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Very weak solutions of boundary value problems for the Laplace operator and the Lamé system on polyhedral domains in $\mathbb{R}^3$

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à la mémoire de Pierre Grisvard

The notion of very weak solutions is introduced in this paper in order to solve the boundary value problems for the Laplace operator and for the Lamé system with nonsmooth data in polyhedral domains. A continuity theorem is given for variational solutions of the above problems. This result may be used to solve problems with concentrated loads.

1. Introduction

The problems to be considered in this paper consist of typical elliptic operators—the Laplace operator Δ and the Lamé system L on nonsmooth domains and with nonsmooth data, or more precisely, on polyhedral domains and with concentrated loads (the Dirac's measure) even on the boundary, i.e.

$$\begin{cases}\Delta u = f & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n} = h & \text{on } \Gamma_N,\\ u = g & \text{on } \Gamma_D;\end{cases} \tag{P1}$$

$$\begin{cases}\boldsymbol{Lu} = \boldsymbol{f} & \text{in } \Omega,\\ \boldsymbol{Tu} = \boldsymbol{h} & \text{on } \Gamma_N,\\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma_D,\end{cases} \tag{P2}$$

where Ω is a polyhedral domain in $\mathbb{R}^3$ with straight faces Γ_j, $j \in \mathfrak{N} = \{1, \ldots, J\}$, $\Gamma_D = \partial\Omega \backslash \cup_{j \in \mathfrak{N}N} \bar{\Gamma}_j$ and $\bar{\Gamma}_N = \partial\Omega \backslash \cup_{j \in \mathfrak{N}_D} \bar{\Gamma}_j$ with $\mathfrak{N}_D \cup \mathfrak{N}_N = \mathfrak{N}$ and $\mathfrak{N}_D \cap \mathfrak{N}_N = \varnothing$, and $f, h, \boldsymbol{f}, \boldsymbol{h}$ may contain the Dirac's measure.

Problems of this kind are often found in physical modelling (even in analysis), where natural domains are often nonsmooth or may be considered as "small perturbations" of some nonregular domains, and sometimes with concentrated loads: forces, thermo-sources, etc. In practice, a concentrated load is generally the idealisation of a load acting on a small area, where one need not determine its distribution, except its total value. In analysis, the concern is with fundamental solutions.

In the literature, many works on boundary value problems are concerned with smooth domains. The classical results can be found, for example, in the paper of Agmon, Douglis and Nirenberg [1] where smooth problems (with both domain and data being smooth) are considered. In [14] Lions and Magenes give some general results on solutions corresponding to smooth and nonsmooth data on smooth domains. In later works, many references may be found on problems with nonsmooth domains, such as the work of Maz'ja and Plamenevskii [16], Grisvard [7] and Dauge [3]. The emphasis of this work is on the regularity or the singularity of the solutions corresponding to relatively smooth data. In [21] Wildenhain treats elliptic boundary value problems in the space of distributions. But some problems remain to be solved.

Problems concerned with concentrated loads (even on the boundary) will be solved in this paper, by the method of transposition, for a bounded polyhedron in $\mathbb{R}^3$. Similar problems are solved by Lions and Magenes [14] for smooth domains. Instead of using the usual Sobolev spaces as in [14] (which are not suitable for our present case), we use the domain of the operator corresponding to variational solutions. There are two major difficulties in solving our problems: one is the *continuity up to the boundary of functions in the domain of the operator corresponding to variational solutions*; another is the problem of traces for nonregular functions. There are some works on the continuity of variational solutions corresponding to relatively smooth data on a three-dimensional nonsmooth domain. Stampacchia gave a continuity result of such solutions for a Dirichlet problem in an H_0^1-domain [20]. Using the potential theory, the present author obtained a continuity result for both Dirichlet and Neumann problems in a Lipschitz domain [6]. (Note that a polyhedron domain is not always Lipschitzian.) In this paper, we give the continuity results for Dirichlet, Neumann and mixed boundary conditions on a polyhedron domain. The full description of the traces of nonregular functions can be found in [4] or [5].

This paper is organised in two sections, one for the Laplace operator and the other for the Lamé system. In each section we first introduce the notion of very weak solutions, then discuss the continuity of variational solutions, the solutions for concentrated loads and the decomposition of very weak solutions.

2. The case of the Laplace operator

Let Ω, Γ_D and Γ_N be as defined in the Introduction. By the variational method (see [18]), we have the following classical result: if $F(v) = \int_\Omega fv\,dx + \int_{\Gamma_N} hv\,d\sigma$ is a continuous linear functional on $H_D^1(\Omega) = \{v \in H^1(\Omega) | v|_{\Gamma_D} = 0\}$, then (P1) will possess a unique solution in the following sense: there exists a unique $u \in H^1(\Omega)$ such that

$$\begin{cases} u = g & \text{on } \Gamma_D, \text{ and} \\ \displaystyle\int_\Omega \nabla u\, \nabla v\, dx = F(v), & \text{for all } v \in H_D^1(\Omega). \end{cases} \tag{2.1}$$

But this does not cover the case with concentrated loads where we no longer have a solution with finite energy. So we need to introduce the notion of very weak solutions. The idea is to increase the regularity of test functions.

The very weak solution

We denote by $E_0(\Omega)$ the set of variational solutions of the Laplace operator corresponding to a square integrable right-hand side with homogeneous boundary conditions, i.e.

$$E_0(\Omega) = \{v \in H^1_D(\Omega) \mid \Delta v \in L^2(\Omega),\ Nv|_{\Gamma_N} = 0\} \tag{2.2}$$

with norm $\|v\|_{E_0} = (\|v\|^2_{H^1} + \|\Delta v\|^2_{L^2})^{\frac{1}{2}}$, where Nv is defined by (see [4]):

Definition 2.1. For $v \in E(\Omega) = \{v \in H^1(\Omega) \mid \Delta v \in L^2(\Omega)\}$ we define $Nv \in (H^1(\Omega))'$ (the dual space of $H^1(\Omega)$) by

$$\langle Nv, u\rangle_{(H^1)' \times H^1(\Omega)} = \int_\Omega \Delta v u \, dx + \int_\Omega \nabla u \nabla v \, dx \quad \forall\, u \in H^1(\Omega).$$

N is a generalisation of the normal derivative and we have formally

$$\langle Nv, u\rangle = \left\langle\!\!\left\langle \int_{\partial\Omega} \frac{\partial v}{\partial n} u \, d\sigma \right\rangle\!\!\right\rangle.$$

Lemma 2.2. *Let $v \in E_0(\Omega)$; then $\langle Nv, w\rangle = 0$, for all $w \in H^1_D(\Omega)$.*

Remark 2.3. For a Lipschitz domain Ω, there exists a continuous extension operator from $H^1(\Omega)$ into $H^1(\mathbb{R}^3)$. Hence for $v \in E(\Omega)$, Nv can be regarded as an element of $H^{-1}(\mathbb{R}^3)$, and the support set of Nv, supp Nv, is contained in $\partial\Omega$.

Remark 2.4. Up to now we have not yet given the proper definition of the restriction of Nv to Γ_N for $v \in E(\Omega)$. The simplest way is to define $Nv|_{\Gamma_N}$ as a linear continuous function on $H^1_D(\Omega)$, and in what follows we shall take this as the definition of the restriction. Then Lemma 2.2 is a simple consequence of the definition, and it will be used to see the coherence of the very weak solution, which is defined below together with the variational solution.

Using the space $E_0(\Omega)$, we can define the very weak solution by transposition.

Definition 2.5. Let $\mathscr{F}$ be in $(E_0(\Omega))'$, the dual space of $E_0(\Omega)$. We say that u is the *very weak solution of the Laplace operator corresponding to* $\mathscr{F}$ if and only if

$$\begin{cases} u \in L^2(\Omega), \\ \displaystyle\int_\Omega u \Delta v \, dx = \mathscr{F}(v), \quad \forall\, v \in E_0(\Omega). \end{cases} \tag{2.3}$$

The following theorem is well known, and it ensures that Definition 2.5 is meaningful.

Theorem 2.6. *The Laplace operator Δ is an isomorphism from $E_0(\Omega)$ onto $L^2(\Omega)$.*

Theorem 2.7. *For every $\mathscr{F} \in (E_0(\Omega))'$, there exists a unique $u \in L^2(\Omega)$ as the solution of* (2.3).

