THE ANALYTICAL STUDY ON THE LASER INDUCED REVERSE-PLUGGING EFFECT BY USING THE CLASSICAL ELASTIC PLATE THEORY (II) — REVERSE-BULGE MOTION

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Abstract

The reverse-bulge motion (RBM) in the metallic foils, which is induced by spatially cylindrical long pulse laser, is examined in order to analyse the newly-discovered reverse-plugging effect (RPE). An uncoupled, thin plate theory is used to determine the induced flexural vibrations. The solution is obtained as the superposition of two displacement fields, representing the quasi-static and the dynamic behaviors. Meanwhile, the equivalent thermal loading and the dimensionless analysis of thin plate motion are presented. Numerical results presented may partially explain the RBM of thin plate at the early stage of laser irradiation.

Key words long-pulsed laser beam, the RPE, the RBM, thermal-elastic thin-plate theory

I. Introduction

A new type of damage, i.e. the RPE in studying the interaction of a single-mode long-pulsed Nd: Glass laser beam with copper and aluminum foils has been reported[1-2]. The RPE is different from the well-known damage types which are melting, vaporization and shock waves in materials. The process of the RPE in metallic foils induced by long-pulsed laser is divided into three macroscopic stages, i.e. the reverse-bulge formation, shear deformation localization and perforation. The RPE is also a typical 3-F (Flow-Fracture-Fragmentation) process. The temperature distribution analysis has shown that the temperature gradient in the axial direction is the key factor to induce the RBM. The discontinuity of temperature and its gradient on the rim of laser spot in the radial direction is the key factor to induce shear deformation localization[3].

Based on the temperature field analysis in[3], the present study explores the characteristics of the RBM in the metallic foils irradiated by a spatially cylindrical type long-pulse laser beam, where the classical thin-plate theory is used. The exact solution is derived as the superposition of two displacement fields, representing the quasi-static and the dynamic behaviors. In Section II, the theoretical considerations and the governing equations are outlined. In Section III, the quasi-static behavior of the transverse motion is obtained. In Section IV, the dynamic behavior of the transverse motion is investigated by using the Hankel transform and Laplace transform. Some numerical results, discussions and dimensionless
II. Governing Equations

In the present study the transverse deflection of a thin plate irradiated by a pulse laser beam can be modelled on the basis of the assumptions:

1. The thermal-mechanical coupling effect is neglected. The neglect of thermoelastic coupling is generally justifiable for the problems in which thermoelastic dissipation is not of primary interest.

2. We shall confine ourselves to the infinitesimal deformation theory. And thus, the effect of membrane force and shear force on the transverse deflection and their coupling effect are ignored.

3. All the material parameters are constant. This implies that the temperature-dependence of parameters is neglected.

However, although the assumptions (2) and (3) may be invalid for the whole process of the PRE, they should be reasonable for the RBM. According to the classical (Kirchhoff) plate theory, the transverse displacement of the plate middle plane, as described in Fig. 3 in [3], is governed by the equations as follows:

\[
D_1 \Delta^2 w + \frac{1}{1-\nu} \Delta M_r + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{2.1}
\]

initial condition

\[
w \bigg|_{t=0} = \frac{\partial w}{\partial t} \bigg|_{t=0} = 0 \tag{2.2}
\]

boundary condition

\[
w \bigg|_{r=b} = \frac{\partial w}{\partial r} \bigg|_{r=b} = 0 \tag{2.3}
\]

where \( \Delta \) is the Laplace operator, \( w \) represents the transverse displacement of the plate middle plane, \( D_1 = E h^3 / 12(1-\nu^2) \) denotes the bending stiffness, and \( E, \nu, \rho \) denote, respectively, the Young’s modulus, Poisson ratio and mass density. Other notations are defined in [3]. The thermal moment or equivalent external loading in Eq. (2.1) is defined in term of the temperature rise \( \theta(r, z, t) \) by

\[
M_r = \alpha_0 E \int_{-h/2}^{h/2} \theta(r, z, t) z \, dz \tag{2.4}
\]

where \( \alpha_0 \) is the coefficient of thermal expansion.

For the convenience in the subsequent analysis, we introduce the following dimensionless variables

\[
\hat{\theta} = \frac{\theta}{h T_m}, \quad \hat{w} = \frac{w}{w_0}, \quad \hat{M}_r = \frac{M_r}{\frac{1}{4} h^2 \rho \sigma_0 M_r},
\]

\[
\hat{M}_t = \frac{M_t}{\frac{1}{4} h^2 \sigma_0 m_0}, \quad \hat{M}_b = \frac{M_b}{\frac{1}{4} h^2 \sigma_0 m_0} \tag{2.5}
\]
Also, the basic dimensionless parameters are defined as

\[ h_1 = \frac{h}{a}, \quad A = \frac{a \alpha^2}{D}, \quad B = \frac{(a + \beta) a^2}{D}, \quad h_5 = \alpha_T \]

\[ h_4 = \frac{I_{max} a}{k_0 T_m}, \quad h_6 = \frac{E}{\alpha_0}, \quad h_8 = \frac{\rho D}{a E} \] \hspace{1cm} (2.6)

