^{n-1}] + i[p_2(u,v)z + q_2(u,v)\bar{z}^{n-1}].$$

Corollary 2.3 When (0, 0) is the singular point of g

(1) $n=1$, $g(0,0)=0$, $(dg)_{(0,0)}=0$.

(2) $n=2$, $p(0,0)=q(0,0)=0$.

(3) $n\geqslant 3$, $p(0,0)=0$.

We denote the set of the polynomial germs by $U_{z,\lambda}$ if the germs vanish at (0, 0), and the set of the general polynomial germs by $e_{z,\lambda}$.

III. Z_n-Equivalent, Restricted Tangent Spaces and Tangent Spaces

We discuss the basic concepts and methods of Z_n-singularity theory in this section.

Definition 3.1 Let g and h in $E(Z_n)$ be Z_n-equivariant bifurcation problems. Then g and h are Z_n-equivalent if there exists an invertible change of coordinates (z, λ) to $(Z(z,\lambda), \Lambda(\lambda))$ and matrix-valued germ $S(z,\lambda)$ such that

$$g(z,\lambda)=S(z,\lambda)h(Z(z,\lambda),\Lambda(\lambda))$$

where $Z(0,0)=0$, $\Lambda(0)=0$, $\Lambda'(0)>0$ and for $r\in Z_n$

$$\left.\begin{array}{ll}\text{(a)} & Z(rz,\lambda)=rZ(z,\lambda)\\ \text{(b)} & S(rz,\lambda)r=rS(z,\lambda)\\ \text{(c)} & \det S(0,0)\neq 0,\ \det(dZ)_{(0,0)}\neq 0\end{array}\right\} \qquad (3.1\text{a,b,c})$$

Lemma 3.1 The submodule of the smooth germs satisfying (3.1b) is generated by the germs:

$$S_1w=w,\quad S_2w=z^2\bar{w},\quad S_3w=\bar{z}^{n-2}\bar{w},\quad S_4w=z^nw.$$

Proof Let the linear mapping

$$S(z,\lambda)w=\sum\alpha_{jk}(\lambda)z^j\bar{z}^kw+\sum\beta_{jr}z^j\bar{z}^k\bar{w}$$

where $\alpha_{jk}(\lambda)$, $\beta_{jk}(\lambda)$ are the complex number. The condition (3.1b) leads to

$$\sum\alpha_{jk}(\lambda)\exp\left[(j-k)\frac{2\pi}{n}i\right]z^j\bar{z}^kw$$
$$+\sum\beta_{jk}(\lambda)\exp\left[(j-k-2)\frac{2\pi}{n}i\right]z^j\bar{z}^k\bar{w}=0$$

Then $\alpha_{jk}\equiv 0$ unless $j=k$ (mod n) and $\beta_{jk}\equiv 0$ unless $j\equiv k+2$ (mod n). It follows that $S(z,\lambda)w$ can be generated by

$$w,\ z^2\bar{w},\ \bar{z}^{ln}w,\ z^{ln+2}\bar{w},\ \bar{z}^{ln+2}\bar{w},\ z^{ln}w.$$

Noting the identities:

$$z^{ln}w=(z^n+\bar{z}^n)z^{(l-1)n}w-(z\bar{z})^nz^{(l-2)n}w$$
$$\bar{z}^{ln}w=(z^{ln}+\bar{z}^{ln})w-z^{ln}w$$
$$z^{ln+2}\bar{w}=(z^n+\bar{z}^n)z^{(l-1)n+2}\bar{w}-(z\bar{z})^nz^{(l-2)n}\bar{w}$$
$$\bar{z}^{(l+1)n-2}\bar{w}=(z^{ln}+\bar{z}^{ln})\bar{z}^{n-2}\bar{w}-(z\bar{z})^{n-2}z^{(l-1)n-2}\bar{w}$$
$$z^{n+2}\bar{w}=(z^n+\bar{z}^n)z^2\bar{w}-(z\bar{z})^2\bar{z}^{n-2}\bar{w}$$

the generators are S_j, $j=1, 2, 3, 4$.

Introducing the invariant coordinates

$$[p,q]=p(u,v)z+q(u,v)z^{n-1}.$$

we express the generators as following:

$$\left.\begin{array}{l} S_1 g = g = [p,\ q] \\ S_2 g[u\bar{p}+v\bar{q},\ -u\bar{q}] \\ S_3 g = [u^{n-2}\bar{q},\ \bar{p}] \\ S_4 g = [vp+u^{n-1}q, -up] \end{array}\right\} \tag{3.2}$$

Now we formulate the Z_n-equivariant restricted tangent space $RT(g,\ Z_n)$. It is a submodule generated by the $S_1 g, \cdots, S_t g, (dg)X_1, \cdots, (dg)X_t$ over $e(Z_n)$ where $S_1 \ldots, S_t$ and $X_1, \cdots, X_t$ satisfy

$$E(Z_n) = e(Z_n)\langle S_1,\ \cdots,\ S_t\rangle$$
$$U(Z_n) = e(Z_n)\langle X_1,\ \cdots,\ X_t\rangle$$