Proof. As Δ is an isomorphism from $E_0(\Omega)$ onto $L^2(\Omega)$, for every $h \in L^2(\Omega)$ there exists a $v \in E_0(\Omega)$ such that $\Delta v = h$. Denote by Δ^{-1} the inverse of Δ in $L^2(\Omega)$ to $E_0(\Omega)$;

then (2.3) is equivalent to

$$\begin{cases} u \in L^2(\Omega), \\ \displaystyle\int_\Omega uh\,dx = \mathscr{F} \circ \Delta^{-1}(h), \quad \forall\, h \in L^2(\Omega). \end{cases}$$

Because $\mathscr{F} \in (E_0(\Omega))'$, $\mathscr{F} \circ \Delta^{-1}(h)$ is a continuous linear functional on $L^2(\Omega)$. So, by a Riesz representation, we have the existence and uniqueness of the very weak solution u. □

REMARK 2.8. It should be pointed out that if u is the solution of

$$\begin{cases} u \in L^2(\Omega), \\ \Delta u = 0 & \text{in } \Omega, \\ u|_{\Gamma_j} = 0 & \text{for } j \in \mathfrak{N}_D, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_j} = 0 \quad \text{for } j \in \mathfrak{N}_N, \end{cases}$$

it does not follow that $\int u\Delta v\,dx = 0$, for any $v \in E_0(\Omega)$. For instance, let $\Omega = D_1 = \{(\rho, \varphi) \mid \rho < 1, 0 < \varphi < 2\pi\}$, $\Gamma_D = \partial D_1$; then $u = \pi(\rho^{-\frac{1}{2}} - \rho^{\frac{1}{2}}) \sin(\varphi/2) \in L^2(D_1)$, $\Delta u = 0$ and $u|_{\Gamma_j} = 0$, $j \in \mathfrak{N}$. But it is known that $\int u\Delta v\,dx = k(v)$ for any $v \in E_0(\Omega)$, where k is the coefficient of the singular part of v, i.e. $v = v_0 + k(v)\rho^{\frac{1}{2}} \sin(\varphi/2)$ with $v_0 \in H^2(\Omega)$.

Using Lemma 2.2, one may prove the following proposition:

PROPOSITION 2.9. *Let $\mathscr{F}(v) = -F(v) + G(v)$ with $F \in (H_D^1(\Omega))'$ and $G(v) = \langle Nv, u_0 \rangle$, where $u_0 \in H^1(\Omega)$. Assume that u is the very weak solution corresponding to $\mathscr{F} \in (E_0(\Omega))'$. Then u is a variational solution, i.e.*

$$\begin{cases} u \in H^1(\Omega) & \text{such that } u - u_0 \in H_D^1(\Omega) \text{ and,} \\ \displaystyle\int_\Omega \nabla u\, \nabla v\, dx = F(v) & \forall\, v \in H_D^1(\Omega). \end{cases} \tag{2.4}$$

Proof. Let u be the unique variational solution of (2.4). As $E_0(\Omega) \subset H_D^1(\Omega)$, so for $v \in E_0(\Omega)$, (2.4) will still hold. Then by the definition of N we have

$$\int_\Omega u\Delta v\,dx = -F(v) + \langle Nv, u \rangle. \tag{*}$$

Since $u - u_0 \in H_D^1(\Omega)$, from Lemma 2.2, we have

$$\langle Nv, u \rangle = \langle Nv, u_0 \rangle \quad \forall\, v \in E_0(\Omega),$$

so we conclude that

$$\int_\Omega u\Delta v\,dx = \mathscr{F}(v) \quad \forall\, v \in E_0(\Omega).$$

Due to the uniqueness of the very weak solution, we have the proposition. □

REMARK 2.10. In the theory of mechanics, equation (*) is a kind of reciprocal principle, i.e. the work done by the forces of state u on the displacement of state v is equal to the work done by the forces of state v on the displacement of state u.

The continuity of variational solutions

THEOREM 2.11. *Let $E_0(\Omega)$ be defined by* (2.2) *and $C_b(\Omega)$ be the set of functions which are continuous up to the boundary (where, in general, $C_b(\Omega)$ is not always equal to $C(\bar{\Omega})$ except when Ω is a Lipschitz domain); then we have*

$$E_0(\Omega) \hookrightarrow C_b(\Omega).$$

By partition of unity, the problem could be localised as

$$\begin{cases} u \in H_D^1(\Omega_C) & (u=0 \text{ for } r>R), \\ \Delta u = f \in L^2(\Omega_C) & (f=0 \text{ for } r>R), \\ u=0 & \text{for } \omega \in \gamma_D, \\ \dfrac{\partial u}{\partial n}=0 & \text{for } \omega \in \gamma_N, \end{cases} \tag{2.5}$$

$$+ \text{ boundary condition on } \{r=R\} \text{ or at infinity,}$$

where Ω_C is the cone generated by a vertex, and $G = \Omega_C \cap S^2$ is a curvilinear polygon on the unit sphere, with $\partial G = \bar{\gamma}_N \cup \bar{\gamma}_D$ and $\gamma_N \cap \gamma_D = \varnothing$. In spherical coordinates we have $\Omega_C = \{(r, \omega) \,|\, r>0,\ \omega = (\vartheta, \varphi) \in G\}$ and

$$\Delta u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \Delta' u\right],$$

with

$$\Delta' = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial u}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}.$$

To prove the theorem, it is sufficient to verify that the solutions of (2.5) are continuous. The proof is rather constructive. We first transform the Poisson's equation in (2.5) by variable transformations, then construct a solution of the transformed equation. Such a solution reduces the problem to a homogeneous equation, which could then be estimated by the eigen-expansion of the Laplace–Beltrami operator Δ'. The details are given in what follows.

The solution of the transformed equation. Let $r = e^t$, $v(t, \omega) = r^{-\alpha} u(r, \omega)|_{r=e^t}$ and $0 < \alpha \leqq \frac{1}{2}$; the Poisson's equation of (2.5) becomes

$$\frac{\partial^2 v}{\partial t^2} + (1+2\alpha)\frac{\partial v}{\partial t} + (\Delta' + \alpha + \alpha^2) v = h, \tag{2.6}$$

where

$$h = e^{(2-\alpha)t} f, \tag{2.6'}$$

and we have

$$\int_{-\infty}^{+\infty}\int_G |h|^2\,d\omega\,dt=\int_0^{+\infty}\int_G r^{1-2\alpha}|f(r)|^2 r^2\,d\omega\,dr$$
$$=\int_0^R\int_G r^{1-2\alpha}|f(r)|^2 r^2\,d\omega\,dr$$
$$\leqq R^{1-2\alpha}\|f\|^2_{L^2(\Omega_C)}. \tag{2.7}$$

In order to estimate equation (2.6), we need the following lemma [2]:

LEMMA 2.12. *The operator $-\Delta'$ is a positive, selfadjoint, and anti-compact operator with domain $D_{\Delta'}$. Moreover, $0\in\sigma(-\Delta')$ (the spectrum) if and only if γ_D is empty, where (with the definition of $N\varphi$ similar to Nv in Definition 2.1)*

$$D_{\Delta'}=\{\varphi\in H^1(G)\,|\,\Delta'\varphi\in L^2(G),\ \varphi|_{\gamma_D}=0,\ N\varphi|_{\gamma_N}=0\}. \tag{2.8}$$

From this lemma, we have the following theorem:

THEOREM 2.13. *Let $h\in L^2(\mathbb{R}\times G)$ and $0<\alpha\leqq\frac{1}{2}$ with $\alpha+\alpha^2\notin\sigma(-\Delta')$; then there exists a unique $w\in H^2(\mathbb{R},L^2(G))\cap L^2(\mathbb{R},D_{\Delta'})$ such that*

$$\frac{\partial^2 w}{\partial t^2}+(1+2\alpha)\frac{\partial w}{\partial t}+(\Delta'+\alpha+\alpha^2)w=h. \tag{2.9}$$

Proof. Put

$$\hat{w}(\tau,\omega)=\int_{-\infty}^{+\infty}e^{-i\tau t}w(t,\omega)\,dt,$$

the Fourier transform of w with respect to t. Then equation (2.9) becomes

$$\{-\tau^2+(1+2\alpha)i\tau+\Delta'+\alpha+\alpha^2)\hat{w}=\hat{h}.$$

By Lemma 2.12, it is seen that $\sigma(-\Delta')$ contains only non-negative, isolated eigenvalues (see e.g. [10]), so there exists a $\beta\notin\sigma(-\Delta')$ such that $0<\beta\leqq\frac{3}{4}$. Fixing

$$\alpha=\tfrac{1}{2}(\sqrt{1+4\beta}-1), \tag{2.9'}$$

then $0<\alpha\leqq\frac{1}{2}$ and $\alpha+\alpha^2=\beta$, hence $-\tau^2+(1+2\alpha)i\tau+\alpha+\alpha^2\in\rho(-\Delta')$ for all $\tau\in\mathbb{R}$, where $\rho(-\Delta')$ is the resolvant of $-\Delta'$. In addition, we have

$$\|[-\Delta'+\tau^2-(1+2\alpha)i\tau-(\alpha+\alpha^2)]^{-1}\|_{\mathscr{L}(L^2(G))}\leqq\frac{C}{|-\tau^2+(1+2\alpha)i\tau+\alpha+\alpha^2|}.$$

Put

$$w=\int_{-\infty}^{+\infty}-e^{it\tau}[-\Delta'+\tau^2-(1+2\alpha)i\tau-(\alpha+\alpha^2)]^{-1}\hat{h}\,d\tau,$$

i.e.