Meanwhile, the following dimensionless parameters are used in the present study

\[ w_s = 24(1 + \nu)h_1 h_4, \quad M_s = 8h_1 h_4 h_6, \quad m_s = 8h_1 h_4 h_6/(1 - \nu) \]

\[ m_1 = 12(1 - \nu^2)h_5/h_1 \] \hspace{1cm} (2.7)

where \( \sigma_0 \) is the yield strength at ambient temperature, \( M_s \), and \( M_6 \) are, respectively, the bending moments in the radial and circumferential directions,

\[ M_r = -D \left( \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \frac{\partial w}{\partial r} \right) - \frac{M_5}{1 - \nu} \] \hspace{1cm} (2.8)

\[ -\nu \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} \right) - \frac{M_6}{1 - \nu} \] \hspace{1cm} (2.9)

For the convenience of writing in the subsequent derivation, the dimensionless variables \( \hat{\theta}, \ w, \ M_1, \ M_r, \ M_s, \ r \) and \( z \) are, respectively, replaced by \( \theta, \ w, \ M_1, \ M_r, \ M_6, \ r \) and \( z \). Therefore, the governing equations of the RBM are expressed in dimensionless form as follows:

\[ \Delta^2 w + \frac{1}{h_1} \Delta M_4 + m_1 \frac{\partial^2 w}{\partial z^2} = 0 \] \hspace{1cm} (2.10)

initial conditions

\[ w \bigg|_{r=0} = \frac{\partial w}{\partial t} \bigg|_{r=0} = 0 \] \hspace{1cm} (2.11)

boundary conditions

\[ w \bigg|_{r=r_2} = \frac{\partial w}{\partial r} \bigg|_{r=r_2} = 0 \] \hspace{1cm} (2.12)

The dimensionless thermal moment \( M_1 \) is rewritten as

\[ M_1 = \frac{1}{2h_1^2} \int_{h_1}^{h_2} \theta dz \] \hspace{1cm} (2.13)

Just as in the previous investigations \( 5-6 \), the deflection \( w \) is regarded as the sum of two terms, namely,

\[ w = w_\theta + w_\delta \] \hspace{1cm} (2.14)

where \( w_\theta \) and \( w_\delta \) represent, respectively, the quasi-static deflection and dynamic deflection. The quasi-static deflection \( w_\theta \) satisfies the differential equation

\[ \Delta^2 w_\theta + \frac{1}{h_1} \Delta M_4 = 0 \] \hspace{1cm} (2.15)
together with the prescribed boundary conditions
\[
    w_\dot{\theta} \bigg|_{r=h_1} = \frac{\partial w_\theta}{\partial r} \bigg|_{r=h_1} = 0
\]  

(2.16)
The dynamic deflection \( w_d \) must then satisfy the equation
\[
    \Delta^2 w_d + m_1 \frac{\partial^2 w_d}{\partial t^2} + m_1 \frac{\partial^2 w_d}{\partial t^2} = 0
\]  

(2.17)
together with the boundary conditions
\[
    w_d \bigg|_{r=h_1} = \frac{\partial w_d}{\partial r} \bigg|_{r=h_1} = 0
\]  

(2.18)

In addition, the initial conditions should be expressed as follows
\[
    w_d \bigg|_{t=0} = -w_\theta \bigg|_{t=0}, \quad \frac{\partial w_d}{\partial t} \bigg|_{t=0} = -\frac{\partial w_\theta}{\partial t} \bigg|_{t=0}
\]  

(2.19)

Consequently, (2.14)−(2.19) are the basic governing equations of dimensionless deflection \( w \). Substituting some dimensionless variables into (2.8) and (2.9), the dimensionless bending moments \( M_\theta \) and \( M_\phi \) are expressed as follows
\[
    M_\theta = -\left( \frac{\partial^2 w}{\partial \theta^2} + \frac{\nu}{r} \frac{\partial w}{\partial \theta} + \frac{M_\theta}{h_1} \right)
\]  

(2.20)
\[
    M_\phi = -\left( \frac{\nu}{r} \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{r} \frac{\partial w}{\partial \phi} + \frac{M_\phi}{h_1} \right)
\]  

(2.21)

III. **Quasi-Static Solution**

Substituting the dimensionless temperature rise expression (44) in [3] into the dimensionless thermal moment expression (2.13), we have
\[
    M_\theta = -\frac{h_1}{h_1^2} \sum_{k_n} \frac{J_n(k_n r)}{J_n^2(k_n h_1)} \cdot g(k_n, t)
\]  

(3.1)

where the \( g(k_n, t) \) is expressed as
\[
    g(k_n, t) = \frac{1}{24} g(t) + 2 \sum_{m=1}^{\infty} \frac{(-1)^m - 1}{(m \pi)^2} \left[ g(t) \right.
\]
\[
    \left. - \left( \frac{m \pi}{h_1} \right)^2 \left( \frac{\exp[-At] - \exp[-Ct]}{C-A} - \exp[-Bt] - \exp[-Ct] \right) \right]
\]  

(3.2)

For Eq. (2.15), we have
\[
    \frac{h_1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial w_\theta}{\partial r} \right] \right\} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial M_\phi}{\partial r} \right) = 0
\]  

(3.3)