Theorem 3.1 $RT(g,\ Z_n)$ is the submodule over the ring $e(Z_n)$, generated finitely by the germs:

$$[p,\ q],\ [u\bar{p}+v\bar{q},\ -u\bar{q}],$$
$$[u^{n-2}\bar{q},\ \bar{p}],\ [vp+u^{n-1}q,\ -up],$$
$$[2up_u+nvp_v+p,\ 2uq_u+nvq_v+(n-1)q]$$
$$[vp_u+2nu^{n-1}p_v+(n-1)u^{n-2}q,\ vq_u+2nu^{n-1}q_v+p]$$

Proof Let $g(z,\ \bar{z}) = p(u,\ v)z + q(u,\ v)\bar{z}^{n-1}$. The first four generators can be obtained directly by (3.2). Noting $X_1 = z,\ X_2 = \bar{z}^{n-1}$ due to Theorem 2.2

$$(g_z)z + (g\bar{z})\bar{z} = (2up_u + p + nvp_v)z + (2uq_u + nvq_v + (n-1)q)\bar{z}^{n-1}$$

$$(g_z)\bar{z}^{n-1} + (g\bar{z})\bar{z}^{n-1} = (2nu^{n-1}p_v + vp_u + (n-1)u^{n-2}q)z + (p + vq_u + 2nu^{n-1}q_v)\bar{z}^{n-1}$$

Rewriting the generators in the form of the invariant coordinates, the proof is completed.

Corollary 3.1 Substracting the first generator from the fifth one, we can reduce the fifth generator to the form

$$[2up_u+nvp_v,\ 2uq_u+nvq_v+(n-2)q],$$

and others are kept unchanged.

Finally we formulate the tangent space $T(g,\ Z_n)$ for the unfolding theorem:

$$T(g,\ Z_n) = RT(g,\ Z_n) + R\{(dg)Y_1,\ \cdots (dg)Y_m,\ g,\ g_\lambda,\ \lambda g_\lambda \cdots\}$$

where Y_j satisfy $E(Z_n) = U_{z,\lambda}(Z_n) \oplus R\{Y_1,\ \cdots,\ Y_m\}$. Hence $\mathrm{Fix}\{Z_n\} = 0$ implies $E(Z_n) = U_{z,\lambda}(Z_n)$.

Theorem 3.2 $T(g,\ Z_n) = RT(g,\ Z_n) \oplus e_\lambda\ \{g_\lambda\}$ where e_λ is the set of the germs about λ near zero.

Corollary 3.2 Let W be a vector subspace such that

$$E(Z_n) = T(g,\ Z_n) + W$$

If $g_1,\ \cdots,\ g_l$ are the basis for W, Then $g + \sum_{j=1}^{l} \alpha_j g_j$ is a universal Z_n-equivariable unfolding, where α_j is the unfolding parameter.

It is readily seen that the only difference between Z_n- and D_n-equivariable singularity is $e(Z_n) \in E(Z_n)$.

IV. n=3: Z_3-Equivariable Singularity Theory

Based on the theory mentioned above, now we are in a position to drive the normal form and universal unfolding of Z_n-singularity, in which $n<3$ corresponds to the subharmonic bifurcation of strong resonance, $n=4$ to critical one and $n>5$ to weak resonance. Only Z_3-singularity theory is studied in detail in the paper. Other cases please refer to [4].

Let $g(z, \lambda): C\times R\to C$ be the Z_3-equivariable bifurcation problem, then $g(z, \lambda)=p(u, v, \lambda)z+q(u, v, \lambda)\bar{z}^2$, where $u=z\bar{z}$, $v=z^3+\bar{z}^3$, $p, q\in e(Z_3)$.

Lemma 4.1 If the following nondegenerate conditions hold:

$$q(0, 0, 0)\neq 0, \quad p_\lambda(0, 0, 0)\neq 0 \tag{4.1}$$

then $RT(g, Z_3)=P_3=[U_{u,v,\lambda}, e_{u,v,\lambda}]$.

Proof It is readily seen from Corollary 3.1 that

$$RT(g, Z_3)\subseteq P_3$$

Now we only need to prove $P_3\subseteq RT(g, Z_3)$, using Nagayama lemma[1] only need to prove

$$P_3\subseteq RT(g, Z_3)+P_3\cdot U_{u,v,\lambda}. \tag{4.2}$$

The generators of $RT(g, Z_3)$ are represented as a linear combination of the generators of P_3 module $U_{u,v,\lambda}\cdot P_3$ in the following table:

	$[u, 0]$	$[v, 0]$	$[\lambda, 0]$	$[0, 1]$
$[p, q]$	$p_u(0)$	$p_v(0)$	$p_\lambda(0)$	$p(0)$
$[u\bar{p}+v\bar{q}, -u\bar{q}]$	0	$\bar{q}(0)$	0	0
$[u\bar{q}, \bar{p}]$	$\bar{q}(0)$	0	0	0
$[vp+vq, -up]$	$q(0)$	0	0	0
$[2up_u+3vp_v, 2uq_u+3vq_v+q]$	$2p_u(0)$	$3p_v(0)$	0	$q(0)$
$[vp_u+6u^2p_v+2uq, vq_u+6u^2q_v+p]$	$2q(0)$	$p_u(0)$	0	0

Leaving out the third and fourth rows, we obtain the submatrix of order 4, whose determinant is not equal to zero according to (4.1). Therefore the corresponding inverse transformation can be conducted. That is, the generators of P_3 can be represented as a linear combination of the generators of $RT(g, Z_3)$ module $U_{u,v,\lambda}$. Hence (4.2) holds.