$$\hat{w}=-[\Delta'+\tau^2-(1+2\alpha)i\tau-(\alpha+\alpha^2)]^{-1}\hat{h};$$

then we could verify that

$$\begin{cases} \|\tau^2 \hat{w}\|_{L^2(G)} \leqq C\|\hat{h}\|_{L^2(G)}, \\ \|\hat{w}\|_{D_{\Delta'}} \leqq C\|\hat{h}\|_{L^2(G)}. \end{cases} \tag{2.10}$$

Using Plancherel's equality, we conclude that

$$w \in H^2(\mathbb{R}, L^2(G)) \cap L^2(\mathbb{R}, D_{\Delta'}).$$

It is obvious that such a w verifies equation (2.9). The uniqueness follows directly from (2.10). □

LEMMA 2.14. *Let $D_{\Delta'}$ be defined by* (2.8); *then*

$$D_{\Delta'} \subset H^{1+\varepsilon}(G)$$

for a certain $\varepsilon > 0$. Furthermore, if $u \in D_{\Delta'}$ then

$$\frac{\partial u}{\partial n} \in L^1(\gamma_D).$$

Proof. Because the property is local, we can estimate in a neighbourhood of a vertex A of G. As the operator Δ' is intrinsic, one could choose $A = \{\vartheta = 0\}$. Taking $\rho = \sin \vartheta$, at the neighbourhood of A we have

$$\Delta' = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\partial \varphi^2} - \rho^2 \frac{\partial^2}{\partial \rho^2} - 2\rho\frac{\partial}{\partial \rho}.$$

According to [7, Theorem 5.2.7, p. 271], we see that if $u \in D_{\Delta'}$ then

$$u = u_R + c\hat{S} \quad \text{with} \quad u_R \in H^2(G) \quad \text{and} \quad \hat{S} = \rho^\alpha \Phi(\varphi), \quad \alpha > 0.$$

Hence there exists $\varepsilon > 0$ such that $\hat{S} \in H^{1+\varepsilon}(G)$; it follows that $u \in H^{1+\varepsilon}(G)$. Since $\partial(u_R)/\partial n \in H^{\frac{1}{2}}(\gamma_D) \hookrightarrow L^1(\gamma_D)$, and $\Phi \in C^\infty$, then

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^1(\gamma_D)} \leqq \|\partial(u_R)/\partial n\|_{L^1(\gamma_D)} + C \int_0^R \rho^{\alpha - 1} d\rho \leqq +\infty,$$

because $\alpha > 0$. □

COROLLARY 2.15. *Let w be the solution of* (2.9) *in Theorem* 2.13; *then* $w \in C_b(\mathbb{R} \times G)$.

Proof. Let w be the solution of (2.9). Using (2.10) and the inequality of convexity (ref. [15]), we have

$$\| |\tau|^s \hat{w} \|_{D_{(-\Delta')^t}} \leqq C \|\hat{h}\|_{L^2(G)}$$

for $s + 2(t-1) = 0$. That is,

$$w \in H^s(\mathbb{R}, D_{(-\Delta')^t})$$

(for a reference for operators $(-\Delta')^t$ see [19]). Choosing $\frac{1}{2} < t < \frac{3}{4}$, we have

$$H^s(\mathbb{R}, D_{(-\Delta')^t}) \hookrightarrow C_b(\mathbb{R}, D_{(-\Delta')^t}).$$

From Lemma 2.14,

$$D_{(-\Delta')} \subset H^{1+\varepsilon}(G), \tag{2.11}$$

for a certain $\varepsilon > 0$. In particular, we have

$$D_{(-\Delta')^{\frac{1}{2}}} \subset H^1(G). \tag{2.12}$$

From (2.11) and (2.12) we conclude (as $t > \frac{1}{2}$) that

$$D_{(-\Delta')^t} \hookrightarrow C_b(G),$$

so that (since $t < \frac{3}{4}$, or $s > \frac{1}{2}$)

$$w \in C_b(\mathbb{R} \times G). \quad \square$$

The proof of Theorem 2.11. Setting $\Omega_R = \Omega_C \cap \{r < R\}$, we have the following lemma:

LEMMA 2.16. *Let $0 < \hat{R} < R$ and u be the variational solution of* (2.5); *then u can be decomposed as*

$$u = u_1 + u_2,$$

with $u_1 \in H^1(\Omega_R) \cap C_b(\Omega_R)$, and $u_2 = \Sigma\, C_k r^{\alpha_k} \Phi_k(\omega)$ for $r < \hat{R}$, where $\Phi_k(\omega)$ are the eigenfunctions of the Laplace–Beltrami operator Δ' and $\alpha_k \geqq 0$.

Proof. Let $w \in C_b(\mathbb{R} \times G)$ be the solution of (2.9) with h defined by (2.6′) and α defined by (2.9′). Setting

$$u_1 = r^\alpha w(\log(r), \omega)|_{\Omega_R}$$

as α is positive, we could verify that

$$u_1 \in H^1(\Omega_R) \cap C_b(\Omega_R)$$

and

$$\begin{cases} \Delta u_1 = f & \text{in } \Omega_R, \\ u_1 = 0 & \text{for } 0 < r < R, \quad \omega \in \gamma_D, \\ \dfrac{\partial u_1}{\partial n} = 0 & \text{for } 0 < r < R, \quad \omega \in \gamma_N. \end{cases} \tag{2.5a}$$

Let u be the variational solution of (2.5) and

$$u_2 = u - u_1.$$

Then $u_2 \in H^1(\Omega_R)$ and

$$\begin{cases} \Delta u_2 = 0 & \text{in } \Omega_R, \\ u_2 = 0 & \text{for } 0 < r < R, \quad \omega \in \gamma_D, \\ \dfrac{\partial u_2}{\partial n} = 0 & \text{for } 0 < r < R, \quad \omega \in \gamma_N, \end{cases} \tag{2.5b}$$

$$+ \text{ boundary condition on } \{r = R\}.$$

Since $-\Delta'$ is a positive, selfadjoint and anti-compact operator, its spectrum $\sigma(-\Delta')$ contains only non-negative isolated eigenvalues, and the corresponding eigenfunctions form a base in $L^2(G)$, [2]. Let $\sigma(-\Delta') = \{\lambda_k\}$ $(0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_k \leqq \ldots,$ $\lambda_k \to \infty$, as $k \to \infty)$ be its spectrum, and $\Phi_k(\omega)$ the eigenfunction corresponding to λ_k, and put

$$u_2 = \sum v_k(r)\Phi_k(\omega).$$

From (2.5b), u_2 must satisfy

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} v_k(r)\right] - \lambda_k v_k(r) = 0.$$

It is easy to solve this equation:

$$v_k(r) = C_k^- r^{\alpha_k^-} + C_k^+ r^{\alpha_k^+},$$

with

$$\alpha_k^\pm = \frac{-1 \pm \sqrt{1+4\lambda_k}}{2}.$$

As $u_2 \in H^1(\Omega_R)$, we must have $C_k^- = 0$ for $\lambda_k \geqq 0$ (for $r^{\alpha_k^\pm} \in H^1(\Omega_R)$, we have $\alpha_k^\pm > -\frac{1}{2}$). So

$$u_2 = \sum C_k^+ r^{\alpha_k^+} \Phi_k(\omega)$$

with $\alpha_k^+ \geqq 0$. □

Proof of Theorem 2.11. From the above lemma we have $u = u_1 + u_2$ and $u_1 \in C_b(\Omega_R)$. Now we prove that we also have $u_2 \in C_b(\Omega_{\hat{R}})$.