Integrating Eq. (3.3) and referring to the finiteness of \( w \) and \( \frac{\partial w_\theta}{\partial r} \) at the point \( r=0 \), we obtain
\[
    w_\theta = \frac{1}{4} C_1 r^2 - \frac{1}{h_1} \sum_{k_n} \frac{J_n(k_n r)}{k_n J_n^2(k_n h_1)} \cdot g(k_n, t) + C_2
\]  

(3.4)
where \( C_1 \) and \( C_2 \) are integral constants which are determined with boundary conditions (2.16). Then we have

\[
\omega_s = -\frac{1}{h_s^2} \sum_{k_n} \frac{f^s(k_n)g(k_n,t)}{k_n J_1^s(k_nh_2)} \left[ J_0(k_nr) + \frac{1}{2h_2} k_n J_1(k_nh_2)(r^2 - h_2^2) \right] \tag{3.5}
\]

In order to understand the characteristics of the quasi-static bending moments distribution, we insert (3.5) into (2.20)-(2.21) and obtain the dimensionless bending moments

\[
M_r = \frac{1}{h_s^2} \sum_{k_n} \frac{f^s(k_n)g(k_n,t)}{k_n J_1^s(k_nh_2)} \left[ \frac{1 + \nu}{h_2} J_1(k_nh_2) + \frac{1 - \nu}{r} J_1(k_nr) \right] \tag{3.6}
\]

\[
M_\theta = \frac{1}{h_s^2} \sum_{k_n} \frac{f^s(k_n)g(k_n,t)}{k_n J_1^s(k_nh_2)} \left[ (1 - \nu)k_n J_0(k_nr) + \frac{1 + \nu}{h_2} J_1(k_nh_2) - \frac{1 - \nu}{r} J_1(k_nr) \right] \tag{3.7}
\]

IV. Dynamic Solution

To obtain an exact solution to Eq. (2.17) together with the prescribed boundary conditions (2.18) and the initial conditions (2.19), the dynamic deflection \( \omega_d \) is expanded in the following form

\[
\omega_d = \sum_{a_n} \omega^s_{a_n}(\alpha_n,t) \left[ I_0(\alpha_nr) - \frac{I_0(\alpha_nh_2)}{I_0(\alpha_nh_2)} J_0(\alpha_nr) \right] \tag{4.1}
\]

where \( \alpha_n \) are the roots of the following equation

\[
I_0(\alpha_nh_2) I_1(\alpha_nh_2) + J_1(\alpha_nh_2) I_0(\alpha_nh_2) = 0 \tag{4.2}
\]

and \( I_n, J_n \) are, respectively, the \( n \)-th order imaginary variable Bessel function and the \( n \)-th order Bessel function. Setting \( y_n \) as

\[
y_n = I_0(\alpha_nr) - \frac{I_0(\alpha_nh_2)}{I_0(\alpha_nh_2)} J_0(\alpha_nr) \tag{4.3}
\]

utilizing repeatedly the Bessel equation and some transforms, we prove readily that the eigenfunctions \( y_n \) are complete and orthogonal, that is,

\[
\int_0^{h_2} y_m y_n r dr = 0 \quad (m \neq n) \tag{4.4}
\]

In order to solve exactly the dynamic deflection \( \omega_d \), the modulus and some transforms are derived in the following.

1. The derivation of modulus \( \int_0^{h_2} y^s \rho r dr \)

Substituting (4.3) into the expression of modulus \( \int_0^{h_2} y^s \rho r dr \) we obtain

\[
\int_0^{h_2} y^s \rho r dr = \int_0^{h_2} \left[ I_0^s(\alpha_nr) + \frac{I_0^s(\alpha_nh_2)}{J_0^s(\alpha_nh_2)} J_0^s(\alpha_nr) - \frac{2I_0^s(\alpha_nh_2) I_0^s(\alpha_nr) J_0^s(\alpha_nr)}{J_0^s(\alpha_nh_2) J_0^s(\alpha_nr)} \right] r dr \tag{4.5}
\]

According to the following expression.
and the imaginary variable Bessel equation,

\[ r^2 \frac{d^2 I_0(\alpha r)}{dr^2} + \frac{d I_0(\alpha r)}{dr} - \alpha_1^2 r I_0(\alpha r) = 0 \]  

utilizing Eq. (4.7) and some transforms, we obtain

\[ \int_0^{h_2} I_0^2(\alpha r) rdr = \frac{1}{2} h_1^2 [I_0^2(\alpha h_2) - I_0^2(\alpha h_3)] \]  

Similarly, utilizing the Bessel Equation

\[ r^2 \frac{d^2 J_0(\alpha r)}{dr^2} + \frac{d J_0(\alpha r)}{dr} + \alpha_2^2 r J_0(\alpha r) = 0 \]

we obtain easily

\[ \int_0^{h_2} J_0^2(\alpha r) rdr = \frac{1}{2} h_1^2 [J_0^2(\alpha h_2) + J_0^2(\alpha h_3)] \]

\[ \int_0^{h_2} J_0(\alpha r) J_0(\alpha r) rdr = 0 \]

Substituting (4.8), (4.10) and (4.11) into (4.5), and referring to Eq. (4.2), we obtain the modulus as follows

\[ \int_0^{h_2} y_1^2 rdr = h_1^2 I_0^2(\alpha h_2) \]  