Theorem 4.1 (Recognition problem) If the nondegenerate conditions hold, $g(z, \lambda)$ is Z_3-equivalent to the normal form

$$N_3(z, \lambda)=(\varepsilon_1+i\varepsilon_2)\lambda z+\bar{z}^2,$$

where ε_1 and ε_2 are real numbers such that $\varepsilon_1^2+\varepsilon_2^2=1$.

Proof Let

$$g_t(z, \lambda)=[P_\lambda(0)+t(p_u(0)u+p_v(0)v+\varphi(u,v,\lambda)), \ q(0)+t\psi(u, v, \lambda)]$$

where $\varphi, \psi\subset P_3\cdot U_{u,v,\lambda}$ · $RT(g_t, Z_3)=P_3$ holds in accordance with Lemma 4.1. Let $t=0$ and $t=1$, then $RT(g_0, Z_3)=RT(g_1, Z_3)$. Noting $g_1=g$ we have shown that g is Z_3-equivalent to $g_0=p_\lambda(0)\lambda z+q(0)\bar{z}^2$. Introducing the scalar transformation: $Z(z, \lambda)=\xi z$, $\Lambda(\lambda)=\theta\lambda$, where ξ is a nonzero complex number and θ is a positive real number. Then

$$g_0(z, \lambda)=(p_\lambda(0)\theta\xi)\lambda z+(q(0)\bar{\xi})\bar{z}^2$$

Let

$$\bar{\xi}^2=1/q(0),\quad \theta=|\sqrt{\bar{q}(0)}/p_\lambda(0)|$$

that is

$$q(0)\bar{\xi}^2=1,\quad p_\lambda(0)\theta\xi=\varepsilon_1+i\varepsilon_2$$

in which

$$\left.\begin{aligned}\varepsilon_1&=\mathrm{Re}\left\{\frac{p_\lambda(0)}{\sqrt{\bar{q}(0)}}\Big/\left|\frac{p_\lambda(0)}{\sqrt{\bar{q}(0)}}\right|\right\}\\ \varepsilon_2&=\mathrm{Im}\left\{\frac{p_\lambda(0)}{\sqrt{\bar{q}(0)}}\Big/\left|\frac{p_\lambda(0)}{\sqrt{\bar{q}(0)}}\right|\right\}\end{aligned}\right\}\tag{4.3}$$

therefore $g(z,\lambda)$ is Z_3-equivalent to the $N_3(z,\lambda)$.

Note ε_1 and ε_2 above is with reference to (4.3).

Theorem 4.2 Z_3-codimension of $N_3(z,\lambda)$ is zero. Hence the universal unfolding of g is $N_3(z,\lambda)$ in the nondegenerate conditions.

Proof It is seen from Theorem 3.2 that

$$\begin{aligned}T(N_3,Z_3)&=RT(N_3,Z_3)+e\lambda\left\{\frac{\partial N_3}{\partial\lambda}\right\}\\&=[U_{u,v,\lambda},\ e_{u,v,\lambda}]+e_\lambda[\varepsilon_1+i\varepsilon_2,\ 0]\\&=E_{z,\lambda}(Z_3)\end{aligned}$$

Thus $\mathrm{Codim}_{Z_3}(N_3)=0$.

V. Summary

The basic concepts, recognition problems and universal unfoldings of Z_n-equivariable singularity are studied in detail in this paper. Especially the Z_3-equivariable normal form and universal unfolding are formulated under the nondegenerate conditions. Of course we can further study the degenerate case and the bifurcation digrams in the polar coordinates or local transfer sets in a similar way[4].

We are able to study the subharmonic resonance bifurcation of the periodic parametric-excited systems based on the above mathematical theory of the Z_n-equivariable singularity, and formulate the all possible local bifurcation diagrams in detail, which reasonably illustrate the phenomena such as hysteresis, catastrophe and multi-equilibrium in practical vibration engineering problems[4].

References

[1] Golubitsky M., I. Stewart and D. G. Scheaffer, *Singularities and Groups in Bifurcation Theory*, Vol. 1, 2, Springer-Verlag, New York (1985, 1988).

[2] Sattinger. D. H., *Group Theoric Methods in Bifurcation Theory*, Springer-Verlag, New York (1979).

[3] Buzano, E., G. Geymonant and T. Poston, Post-bucking behavior of non-linear hyperelastic thin rod with cross-section invariant under the dihedral group D_n, *Arch. Rational. Mech. Anal.*, **89,** 4 (1985).

[4] He Guo-wei, The Subharmonic Bifurcation of The Periodic Parameter-Excited System, Dissertation, Northwestern Polytechnical University (1991).