As $u_2 \in H^1(\Omega_R)$, the theory of traces [7] gives

$$u_2(r, \omega)|_{r=R} \in L^2(G),$$

i.e.

$$\sum (C_k^+ R^{\alpha_k^+})^2 < +\infty.$$

Then for all $\hat{R} < R$ and all integer n,

$$\sum [C_k^+ (\alpha_k^+)^n \hat{R}^{\alpha_k^+}]^2 < +\infty$$

holds. Using the property

$$\Delta'\Phi_k = \lambda_k \Phi_k \quad \text{and} \quad \lambda_k \sim O((\alpha_k^+)^2)$$

and the previous inequalities, we can prove that for all $s > 0$ there exists a K such that

$$\sum_{k>k} C_k^+ r^{\alpha_k^+} \Phi_k(\omega) \in H^s(]0, \hat{R}[, D_{\Delta'}),$$

which is continuous for s large enough. Since $\alpha_k^+ \geqq 0$ and $\Phi_k \in D_{\Delta'} \hookrightarrow C_b(G)$ (when $\alpha_k^+ = 0$, we have $\lambda_k = 0$ so that $\Phi_k = \text{constant}$) for all k, we could conclude that

$$u_2 \in C_b(\Omega_{\hat{R}}),$$

so

$$u \in C_b(\Omega_{\hat{R}}).$$

By partition of unity, we obtain Theorem 2.11. □

Solution for concentrated loads

Existence. Suppose that we have concentrated loads P_i on points $A_i \in \Omega \cup \Gamma_N$, $i = 1, 2, \ldots, n_i$, and concentrated loads φ_α on rectifiable curves $\gamma_\alpha \subset \Omega \cup \Gamma_N$, $\alpha = 1, 2, \ldots, n_\alpha$; then f and h will take the following form

$$\sum_i A_i \delta_{A_i} + \sum_\alpha \varphi_\alpha \delta_{\gamma_\alpha},$$

where δ_{A_i} and δ_{γ_α} are the Dirac's measures with supports on A_i and on γ_α, respectively. Here the dimension of γ_α is 1. With the help of Remark 2.10, we can now construct the very weak solution of (P1) with these concentrated loads. The effects of these loads on the field v can be put formally in the following expression:

$$\text{“} \int_\Omega fv\, dx - \int_{\Gamma_N} gv\, d\sigma\text{”} = \sum_i P_1 v(A_i) + \sum_\alpha \int_{\gamma_\alpha} \varphi_\alpha v\, ds,$$

so the very weak form for (P1) with such loads should be

$$\int_\Omega u\Delta v\, dx = -\sum_i P_i v(A_i) - \sum_\alpha \int_{\gamma_\alpha} \varphi_\alpha v\, ds + \langle Nv, u_0 \rangle, \quad \forall\, v \in E_0(\Omega), \qquad (2.13)$$

where $u_0 \in H^1(\Omega)$ and $u_0|_{\Gamma_D}$ is the imposed displacement on Γ_D. In order to use Theorem 2.7 to establish the existence and uniqueness of the solution for the above problem, it is necessary and sufficient to show that

$$\mathscr{F}(v) = -\sum_i P_i v(A_i) - \sum_\alpha \int_{\gamma_\alpha} \varphi_\alpha v\, ds + \langle Nv, u_0 \rangle \qquad (2.14)$$

is a continuous linear functional on $E_0(\Omega)$. This is ensured by Theorem 2.11. Thus we have:

COROLLARY 2.17. *Let $\mathscr{F}$ be defined by* (2.14) *and $\varphi_\alpha \in L^1(\gamma_\alpha)$; then $\mathscr{F} \in (E_0(\Omega))'$.* As a consequence we have:

COROLLARY 2.18. (P1) *possesses a unique* (*very weak*) *solution $u \in L^2(\Omega)$, corresponding to $\mathscr{F}$ defined by* (2.14), *in the sense of* (2.13).

For numerical calculation, the more singular the solution is, the less the accuracy. So, in general, using the usual numerical methods to solve the problems with concentrated loads will cause significant error. But, if we know exactly the singular part of the solution, we can transform the problem to a regular one; then we can obtain better accuracy for the numerical calculation. This is the purpose of what follows.

The decomposition of the solution. We shall treat the pointwise concentrated loads and curve supported loads separately. We shall assume that γ_α are segments in $\bar{\Omega} \backslash \bar{\Gamma}_D$. First, let us assume that one of the vertices of Ω is at the origin $\mathbf{0} = (0, 0, 0)$. We define

$$S^1 = \frac{1}{\sigma \cdot r},$$

where r is the spherical coordinate in the radial direction of the spherical coordinate system ($r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, (x, y, z) is the Cartesian coordinate system), and σ is the measure of the cubic angle of this vertex, i.e. (for a certain R_0)

$$\sigma = \text{mes}\,(\partial B(R) \cap \Omega)/R^2 = \text{mes}\,(G), \quad R < R_0,$$

where $B(R)$ is the ball centred at the origin with radius R and G is the intersection of the unit sphere and the polyhedral cone generated by this vertex of Ω.

It is easy to see that S^1 is regular outside a neighbourhood of the origin $\mathbf{0}$, and that the following lemma holds:

LEMMA 2.19. *Let P be any plane passing through the origin and $\boldsymbol{n}$ be its normal vector; then*

$$\frac{\partial S^1}{\partial n} = 0, \quad \text{on } P \backslash \{\mathbf{0}\}.$$

We also have

$$\Delta S^1 = 0, \quad \text{in } \mathbb{R}^3 \backslash \{\mathbf{0}\}.$$

HYPOTHESIS H_e. *Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$, and e be a point or a segment in $\mathbb{R}^3$; we say that e satisfies Hypothesis H_e in Ω if*: (i) *$e \cap \bar{\Gamma}_D$ is empty and,* (ii) *for any $v \in C_b(\Omega)$, $x \in e$, we have*

$$\lim_{i \to \infty} v(z_i) = \lim_{i \to \infty} v(y_i),$$

where $z_i, y_i \in \Omega$ such that $\lim_{i \to \infty} z_i = \lim_{i \to \infty} y_i = x$.

REMARK 2.20. If e is on one face of a crack of Ω, then e does not satisfy Hypothesis H_e. Hypothesis H_e is introduced to ensure the validity of the decomposition of solutions corresponding to concentrated loads in what follows. If Hypothesis H_e is violated, we should modify the definitions of S^1 and S^2 (see later), and in general we could no longer have their explicit forms.

LEMMA 2.21. *Let S^1 be defined as above and the origin $\mathbf{0}$ satisfy Hypothesis H_0; then for any $v \in E_0(\Omega)$ we have*

$$\int_{G_\varepsilon} v \frac{\partial S^1}{\partial n}\, ds \to v(0) \quad \text{as } \varepsilon \to 0,$$

$$\int_{G_\varepsilon} S^1 \frac{\partial v}{\partial n}\, ds \to 0 \quad \text{as } \varepsilon \to 0,$$

where $G_\varepsilon = \partial B(\varepsilon) \cap \Omega$.

Proof. As a convention, we use C to denote any constant that does not depend on ε.

Since $v \in E(\Omega)$, using the decomposition of the functions of $E_0(\Omega)$ in the subsection 'The continuity of variational solutions', above, we can obtain that

$$v = r^\alpha w(\log \Gamma, \omega) + \sum_{k=1}^{K} C_k \cdot r^{\alpha_k} \Phi_k(\omega) + \sum_{k > K} C_k \cdot r^{\alpha_k} \Phi_k(\omega),$$

where $0 < \alpha \leqq \frac{1}{2}$, $\alpha_k \geqq 0$, $\Phi_k \in D_\Delta$, and $w \in H^s(\mathbb{R}, D_{(-\Delta')^t})$ for $s + 2(t-1) = 0$, and

$$w^K(t, \omega) = r^{-1} \sum_{k > K} C_k \cdot r^{\alpha_k} \Phi_k(\omega)|_{r = e^{-1}} \in H^2(\mathbb{R}, D_{\Delta'}).$$

Because for $\alpha_k = 0$ we have $\partial r^{\alpha_k}/\partial r = 0$, then

$$\left| \int_{G_\varepsilon} S^1 \frac{\partial v}{\partial n} ds \right| = \left| \int_G \frac{1}{\sigma \cdot \varepsilon} \cdot \frac{\partial v}{\partial r} \cdot \varepsilon^2 \, d\omega \right|$$

$$\leqq C \cdot \varepsilon \int_G \left| \varepsilon^{\alpha - 1}(w_t + \alpha \cdot w) + \cdot w_t^K + w^K + \sum_{k=1}^{K} C_k \cdot \varepsilon^{\alpha_k - 1} \Phi_k(\omega) \right| d\omega$$

$$\leqq C \cdot \varepsilon^\delta \int_G \left\{ |w_t| + |w| + |w^K| + |w_t^K| + \sum_{k=1}^{K} |C_k \Phi_k(\omega)| \right\} d\omega,$$

where $\delta = \min\{\alpha, \alpha_k; \alpha_k \neq 0, k = 1, \ldots, K\}$. As $w \in H^{3/2+h}(\mathbb{R}, D_{(-\Delta')^{1/4-h/2}})$, then for h small enough there exists a certain $\lambda > 0$ such that

$$w, w_t \in C_b(\mathbb{R}, H^\lambda) \hookrightarrow C_b(\mathbb{R}, L^2(G)).$$

So

$$\sup_t |w|, \sup_t |w_t| \in H^\lambda(G) \hookrightarrow L^2(G) \hookrightarrow L^1(G).$$

Similarly, we can obtain

$$\sup_t |w^K|, \sup_t |w_t^K|, \Phi_k \in L^1(G).$$

Hence

$$\lim_{\varepsilon \to 0} \int_{G_\varepsilon} S^1 \frac{\partial v}{\partial n} ds = 0.$$