2. The transform coefficients

(1) \[ \int_0^{h_2} J_0(k \alpha r) y_n(\alpha r) rdr \]

Substituting (4.3) into the above expression, we have

\[ \int_0^{h_2} J_0(k \alpha r) y_n(\alpha r) rdr = \int_0^{h_2} J_0(k \alpha r) \left[ I_0(\alpha r) \frac{I_0(\alpha h_2)}{J_0(\alpha h_2)} \right] rdr \]

Utilizing the Bessel Eq. (4.7) and the following equation,

\[ r^2 \frac{d^2 I_0(k \alpha r)}{dr^2} + \frac{d I_0(k \alpha r)}{dr} + k_1^2 r I_0(k \alpha r) = 0 \]

we have

\[- (k_1^2 + \alpha_1^2) r I_0(\alpha r) J_0(k \alpha r) = \frac{d}{dr} \left[ r I_0(\alpha r) \frac{d J_0(k \alpha r)}{dr} - r \frac{d I_0(\alpha r)}{dr} J_0(k \alpha r) \right] \]

Consequently, we obtain

\[ \int_0^{h_2} r I_0(\alpha r) J_0(k \alpha r) dr = - \frac{k_1 \alpha_1}{k_1^2 + \alpha_1^2} I_0(\alpha h_2) J_1(k h_1) \]
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where the identity \( J_\nu(k_n h_2) = 0 \) is used. Similarly, we obtain the following expression

\[
\int_0^{h_1} r J_0(\alpha_n r) J_0(\alpha_n r) r dr = \frac{h_1^2 k_n}{k_1^2 - \alpha_n^2} J_0(\alpha_n h_2) J_1(\alpha_n h_2)
\]

Finally, we have the following transform coefficient,

\[
\int_0^{h_1} J_0(\alpha_n r) y_n(\alpha_n r) r dr = \frac{2 h_1 k_n \alpha_n^2}{k_1^2 - \alpha_n^2} J_0(\alpha_n h_2) J_1(\alpha_n h_2)
\]

According to the following expressions

\[
\int_0^{h_1} I_0(\alpha_n r) r dr = \frac{h_1 I_1(\alpha_n h_2)}{\alpha_n}, \quad \int_0^{h_1} J_0(\alpha_n r) r dr = \frac{h_1 I_1(\alpha_n h_2)}{\alpha_n}
\]

we have

\[
\int_0^{h_1} y_n(\alpha_n r) r dr = \frac{2 h_1^2}{\alpha_n} I_1(\alpha_n h_2)
\]

\[
\int_0^{h_1} I_0(\alpha_n r) r^2 dr = \frac{h_1^2}{\alpha_n} I_1(\alpha_n h_2) - \frac{2 h_1^2}{\alpha_n^2} I_0(\alpha_n h_2) + \frac{4 h_1^2}{\alpha_n^3} I_1(\alpha_n h_2)
\]

\[
\int_0^{h_1} J_0(\alpha_n r) r^2 dr = \frac{h_1^2}{\alpha_n} J_1(\alpha_n h_2) + \frac{2 h_1^2}{\alpha_n^2} J_0(\alpha_n h_2) - \frac{4 h_1^2}{\alpha_n^3} J_1(\alpha_n h_2)
\]

Finally, we have the following transform coefficient,

\[
\int_0^{h_1} (h_1^2 - r^2) y_n(\alpha_n r) r dr = \frac{4 h_1^2}{\alpha_n^2} I_0(\alpha_n h_2)
\]

3. Transform coefficient \( w_\delta^*(\alpha_n, t) \)

Substituting expression (4.1) into Eq. (2.17), we have the following equation on the unknown variable \( w_\delta^*(\alpha_n, t) \)

\[
\sum_{\alpha_n} \alpha_n^4 w_\delta^*(\alpha_n, t) y_n(\alpha_n r) + m_1 \sum_{\alpha_n} \frac{d^2 w_\delta^*(\alpha_n, t)}{dt^2} y_n(\alpha_n r) + m_1 \frac{\partial^2 w_\delta^*}{\partial t^2} = 0
\]

Referring to the orthogonality of the eigenfunction \( y_n \), the right side and the left side of Eq. (4.24) times, simultaneously, \( y_n(\alpha_n r) r \) and integrating them, meanwhile substituting the modulus (4.12) into the new equation, finally, we obtain the ordinary differential equation on the unknown variable \( w_\delta^*(\alpha_n, t) \) as follows

\[
\alpha_n^4 h_1^2 I_0(\alpha_n h_2) w_\delta^*(\alpha_n, t) + m_1 h_1^2 I_0(\alpha_n h_2) \frac{d^2 w_\delta^*(\alpha_n, t)}{dt^2} + m_1 \int_0^{h_1} \partial^2 w_\delta^* y_n r dr = 0
\]

Substituting (4.18) and (4.23) into the last term on the left side of Eq. (4.25), we have

\[
\int_0^{h_1} \frac{d^2 y_n}{dt^2} y_n r dr = \frac{2}{h_1^2} \sum_{k_\delta} \frac{f_\delta(k_\delta)}{J_1(k_\delta h_2)} \frac{k_\delta^2 I_0(\alpha_n h_2)}{(k_1^2 - \alpha_n^2)^2} \frac{d^2 g(k_\delta, t)}{dt^2}
\]
Furthermore, we obtain the ordinary differential equation as follows

$$\frac{d^2w}{dt^2} + \frac{a^2}{m_1} w + \sum_{k_n} \frac{a^2}{\sqrt{m_1}} F(k_n, \alpha_n) \frac{d^2g(k_n, t)}{dt^2} = 0$$  \hspace{1cm} (4.27)$$

where

$$F(k_n, \alpha_n) = \frac{2}{h^2 \sqrt{m_1}} \frac{k^2 f^*(k_n)}{J_1(k_n h_2) I_0(\alpha_n h_2)} (k_n^2 - \alpha_n^2) \alpha_n^2$$ \hspace{1cm} (4.28)$$

Using Eq. (3.2) and the following formula

$$\frac{1}{24} + 2 \sum_{m=1}^{\infty} \frac{((m^2 - 1)}{(m\pi)^4} = 0$$ \hspace{1cm} (4.29)$$

we have

$$g \bigg|_{t=0} = 0 \quad \frac{dg}{dt} \bigg|_{t=0} = 0$$ \hspace{1cm} (4.30)$$

Consequently, referring to Eq. (4.27) we have the transform coefficient \( w^*(\alpha_n, t) \) as follows

$$w^*(\alpha_n, t) = C_n(t) \cos\left(\frac{a^2}{\sqrt{m_1}} t\right) + D_n(t) \sin\left(\frac{a^2}{\sqrt{m_1}} t\right)$$ \hspace{1cm} (4.31)$$

where

$$C_n(t) = \int_0^t \sum_{k_n} F(k_n, \alpha_n) \frac{d^2g(k_n, t')}{dt'^2} \sin\left(\frac{a^2}{\sqrt{m_1}} t'\right) dt'$$ \hspace{1cm} (4.32)$$

$$D_n(t) = -\int_0^t \sum_{k_n} F(k_n, \alpha_n) \frac{d^2g(k_n, t')}{dt'^2} \cos\left(\frac{a^2}{\sqrt{m_1}} t'\right) dt'$$ \hspace{1cm} (4.33)$$

Setting the notations \( L_1(\eta) \) and \( L_2(\eta) \) as

$$L_1(\eta) = \int_0^t \exp[-\eta t'] \sin\left(\frac{a^2}{\sqrt{m_1}} t'\right) dt'$$ \hspace{1cm} (4.34)$$

$$L_2(\eta) = \int_0^t \exp[-\eta t'] \cos\left(\frac{a^2}{\sqrt{m_1}} t'\right) dt'$$ \hspace{1cm} (4.35)$$

we obtain

$$L_1(\eta) = \frac{1}{1 + \frac{1}{\eta}} \left[ \frac{1}{\eta} \frac{a^2}{\sqrt{m_1}} - \frac{1}{\eta} \exp[-\eta t] \sin\left(\frac{a^2}{\sqrt{m_1}} t\right) \right]$$ \hspace{1cm} (4.36)$$

$$L_2(\eta) = \frac{1}{1 + \frac{1}{\eta}} \left[ \frac{1}{\eta} \frac{a^2}{\sqrt{m_1}} - \frac{1}{\eta} \exp[-\eta t] \cos\left(\frac{a^2}{\sqrt{m_1}} t\right) \right]$$ \hspace{1cm} (4.37)$$

Substituting (3.2) into (4.32) and (4.33), we obtain, respectively, the following expressions...
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\[ E_1(\alpha_n, k_n, t) = \int_0^t \frac{d^2 g(\alpha_n, t')}{dt'^2} \sin \left( \frac{\alpha_n}{\sqrt{m_1}} t' \right) dt' \]

\[ = \frac{1}{24} \left[ A_t^2 L_1(A) - B_t^2 L_1(B) \right] + 2 \sum_{m=1}^\infty \left[ \frac{(-1)^{m-1}}{(m\pi)^4} \right] \left[ A_t^2 L_1(A) - B_t^2 L_1(B) \right] \]

\[ - \left( \frac{m\pi}{h_1} \right)^2 \left[ \frac{A_t^2 L_1(A) - C_t^2 L_1(C)}{C - A} \right] - \frac{B_t^2 L_1(B) - C_t^2 L_1(C)}{C - B} \]  \hspace{1cm} (4.38)

\[ E_2(\alpha_n, k_n, t) = \int_0^t \frac{d^2 g(\alpha_n, t')}{dt'^2} \cos \left( \frac{\alpha_n}{\sqrt{m_1}} t' \right) dt' \]

\[ = \frac{1}{24} \left[ A_t^2 L_2(A) - B_t^2 L_2(B) \right] + 2 \sum_{m=1}^\infty \left[ \frac{(-1)^{m-1}}{(m\pi)^4} \right] \left[ A_t^2 L_2(A) - B_t^2 L_2(B) \right] \]

\[ - \left( \frac{m\pi}{h_1} \right)^2 \left[ \frac{A_t^2 L_2(A) - C_t^2 L_2(C)}{C - A} \right] - \frac{B_t^2 L_2(B) - C_t^2 L_2(C)}{C - B} \]  \hspace{1cm} (4.39)