Because for $v \in E_0(\Omega)$ we have $v \in C_b(\Omega)$ (Theorem 2.11), Hypothesis H_0 ensures that

$$\int_{G_\varepsilon} v \frac{\partial S^1}{\partial n} ds = -\int_{G_\varepsilon} v \frac{\partial S^1}{\partial r} ds = \int_G v(\varepsilon, \omega) \cdot \frac{1}{\sigma \cdot \varepsilon^2} \cdot \varepsilon^2 \, d\omega$$

$$= \frac{1}{\sigma} \cdot \int_G v(\varepsilon, \omega) \cdot d\omega \to v(\mathbf{0}) \quad \text{as } \varepsilon \to 0. \quad \Box$$

Now let (ρ, φ, z) be the cylindrical coordinate system in $\mathbb{R}^3$ and let $a, b \geqq 0$ and $\gamma = \{(0, 0, z) \mid a < z < b\}$ be a segment on an edge of Ω which joins a vertex of Ω at the origin $\mathbf{0}$. Taking a function $\psi \in L^1(\gamma)$, we define

$$S^2 = \frac{1}{2 \cdot \phi} \int_\gamma \frac{1}{|X - Y|} \psi(Y) \, d\gamma = \frac{1}{2 \cdot \phi} \int_a^b \frac{\psi(\mathfrak{z})}{(\rho^2 + (z - \mathfrak{z})^2)^{\frac{1}{2}}} d\mathfrak{z},$$

where ϕ is the measure of the angle of this edge. It is easy to see that S^2 is regular outside a neighbourhood of γ, and the following lemma holds:

Lemma 2.22. *Let S^2 be defined as above; then*

$$\Delta S^2 = 0 \quad \text{in } \mathbb{R}^3 \backslash \gamma.$$

For any plane P passing through γ, we have

$$\frac{\partial S^2}{\partial n} = 0 \quad \text{on } P \backslash \gamma.$$

LEMMA 2.23. *Let* $\mathbb{D} = \{(\rho, \varphi, z) \mid z < k\rho, (\rho^2 + z^2)^{\frac{1}{2}} < R\}$, *with* k, R *being two positive constants, and let* P *be a plane passing through the origin such that* $\gamma \cap P$ *is empty. Then*

$$S^2 \in H^1(\mathbb{D}), \quad \frac{\partial S^2}{\partial n} \in L^{4/3}(P_R) \hookrightarrow (H^{\frac{1}{2}}(P_R))',$$

with $\psi \in L^p(\gamma)$ *such that* $p = 1$ *for* $a > 0$, *and* $p > 2$ *for* $a = 0$, *and* $P_R = P \cap \{(\rho, \varphi, z) \mid (\rho^2 + z^2)^{\frac{1}{2}} < R\}$.

Proof. A simple calculation shows that

$$|\nabla S^2| \leqq C \cdot \int_a^b \frac{|\psi(\mathfrak{z})|}{\rho^2 + (z - \mathfrak{z})^2} \, d\mathfrak{z}.$$

If $a > 0$ then the distance from γ to $\mathbb{D}$ is no less than $ka/(1 + k^2)^{\frac{1}{2}}$, so that

$$|\nabla S^2| \leqq C \cdot \frac{(1 + k^2)^{\frac{1}{2}}}{k \cdot a} \int_a^b |\psi(\mathfrak{z})| \, d\mathfrak{z} \quad \text{in } \mathbb{D}.$$

As $\mathbb{D}$ is bounded, we have

$$\| \nabla S^2 \|_{L^2(\mathbb{D})} \leqq C \cdot \| \psi \|_{L^1(\gamma)}.$$

If $a = 0$ then $(1/p + 1/q = 1)$

$$|\nabla S^2| \leqq C \cdot \left(\int_a^b |\psi(\mathfrak{z})|^p \, d\mathfrak{z} \right)^{1/p} \left(\int_a^b \frac{d\mathfrak{z}}{(\rho^2 + (z - \mathfrak{z})^2)^q} \right)^{1/q}$$

$$\leqq C \cdot \| \psi \|_{L^p(\gamma)} \cdot \begin{cases} (\rho^2)^{\frac{1}{2}q - 1} & z > 0 \\ (\rho^2 + z^2)^{\frac{1}{2}q - 1} & z \leqq 0 \end{cases} \quad \text{in } \mathbb{D}.$$

Since $z < k\rho$, we can obtain, for $z > 0$,

$$\rho^{-2} \leqq C \cdot (\rho^2 + z^2)^{-1} = C \cdot r^{-2}.$$

So

$$|\nabla S^2| \leqq C \cdot \| \psi \|_{L^p(\gamma)} \cdot r^{1/q - 2} = C \cdot \| \psi \|_{L^p(\gamma)} \cdot r^{-1 - 1/p} \quad \text{in } \mathbb{D}.$$

If $p > 2$ we have $-1 - 1/p > -3/2$, which means (as $\mathbb{D}$ is bounded) that

$$\| \nabla S^2 \|_{L^2(\mathbb{D})} \leqq C \cdot \| \psi \|_{L^p(\gamma)}, \quad p > 2.$$

Similarly, we can verify that S^2 is square integrable in $\mathbb{D}$. Therefore

$$S^2 \in H^1(\mathbb{D}),$$

with $\psi \in L^p(\gamma)$ such that $p = 1$ for $a > 0$, and $p > 2$ for $a = 0$.

As γ is a segment with $\gamma \cap P = \emptyset$, then there exists a positive number k such that for any $(\rho, \varphi, z) \in P$ we have $z < k\rho$. So

$$\frac{\partial S^2}{\partial n} \leqq |\nabla S^2| \leqq C \cdot r^{-1-1/p}, \quad \text{for } a = 0,$$

$$\frac{\partial S^2}{\partial n} \leqq |\nabla S^2| \leqq C, \quad \text{for } a > 0,$$

$$\int_{P_R} |r^{-1-1/p}|^{4/3}\, ds \leqq 2\pi \int_0^R |r^{-1-1/p}|^{4/3} r\, dr \leqq +\infty$$

if $(-1 - 1/p) \cdot \frac{4}{3} + 2 > 0$, i.e. $p > 2$. Then from the Sobolev embedding theorem we conclude that

$$\frac{\partial S^2}{\partial n} \in L^{4/3}(P_R) \hookrightarrow (H^{\frac{1}{2}}(P_R))'. \quad \square$$

LEMMA 2.24. *Let* $\Gamma_\varepsilon = \{\rho = \varepsilon, 0 < z < b\} \cap \Omega \backslash B(2\varepsilon)$, $S_\varepsilon = \partial B(2\varepsilon) \cap \Omega \backslash \{\rho \leqq \varepsilon\}$, *and Hypothesis* H_γ *be satisfied. Then, for any* $v \in E_0(\Omega)$, we have

$$\int_{S_\varepsilon} v \frac{\partial S^2}{\partial n}\, ds + \int_{S_\varepsilon} S^2 \frac{\partial v}{\partial n}\, ds \to 0 \quad \textit{as } \varepsilon \to 0,$$

$$\int_{\Gamma_\varepsilon} v \frac{\partial S^2}{\partial n}\, ds \to \int_a^b \psi(z) \cdot v_0(z)\, dz \quad \textit{as } \varepsilon \to 0,$$

$$\int_{\Gamma_\varepsilon} S^2 \frac{\partial v}{\partial n}\, ds \to 0 \quad \textit{as } \varepsilon \to 0,$$

where S^2 *is as defined previously with* $\psi \in L^p(\gamma)$ *under the conditions*:

(i) $p = 1$ *for* $a > 0$, *and* $\phi < 2\pi$ *or* $\gamma \subset \Omega$;
(ii) $p > 1$ *for* $a > 0$ *and* $\phi = 2\pi$ *with* $\gamma \subset \partial\Omega$;
(iii) $p > 2$ *for* $a = 0$.

Also, $v_0(z) = \lim_{\varepsilon \to 0} v(\varepsilon, \varphi, z)$.