Finally, we have

\[ \omega_0^0(\alpha_n, t) = \left[ \sum_{k_n} F(k_n, \alpha_n) E_1(\alpha_n, k_n, t) \right] \cos \left( \frac{\alpha_n}{\sqrt{m_1}} t \right) \]

\[ - \left[ \sum_{k_n} F(k_n, \alpha_n) E_2(\alpha_n, k_n, t) \right] \sin \left( \frac{\alpha_n}{\sqrt{m_1}} t \right) \]  \hspace{1cm} (4.40)

Consequently, inserting (4.40) into (4.1), we obtain the exact solution of dynamical deflection \( w_d \) in the following

\[ w_d = \sum_{\alpha_n} \left\{ F(k_n, \alpha_n) E_1(\alpha_n, k_n, t) \right\} \cos \left( \frac{\alpha_n}{\sqrt{m_1}} t \right) - \left[ \sum_{k_n} F(k_n, \alpha_n) E_2(\alpha_n, k_n, t) \right] \]

\[ \sin \left( \frac{\alpha_n}{\sqrt{m_1}} t \right) \left[ I_0(\alpha_n h_1) - \frac{I_\theta(\alpha_n h_1)}{I_0(\alpha_n h_1)} I_\theta(\alpha_n r) \right] \]  \hspace{1cm} (4.41)

Finally, substituting (4.1) and (4.41) into (2.14) we obtain the exact solution of dimensionless deflection \( w \).

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Fig. 1 Histories of \( M_1 \) at point \( r=0 \) with different values of \( h_1 \)

Fig. 2 The variation of \( T_{\text{max}_1} \) and \( T_{\text{max}_2} \) with \( h_1 \), where \( T_{\text{max}_1} \) and \( T_{\text{max}_2} \) are, respectively, the time, when \( M_1 \) at point \( (r=0) \) and \( \partial M_1 / \partial r \) on the rim of laser spot reach their maximum values.
Fig. 3  The variation of $M_{\text{max}}$ with $h^{-1}$, where $M_{\text{max}}$ is the maximum value of $M$ at the point $r=0$.

Fig. 4  The variation of $M_r$ with $r$, where $r=3.7 \times 10^{-4}$ and $h^{-1}=25$.

Fig. 5  The variation of $M_r$ and $M_{r'}$ with $r$, where $r=1.5 \times 10^{-4}$ and $h^{-1}=25$.

(a) two-dimensional shape

(b) three-dimensional shape

Fig. 6  Deflected shape of the middle plane of a thin plate, where $r=3.0 \times 10^{-4}$ and $h^{-1}=25$.

Fig. 7  The histories of $w, w_r$ and $w$ at the point $r=0$, where $h^{-1}=25$.

Fig. 8  The variations of $w_{\text{max}}, w_{r\text{max}}$, and $w_{r''}$ with $h^{-1}$, where $w_{\text{max}}, w_{r\text{max}}$, and $w_{r''}$ are, respectively, the maximum values of $w, w_r$, and $w$ at the point $r=0$. 
V. Calculated Results and Discussions

Experimental results\textsuperscript{11-21} and the analysis of temperature fields show that only the spatially cylindrical distribution of laser offers a formidable potential for the RPE. Consequently, the spatially cylindrical laser beam, which was described in Fig. 2 in [3], is supposed in the present study. Meanwhile, the temporal shape is expressed as formula (22) in [3] and the target material is H65 copper alloy foils. The characteristics of the thermal moments, the transverse deflection and the quasi-static bending moments are analyzed in the following.

1. Dimensionless analysis

The characteristics of the RBM, i.e., the motion state at the early stage of laser irradiation, are determined by the laser parameters, the material parameters and the geometric parameters. In other words, there is the following implicit function

$$\Psi[h, a, \alpha, D, (\alpha + \beta), \alpha_0, T_m, I_{\text{max}}/k_0, E, \sigma_0, \rho] = 0$$  \hspace{1cm} \text{(5.1)}$$

where $I_{\text{max}} = (1 - R_0) P_{\text{max}}/\pi a^2$ is the maximum laser intensity absorbed by the target. If the dimensions of distance $[L]$, time $[T]$, mass $[M]$ and temperature $[K]$ are selected as basic dimensions, the dimensions of these variables are:

$$[h] = [L], \quad [a] = [L], \quad [\alpha] = [T^{-1}], \quad [D] = [L^2 T^{-1}], \quad [(\alpha + \beta)] = [T^{-1}], \quad [a_0] = [K], \quad [T_m] = [K], \quad [I_{\text{max}}/k_0] = [K L^{-1}], \quad [E] = [M L^{-1} T^{-2}],$$

$$[\sigma_0] = [M L^{-1} T^{-2}], \quad [\rho] = [M L^{-3}].$$  \hspace{1cm} \text{(5.2)}$$

According to the $\Pi$ theorem, we have

$$\Phi[h^s a^s \alpha^s D^s (\alpha + \beta)^s \alpha_0^s T_m^s I_{\text{max}}/k_0]^s E^* \sigma_0^s \rho^s = 0$$  \hspace{1cm} \text{(5.3)}$$