Proof. (1) Similarly to the proof of Lemma 2.23, we can obtain the following estimates (with $k = 3^{\frac{1}{2}}$):

$$|S^2| \leqq C \cdot \|\psi\|_{L^p(\gamma)} \cdot r^{-1/p} \quad \textit{on } S_\varepsilon,$$

$$|\nabla S^2| \leqq C \cdot \|\psi_{L^p(\gamma)} \cdot r^{-1-1/p} \quad \text{on } S_\varepsilon$$

if $a = 0$, and

$$|S^2| \leqq C \cdot \|\psi\|_{L^1(\gamma)}, \quad |\nabla S^2| \leqq C \cdot \|\psi\|_{L^1(\gamma)}$$

if $a > 0$. Because when $a = 0$ we have assumed that $p > 2$, then in all cases we have, for $r = 2\varepsilon$ (and with γ being bounded),

$$|S^2| \leqq C \cdot \varepsilon^{-\frac{1}{2}}, \quad |\nabla S^2| \leqq C \cdot \varepsilon^{-3/2}.$$

Hence

$$\int_{S_\varepsilon}\left|v\frac{\partial S^2}{\partial n}+S^2\frac{\partial v}{\partial n}\right|ds=\int_G\left|v\frac{\partial S^2}{\partial n}+S^2\frac{\partial v}{\partial n}\right|\varepsilon^2\,d\omega$$
$$\leqq C\cdot\int_G|v|\varepsilon^{\frac{1}{2}}+\left|\frac{\partial v}{\partial n}\right|\varepsilon^{3/2}\,d\omega.$$

Using the decomposition of $v\in E_0(\Omega)$ as in the proof of Lemma 2.21, we have

$$\int_{S_\varepsilon}\left|v\frac{\partial S^2}{\partial n}+S^2\frac{\partial v}{\partial n}\right|ds\leqq C\cdot\varepsilon^{\frac{1}{2}}\to 0\quad\text{as }\varepsilon\to 0.$$

(2) Let

$$\bar{\psi}(\mathfrak{z})=\begin{cases}\psi(\mathfrak{z}) & \mathfrak{z}\in(a,b),\\ 0 & \mathfrak{z}\in\mathbb{R}\backslash(a,b);\end{cases}$$

then

$$\int_{\Gamma_\varepsilon}v\frac{\partial S^2}{\partial n}ds=\frac{\varepsilon^2}{2\cdot\phi}\int_0^\phi\int_g^b\int_{-\infty}^{+\infty}\frac{\bar{\psi}(\mathfrak{z})v(\varepsilon,\varphi,z)}{(\varepsilon^2+(z-\mathfrak{z})^2)^{3/2}}d\mathfrak{z}\,dz\,d\varphi$$
$$=\frac{\varepsilon^2}{2\cdot\phi}\int_0^\phi\int_g^b\int_{-\infty}^{+\infty}\frac{\bar{\psi}(z-\mathfrak{z})v(\varepsilon,\varphi,z)}{(\varepsilon^2+\mathfrak{z}^2)^{3/2}}d\mathfrak{z}\,dz\,d\varphi$$
$$=\frac{1}{2\cdot\phi}\int_0^\phi\int_g^b\int_{-\infty}^{+\infty}\frac{\bar{\psi}(z-\varepsilon h)v(\varepsilon,\varphi,z)}{(1+h^2)^{3/2}}dh\,dz\,d\varphi,$$

where

$$g(\varphi)=\max\{z,3^{\frac{1}{2}}\varepsilon\}\quad\text{if }(\varepsilon,\varphi,z)\in\partial\Omega.$$

It is easy to see that $g(\varphi)\sim O(\varepsilon)$. Now we shall show that we can replace $\bar{\psi}(z-\varepsilon h)$ in the last formula by $\bar{\psi}(z)$ in the limit case. In fact, as $v\in C_b(\Omega)$,

$$I(\varepsilon)=\left|\int_g^b\int_{-\infty}^{+\infty}\frac{[\bar{\psi}(z-\varepsilon h)-\bar{\psi}(z)]}{(1+h^2)^{3/2}}v\,dh\,dz\right|$$
$$=\sup|v|\int_g^b\int_{-\infty}^{+\infty}\frac{|\bar{\psi}(z-\varepsilon h)-\bar{\psi}(z)|}{(1+h^2)^{3/2}}dh\,dz.$$

Combining the integrability of $(1+h^2)^{-3/2}$ over $(-\infty,+\infty)$ and the average convergence of L^1 **[18]** we can conclude that

$$I(\varepsilon)\to 0\quad\text{as }\varepsilon\to 0.$$

Therefore, we have (since $\int_{-\infty}^{+\infty}(1+h^2)^{-3/2}\,dh=2$)

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} v\frac{\partial S^2}{\partial n}\,ds=\lim_{\varepsilon\to 0}\frac{1}{2\cdot\phi}\int_0^{\phi}\int_g^b\int_{-\infty}^{+\infty}\frac{\bar{\psi}(z)v(\varepsilon,\varphi,z)}{(1+h^2)^{3/2}}\,dh\,dz\,d\varphi$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\phi}\int_0^{\phi}\int_g^b\bar{\psi}(z)v(\varepsilon,\varphi,z)\,dz\,d\varphi$$

$$=\int_0^b\bar{\psi}(z)v_0(z)\,dz=\int_a^b\psi(z)v_0(z)\,dz.$$

Note that the last limit is ensured by the continuity of v and Hypothesis H_γ.

(3) Now let us prove the third part of the lemma. First we argue the case where $a>0$. We always assume that ε is small enough.

$$\left|\int_{\Gamma_\varepsilon} S^2\frac{\partial v}{\partial n}\,ds\right|=\left|\varepsilon\int_0^{\phi}\int_g^b S^2\frac{\partial v}{\partial \rho}\,dz\,d\varphi\right|$$

$$\leqq\left|\varepsilon\int_0^{\phi}\int_{a/2}^b S^2\frac{\partial v}{\partial \rho}\,dz\,d\varphi\right|+\left|\varepsilon\int_0^{\phi}\int_g^{a/2} S^2\frac{\partial v}{\partial \rho}\,dz\,d\varphi\right|$$

$$\leqq\varepsilon\left|\int_{a/2}^b(S^2)^2\,dz\right|^{\frac12}\cdot\int_0^{\phi}\left|\int_{a/2}^b\left|\frac{\partial v}{\partial\rho}\right|^2dz\right|^{\frac12}d\varphi$$

$$+\varepsilon\left|\int_g^{a/2}(S^2)^2\,dz\right|^{\frac12}\cdot\int_0^{\phi}\left|\int_g^{a/2}\left|\frac{\partial v}{\partial\rho}\right|^2dz\right|^{\frac12}d\varphi. \tag{2.15}$$

Using the properties of convolution we have, for $1\leqq p\leqq 2$, $\frac12=1/p+1/h-1$,

$$\left|\int_{a/2}^b(S^2)^2\,dz\right|^{\frac12}\leqq C\cdot\|\psi\|_{L^p(\gamma)}\|(\varepsilon^2+z^2)^{-\frac12}\|_{L^h(-b,b)}$$

$$\leqq C\cdot\|\psi\|_{L^p(\gamma)}\begin{cases}\varepsilon^{\frac12-1/p} & 1\leqq p<2,\\ |\log\varepsilon| & p=2.\end{cases}$$

As $a>0$, therefore

$$\left|\int_g^{a/2}(S^2)^2\,dz\right|^{\frac12}\leqq C\cdot\left|\int_g^{a/2}\left|\int_a^b\frac{\psi(\mathfrak{z})}{(\rho^2+(z-\mathfrak{z})^2)^{\frac12}}\,d\mathfrak{z}\right|^2dz\right|^{\frac12}$$

$$\leqq C\cdot\left|\int_g^{a/2}\left|\int_a^b\frac{\psi(\mathfrak{z})}{(a/2)}\,d\mathfrak{z}\right|^2dz\right|^{\frac12}$$

$$\leqq C\cdot\|\psi\|_{L^1(\gamma)}.$$

So for $a>0$ we have $(1\leqq p<2)$

$$\left|\int_{\Gamma_\varepsilon}S^2\frac{\partial v}{\partial n}\,ds\right|\leqq C\cdot\varepsilon^{3/2-1/p}\int_0^{\phi}\left|\int_{a/2}^b\left|\frac{\partial v}{\partial\rho}\right|^2dz\right|^{\frac12}d\varphi$$

$$+C\cdot\varepsilon\int_0^{\phi}\left|\int_g^{a/2}\left|\frac{\partial v}{\partial\rho}\right|^2dz\right|^{\frac12}d\varphi.$$

Because $\partial v/\partial\rho = \sin\vartheta\cdot\partial v/\partial r + \cos\vartheta\cdot(\partial v/\partial\vartheta)/r$, we can make estimates for the integrals in each term. We first do so for $\sin\vartheta\cdot\partial v/\partial r$.

$$\left|\int_0^{\Phi}\left|\int_g^{a/2}\left|\sin\vartheta\frac{\partial v}{\partial r}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\right|^2 \leqq C\cdot\int_0^{\Phi}\int_g^{b}\left|\sin\vartheta\frac{\partial v}{\partial r}\right|^2 dz\,d\varphi$$
$$= C\cdot\left|\int_0^{\Phi}\int_{\vartheta g}^{\vartheta b}\frac{\varepsilon}{\sin\vartheta}\left|\frac{\partial v}{\partial r}\right|^2 \sin\vartheta\,d\vartheta\,d\varphi\right|.$$

Here we have used the variable transformation: $z=\varepsilon\cdot ctg\,\vartheta$. Using the decomposition of $v\in E_0(\Omega)$ as in the proof of Lemma 2.21, we can obtain

$$\left|\int_0^{\Phi}\left|\int_g^{a/2}\left|\sin\vartheta\frac{\partial v}{\partial r}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\right|^2$$
$$\leqq C\cdot\left|\int_0^{\Phi}\int_{\vartheta g}^{\vartheta b}\frac{\varepsilon}{\sin\vartheta}\left|r^{\alpha-1}(\alpha w+w_t)+w_t^K+w^K\right.\right.$$
$$\left.\left.+\sum_{k=1}^{K}C_k\cdot r^{\alpha_k-1}\Phi_k(\omega)\right|^2 \sin\vartheta\,d\vartheta\,d\varphi\right|$$
$$\leqq C\cdot\varepsilon^{2\delta-1}\left|\int_G\left\{|w|+|w_t|+|w_t^K|+|w^K|+\sum_{k=1}^{K}|C_k\cdot\Phi_k(\omega)|\right\}^2 d\omega\right|,$$

where we have used the relation $r=\varepsilon/\sin\vartheta$, and $|(\sin\vartheta)^{\mu}|\leqq 1$ for $\mu\geqq 0$. With regard to the fact that for $\alpha_k=0$ we have $r^{\alpha_k}\Phi_k=$ constant, δ is defined by

$$\delta=\min\{\alpha,\alpha_k;\alpha_k\neq 0, 1\leqq k\leqq K\},\quad(\delta>0).$$

Since

$$\sup_t|w|,\ \sup_t|w_t|,\ \sup_t|w^K|,\ \sup_t|w_t^K|,\ |\Phi_k|\in L^2(G),$$

we can conclude that

$$\int_0^{\Phi}\left|\int_g^{a/2}\left|\sin\vartheta\frac{\partial v}{\partial r}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\leqq C\cdot\varepsilon^{\delta-\frac{1}{2}}.$$

Similarly, we have

$$\int_0^{\Phi}\left|\int_{a/2}^{b}\left|\sin\vartheta\frac{\partial v}{\partial r}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\leqq\int_0^{\Phi}\left|\int_g^{b}\left|\sin\vartheta\frac{\partial v}{\partial r}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\leqq C\cdot\varepsilon^{\delta-\frac{1}{2}};$$

i.e. for any p such that $1\leqq p<2$

$$\left|\int_{\Gamma_\varepsilon}S^2\sin\vartheta\frac{\partial v}{\partial r}ds\right|\leqq C\cdot(\varepsilon^{\frac{3}{2}-1/p+\delta-\frac{1}{2}}+\varepsilon^{1+\delta-\frac{1}{2}})$$
$$\leqq C\cdot\varepsilon^{\delta}\to 0\quad\text{as }\varepsilon\to 0.$$

Using the variable transformation $z=(r^2-\varepsilon^2)^{\frac{1}{2}}$, we have

$$\left|\int_0^{\Phi}\left|\int_{a/2}^{b}\left|\cos\vartheta\frac{\partial v}{r\partial\vartheta}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\right|^2\leqq C\cdot\int_0^{\Phi}\int_{r_{a/2}}^{r_b}\left|\frac{\partial v}{r\partial\vartheta}\right|^2 dr\,d\varphi.$$

Let $v = r^\alpha w$ with $0 < \alpha \leqq \frac{1}{2}$, and $\vartheta_1 > 0$ fixed and small enough. We have

$$\int_0^\phi \int_{r_{a/2}}^{r_b} \left|\frac{\partial(r^\alpha w)}{r\partial\vartheta}\right|^2 dr\,d\varphi \leqq (r_{a/2})^{2\alpha-1} \int_0^\phi \int_{r_{a/2}}^{r_b} \left|\frac{\partial w}{\partial\vartheta}\right|^2 \frac{1}{r}\,dr\,d\varphi$$
$$\leqq C\cdot \int_0^\phi \int_{t_{a/2}}^{t_b} \left|\frac{\partial w}{\partial\vartheta}\right|^2 dt\,d\varphi$$
$$\leqq C\cdot \int_0^\phi \int_{t_{a/2}}^{t_b} \left\{\frac{\partial w}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1} + \int_{\vartheta_1}^{\vartheta_t} \frac{\partial^2 w}{\partial^2\vartheta}\,d\vartheta\right\}^2 dt\,d\varphi,$$

where $\sin\vartheta_t = \varepsilon\cdot\exp(-t)$. Since $w \in L^2(\mathbb{R}, D_\Delta{}')$ (see Theorem 2.13), using the decomposition of functions in $D_\Delta{}'$ [7] a simple calculation shows that

$$\frac{\partial w}{\partial\vartheta} \in L^2(\mathbb{R}, H^1(O(\vartheta_1))), \quad (\sin\vartheta)^\mu \frac{\partial^2 w}{\partial^2\vartheta} \in L^2(\mathbb{R}, L^2(G)),$$

where $O(\vartheta_1)$ is a neighbourhood of $\{\vartheta = \vartheta_1\}$ in G, and μ is a positive number such that if $\phi < 2\pi$ or $\gamma \subset \Omega$ then $\mu < \frac{1}{2}$ and if $\phi = 2\pi$ and $\gamma \subset \partial\Omega$ then μ can be any number greater than $\frac{1}{2}$. So

$$\frac{\partial w}{\partial\vartheta}\bigg|_{\vartheta=\vartheta_1} \in L^2(\mathbb{R}, H^{\frac{1}{2}}(\{\vartheta=\vartheta_1\}\cap G)) \hookrightarrow L^2(\mathbb{R}, L^2(\{\vartheta=\vartheta_1\}\cap G)).$$

Also, (as $\min\{|\cos\vartheta_t|, |\cos\vartheta|\} > 0$)

$$\int_0^\phi \int_{t_{a/2}}^{t_b} \left\{\int_{\vartheta_1}^{\vartheta_t} \frac{\partial^2 w}{\partial^2\vartheta}\,d\vartheta\right\}^2 dt\,d\varphi$$
$$\leqq \int_0^\phi \int_{t_{a/2}}^{t_b} \left\{\int_{\vartheta_1}^{\vartheta_t} (\sin\vartheta)^{-2\mu-1}\,d\vartheta\right\}\cdot\left\{\int_{\vartheta_1}^{\vartheta_t} (\sin\vartheta)^{2\mu+1}\left|\frac{\partial^2 w}{\partial^2\vartheta}\right|^2 d\vartheta\right\} dt\,d\varphi$$
$$\leqq C\cdot\varepsilon^{-2\mu} \int_{\mathbb{R}} \int_G \left|(\sin\vartheta)^\mu \frac{\partial^2 w}{\partial^2\vartheta}\right|^2 d\omega\,dt$$

i.e.

$$\left|\int_0^\phi \left|\int_{a/2}^b \left|\cos\vartheta\frac{\partial(r^\alpha w)}{r\partial\vartheta}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\right|^2 \leqq C\cdot(C + C\cdot\varepsilon^{-2\mu}).$$

Taking into consideration that $(r_g)^{2\alpha-1} \leqq C\cdot\varepsilon^{2\alpha-1}$, we obtain

$$\left|\int_0^\phi \left|\int_g^b \left|\cos\vartheta\frac{\partial(r^\alpha w)}{r\partial\vartheta}\right|^2 dz\right|^{\frac{1}{2}} d\varphi\right|^2 \leqq C\cdot\varepsilon^{2\alpha-1}\cdot(C + C\cdot\varepsilon^{-2\mu}).$$

Using the conditions on μ, p and α, we conclude that, as $\varepsilon \to 0$,

$$\left|\int_{\Gamma_\varepsilon} S^2\cdot\cos\vartheta\cdot\frac{\partial(r^\alpha w)}{r\partial\vartheta}\,ds\right| \leqq C\cdot(\varepsilon^{\frac{3}{2}-1/p-\mu} + \varepsilon^{\frac{1}{2}+\alpha-\mu}) \to 0$$

holds. We can estimate the integral for $v = rw^K$ and $\sum_{k=1}^K C_k\cdot r^{\alpha_k}\Phi_k(\omega)$ in a similar way. Taking into account the fact that $\Phi_k = \text{constant}$ for $\alpha_k = 0$, we conclude that

$$\left| \int_{\Gamma_\varepsilon} S^2 \cdot \cos \vartheta \cdot \frac{\partial v}{r \partial \vartheta} \, ds \right| \leqq C \cdot (\varepsilon^{\frac{3}{2} - 1/p - \mu} + \varepsilon^{\frac{1}{2} + \delta - \mu}) \to 0,$$

where $\delta = \min \{\alpha, \alpha_k; \alpha_k \neq 0, 1 \leqq k \leqq K\}$ and $\delta > 0$.