Substituting the dimension of each variable in (5.2) into Eq. (5.3), and let the factors of $[L]$, $[T]$, $[M]$ and $[K]$ on the left part of Eq. (5.3), respectively, be zero, we obtain the following four equations:

$$x_1 + x_2 + 2x_4 - x_6 - x_8 - x_{10} - 3x_{11} = 0$$  \hspace{1cm} \text{(5.4)}$$

$$x_2 + x_4 + x_6 + 2x_8 + 2x_{10} = 0$$  \hspace{1cm} \text{(5.5)}$$

$$x_8 + x_{10} + x_{11} = 0$$  \hspace{1cm} \text{(5.6)}$$

$$-x_8 + x_1 + x_9 = 0$$  \hspace{1cm} \text{(5.7)}$$

From Eqs. (5.4) - (5.7), eliminating $x_3$, $x_5$, $x_7$ and $x_n$, we have the following expression
Finally, we obtain seven basic dimensionless parameters which are the same as that in expression (2.6) in the following:

\[
\begin{align*}
\Psi \{(h/a)^* &\left(aa^2/D\right)^* \left[(\alpha + \beta)a^2/D\right)^* \left(a_0T_m\right)^* \left(I_{\text{max}}a/k_0T_m\right)^* \\
(\sigma_0/E)^* &\left(\rho D^2/a^2E\right)^* \} = 0
\end{align*}
\]

(5.8)

In the above expressions, \(h_1\) is the laser-target geometric dimensionless parameter. \(A\) and \(B\) are the coupled dimensionless parameters which are concerned with the temporal-spatial shape of laser beam and the thermophysical properties of materials. \(h_1\) is the maximum deformable quantity of solid target. \(h_2\) is the coupled dimensionless-parameter which is concerned with the laser intensity, geometric parameter and the thermophysical properties of materials. \(h_3\) is the dimensionless parameter of mechanical property of materials. \(h_4\) is the coupled dimensionless parameter which is concerned with the mechanical-thermal properties of materials. From (2.5) and (2.7), we see easily that the temperature rise \(0\), deflection \(w\), equivalent thermal loading \(M_\alpha\), quasi-static bending moments \(M\), and \(M_\beta\) all depend linearly on \(h_1\). This implies that the above physical variables depend linearly on the intensity of the incident laser beam. Meanwhile, \(w\), \(M_\alpha\), \(M\), and \(M_\beta\) all depend linearly on \(h_1\). The reason is that \(M_\alpha\), \(M\), and \(M_\beta\) result in target deformation, and \(w\) reflects the deformable quantity of target. Also, we see that \(M_\alpha\), \(M\), and \(M_\beta\) all depend linearly on \(h_1\). In other words, the larger Young’s modulus \(E\) is and the less yield strength \(\sigma_0\) is, the more easily the target approaches the yield threshold. From (2.5) and (2.7), we also see that each variable depends nonlinearly on \(h_1\), \(A\), \(B\) and \(h_1\) respectively.

Now supposing that the laser parameters are fixed, we only study the dependence of some physical variables on \(h_1\). The basic parameters of H65 copper alloy for our typical experiment are: \(D=0.335\text{cm}^2/\text{s}, a=0.25\text{cm}\). Consequently, we have \(t_0=0.187\text{s}, A=2800, B=1.77 \times 10^4, h_5 = 1.562 \times 10^{-11}\) and \(r=0.163\).

2. Thermal moment and quasi-static bending moments

[3] shows that the thermal moment \(M_i\) is the key factor to induce the early motion, i. e., the RBM. Fig. 1 displays histories of \(M_i\) at the point \(r=0\) with different values of \(h_1\). A comparison of Fig. 1 with Fig. 3 and Fig. 4 in [3] shows that the temperature difference of both surfaces and the thermal moment reach simultaneously maximum values. Additionally, when the temperature distribution in axial direction comes into close agreement, the thermal moments diminish to zero. Meanwhile, the less \(h_1\) is, the shorter \(t_e\) is, where \(t_e\) is the time, when \(M_i\) takes effect on the RBM. The more \(h_1\) is the longer \(t_e\) is.

Fig. 2 shows the variations of \(T_{\text{max}}\) and \(T_{\text{max}}\) with \(h_1^{-1}\), where \(T_{\text{max}}\) and \(T_{\text{max}}\) are, respectively, the time when \(M_i\) at the point \(r=0\) and \(\frac{\partial M_i}{\partial r}\) on the rim of laser spot reach their maximum values. It is seen easily that \(T_{\text{max}}\) and \(T_{\text{max}}\) are inversely proportional to \(h_1^{-1}\). Fig. 3 shows the variation of \(M_{\text{max}}\) with \(h_1^{-1}\), where \(M_{\text{max}}\) is the maximum value of \(M_i\) at the point \(r=0\). From this figure, we see that \(M_i\) increases with the increasing of \(h_1^{-1}\) when \(h_1^{-1} < 15\). However, \(M_i\) decreases with the increasing of \(h_1^{-1}\) when \(h_1^{-1} > 15\). There is a extreme value point of \(h_1^{-1}\) for which the incident laser damages the target most effectively.

Fig. 4 displays the variation of \(M_i\), with \(r\), where \(r=3.7 \times 10^{-4}\) and \(h_1^{-1}=25\). As expected,
the response of $M_i$ across the irradiated target generally follows the temperature distribution and the laser profile. $M_i$ is uniform within the laser spot and drops sharply near the edge of the laser profile. The special distribution of $M_i$ implies that the spatially cylindrical type pulse laser offers a formidable potential for the exhibition of the RPE.

Fig. 5 displays the variation of $M_i$ and $M_o$ with $r$, where $t=1.5 \times 10^{-4}$ and $h_i^{-1}=25$. $M_i$ is uniform within the laser spot and decrease gradually to zero near the edge of laser spot. However, the characteristic of $M_o$ is very different from that of $M_i$. $M_i$ is positive and uniform within the laser spot. However, near the edge of laser spot, $M_o$ drops sharply to minimum negative value and then increases gradually to zero. The extraordinary feature of $M_i$ and $M_o$ demand us to investigate emphatically the characteristics of deformation near the edge of the laser spot.

3. The analysis of deformation

The results of numerical analysis of the RBM are graphically displayed in the figures which follow. The analysis of the RBM include quasi-static deformation $w_s$, dynamical deflection $w_o$, total deflection $w$ and their velocity $\frac{\partial w_s}{\partial t}$, $\frac{\partial w_o}{\partial t}$, $\frac{\partial w}{\partial t}$ and gradient $\frac{\partial w_s}{\partial r}$, $\frac{\partial w_o}{\partial r}$.

Fig. 6 (a) and (b) display, respectively, two and three dimensional deflected shape of the middle plane of a thin plate, where $h_i^{-1}=25$. In the geometric configuration of the structure, described in Fig. 3 in [3], the deflection value is positive if the thin plate deflects in the same direction as the laser incident direction. The negative values of deflection in the early stage of laser irradiation show that the middle plane of the thin plate bulges in the direction opposite to the laser incident direction. The maximum value of $w$ is 0.26, obtained from Fig. 6 (a). The corresponding factual deflection is $1.84h$ in the case of $h_i=0.0155$ and $h_s=66.4$, where, we observed that the specimen had exhibited the RPE. The prediction of the RBM presented here agrees qualitatively with the experimental observation.

Fig. 7 displays the histories of $w_s$, $w_o$ and $w$ at the point $r=0$, where $h_i^{-1}=25$. It is seen that the contribution of $w_s$ to the RBM takes positive effect during the whole period of laser irradiation. However, the contribution of $w_o$ to the RBM takes negative effect at the beginning of laser irradiation. Meanwhile, the dynamical deflection induces flexural vibration. Fig. 8 shows the variation of $w_{\text{max}}, w_{\text{dmax}}$ and $w_{\text{max}}$ with $h_i^{-1}$, where $w_{\text{max}}, w_{\text{dmax}}$ and $w_{\text{max}}$ are, respectively, the maximum values at point $r=0$. $w_{\text{max}}$, increase rapidly with the increasing of $h_i^{-1}$. $w_{\text{dmax}}$ increase slowly with the increasing of $h_i^{-1}$. Therefore, $h_i$ has important effect on the RBM.

Fig. 9 displays the histories of $\frac{\partial w}{\partial t}$, $\frac{\partial w_s}{\partial t}$ and $\frac{\partial w_o}{\partial t}$ at the point $r=0$. Note that the characteristics of the velocity shown in Fig. 9, and the deflection shown in Fig. 8 are identical. Fig. 9 shows that $\frac{\partial w_{\text{max}}}{\partial t}=16.5$ and $\frac{\partial^2 w_{\text{max}}}{\partial t^2}=2.2 \times 10^4$. The corresponding factual velocity and acceleration are, respectively, $6.2 m/s$ and $4.53 \times 10^3 m/s^2$ in the case of $h_i=0.0155$ and $h_s=66.4$, where, we observed that the specimen had exhibited the RPE.

Fig. 10 displays the variations of $\frac{\partial w}{\partial r}$, $\frac{\partial w_s}{\partial r}$ and $\frac{\partial w_o}{\partial r}$ with $r$. Because of the spatially cylindrical distribution of the laser shape, temperature and equivalent thermal
loading, the deflection gradients have a point of inflection which is on the rim of laser spot. Although the above numerical results and experimental observation show that the deflection is not quite large, the discontinuities of \( \frac{\partial^2 w}{\partial r^2} \) on the rim of laser spot demand us to consider the effect of membrane forces.

VI. Concluding Remarks

The RBM on the newly-discovered RPE is examined by using the uncoupled thin plate theory. Meanwhile, the equivalent thermal loading was presented. The dimensionless analysis of the thin plate motion at the early stage of laser irradiation is given. Numerical results presented show that the rim of laser spot is an extraordinary region which we should emphatically study.

Whereas, the plate material properties have been assumed to be independent of temperature in this study. For the situation with large temperature variation, especially on the rim of laser spot, it will be necessary to account for temperature-dependent properties. This effect, along with the inclusion of thermomechanical coupling in the heat conduction equation and the membrane forces and the shear forces, will be dealt with in forthcoming papers.

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References