For the case where $a = 0$, the estimate is almost the same as the estimate of the second part of (2.15). The only difference is about the estimate of S^2, i.e.

$$\begin{aligned}
\left| \int_{\Gamma_\varepsilon} S^2 \frac{\partial v}{\partial n} \, ds \right| &= \left| \varepsilon \int_0^{\Phi} \int_g^b S^2 \frac{\partial v}{\partial \rho} \, dz \, d\varphi \right| \\
&\leqq \varepsilon \left| \int_g^b (S^2)^2 \, dz \right|^{\frac{1}{2}} \cdot \int_0^{\Phi} \left| \int_g^b \left| \frac{\partial v}{\partial \rho} \right|^2 dz \right|^{\frac{1}{2}} d\varphi \\
&\leqq C \cdot \varepsilon |\log \varepsilon| \cdot \int_0^{\Phi} \left| \int_g^b \left| \frac{\partial v}{\partial \rho} \right|^2 dz \right|^{\frac{1}{2}} d\varphi.
\end{aligned}$$

The estimate of the integral

$$\int_0^{\Phi} \left| \int_g^b \left| \frac{\partial v}{\partial \rho} \right|^2 dz \right|^{\frac{1}{2}} d\varphi$$

is exactly the same as the estimate of the integral

$$\int_0^{\Phi} \left| \int_g^{a/2} \left| \frac{\partial v}{\partial \rho} \right|^2 dz \right|^{\frac{1}{2}} d\varphi \quad \text{for } a > 0.$$

Therefore

$$\left| \int_{\Gamma_\varepsilon} S^2 \sin \vartheta \frac{\partial v}{\partial r} \, ds \right| \leqq C \cdot \varepsilon^{\delta} |\log \varepsilon|,$$

$$\left| \int_{\Gamma_\varepsilon} S^2 \cdot \cos \vartheta \cdot \frac{\partial v}{r \partial \vartheta} \, ds \right| \leqq C \cdot \varepsilon^{\frac{1}{2} + \delta - \mu} |\log \varepsilon|.$$

This completes the proof. □

PROPOSITION 2.25. *Let S^1 and S^2 be defined as previously, with $\gamma = \{(0, 0, z) \mid a < z < b\}$, and let the conditions in Lemma 2.24 be satisfied. We assume that $\mathbf{0} = (0, 0, 0)$ and $\boldsymbol{f} = (0, 0, f)$, $0 \leqq a < b \leqq f$, are the two ends of an edge of Ω, and the origin $\mathbf{0}$ (respectively γ) satisfies Hypothesis H_0 (respectively H_γ). Then there exists a function $u_1 \in H^1(\Omega)$ (respectively $u_2 \in H^1(\Omega)$) such that for any $v \in E_0(\Omega)$*

$$\int_\Omega (S^1 - u_1) \Delta v \, dx = -v(\mathbf{0})$$

(respectively

$$\int_\Omega (S^2 - u_2) \Delta v \, dx = - \int_\gamma \psi \cdot v \, ds).$$

Proof. We prove the proposition for S^2, while the proof for S^1 is analogous and easier.

Let $\Gamma_{N/e} = \Gamma_N \setminus \cup_{i \in \mathfrak{N}_e} \bar{\Gamma}_i$ where $\mathfrak{N}_e = \{i \mid e \subset \bar{\Gamma}_i\}$. Because S^2 is regular outside a neighbourhood of γ and $\gamma \subset \Omega \cap \Gamma_N$, Hypothesis H_γ and Lemma 2.23 allow us to conclude that

$$S^2|_{\Gamma_D} \in H^{\frac{1}{2}}(\Gamma_D), \quad \left.\frac{\partial S^2}{\partial n}\right|_{\Gamma_{N\setminus\gamma}} \in L^{4/3}(\Gamma_{N\setminus\gamma}).$$

Then from the variational formulation we can see that there exists a unique $u \in H^1(\Omega)$ such that $u|_{\Gamma_D} = S^2$ and

$$\int_\Omega \nabla u \nabla v \, dx = \int_{\Gamma_{N\setminus\gamma}} \frac{\partial S^2}{\partial n} v \, ds \quad \text{for any } v \in H^1_D(\Omega).$$

So we have $\Delta u = 0$, $u|_{\Gamma_D} = S^2|_{\Gamma_D}$ and $\partial u/\partial n|_{\Gamma_{N\setminus\gamma}} = \partial(S^2)/\partial n|_{\Gamma_{N\setminus\gamma}}$, $\partial u/\partial n|_{\Gamma_j} = 0$, $j \in \mathfrak{N}_\gamma$.

Denote by $B(\boldsymbol{p}, \varepsilon)$ the ball centred at the point $\boldsymbol{p}$ with radius ε, and by $Z(\boldsymbol{E}, \varepsilon)$ the cylinder with radius ε whose axis passes through segment $\boldsymbol{E}$. We assume that $\{\boldsymbol{A}_i\}$ and $\{\boldsymbol{E}_j\}$ are the set of the vertices and the set of edges of Ω such that $\boldsymbol{A}_i \neq \boldsymbol{0}$, $\boldsymbol{A}_i \neq \boldsymbol{f}$ for any i and $\boldsymbol{E}_j \cap \overline{\boldsymbol{0f}}$ is empty for any j ($\overline{\boldsymbol{0f}}$ is the segment with ends $\boldsymbol{0}$ and $\boldsymbol{f}$). Then we define

$$\Gamma_\varepsilon = \partial Z(\gamma, \varepsilon) \cap \Omega \setminus (B(\boldsymbol{0}, 2\varepsilon) \cup B(\boldsymbol{f}, 2\varepsilon)),$$

$$S_\varepsilon = (\partial B(\boldsymbol{0}, 2\varepsilon) \cup \partial B(\boldsymbol{f}, 2\varepsilon)) \cap \Omega \setminus Z(\gamma, \varepsilon),$$

$$\Gamma'_\varepsilon = \bigcup_j \left\{ \partial Z(E_j, \varepsilon) \cap \Omega \Big\backslash \left(\bigcup_{A_i \in \bar{E}_j} B(A_i, 2\varepsilon) \right) \right\},$$

$$S'_\varepsilon = \left\{ \bigcup_i \partial B(A_i, 2\varepsilon) \right\} \cap \bar{\Omega} \Big\backslash \left\{ \bigcup_j Z(E_j, \varepsilon) \right\},$$

$$\Omega_\varepsilon = \Omega \Big\backslash \left\{ \left(\bigcup_j Z(E_j, \varepsilon) \right) \cup \left(\bigcup_i B(A_i, 2\varepsilon) \right) \cup Z(\gamma, \varepsilon) \cup B(\boldsymbol{0}, 2\varepsilon) \cup B(\boldsymbol{f}, 2\varepsilon) \right\}.$$

Let $w = S^2 - u$. Considering that S^2 is regular everywhere except in the neighbourhood of γ, we can prove (in the same way as in the proof of Lemma 2.24) that for every $v \in E_0(\Omega)$

$$\lim_{\varepsilon \to 0} \int_{\Gamma'_\varepsilon \cup S'_\varepsilon} \left(\frac{\partial v}{\partial n} S^2 - v \frac{\partial S^2}{\partial n} \right) ds = 0$$

holds. As $v(\partial S^2/\partial n)$ is integrable on $\Gamma_{N\setminus\gamma}$, we have

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{N\setminus\gamma} \cap \partial\Omega_\varepsilon} \frac{\partial S^2}{\partial n} v \, ds = \int_{\Gamma_{N\setminus\gamma}} \frac{\partial S^2}{\partial n} v \, ds \quad \text{for any } v \in E_0(\Omega).$$

Because

$$\lim_{\varepsilon \to 0} \int_{\Omega \setminus \Omega_\varepsilon} \nabla u \nabla v \, dx = 0,$$

for $u \in H^1(\Omega)$ and $v \in E_0(\Omega) \subset \mathrm{H}^1(\Omega)$, we obtain

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Omega_\varepsilon} \nabla u \nabla v \, dx - \int_{\Gamma_{N\setminus\gamma} \cap \partial\Omega_\varepsilon} \frac{\partial S^2}{\partial n} v \, ds \right\} = 0.$$

As $u \in H^1(\Omega)$, we know that $\int_{\Gamma'_\varepsilon \cup S'_\varepsilon \cup \Gamma_\varepsilon \cup S_\varepsilon} |u|^2 \, ds$ is bounded by $\|u\|^2_{H^1(\Omega)}$. Using the Cauchy–Schwartz inequality and the estimates for v in $E_0(\Omega)$ in the proof of Lemma 2.24, we have

$$\lim_{\varepsilon \to 0} \int_{\Gamma'_\varepsilon \cup S'_\varepsilon \cup \Gamma_\varepsilon \cup S_\varepsilon} \frac{\partial v}{\partial n} u \, ds = 0.$$

Let us approximate the integral $\int_\Omega w \Delta v \, dx$ by

$$\int_{\Omega_\varepsilon} w \Delta v \, dx = \int_{\Omega_\varepsilon} S^2 \Delta v \, dx - \int_{\Omega_\varepsilon} u \Delta v \, dx$$
$$= \int_{\partial\Omega_\varepsilon} \left(\frac{\partial v}{\partial n} S^2 - v \frac{\partial S^2}{\partial n} \right) ds + \int_{\Omega_\varepsilon} \nabla u \nabla v \, dx - \int_{\partial\Omega_\varepsilon} \frac{\partial v}{\partial n} u \, ds.$$

By virtue of the boundary conditions satisfied by S^2, u and v, we obtain

$$\int_{\Omega_\varepsilon} w \Delta v \, dx = \int_{\Gamma_\varepsilon \cup S_\varepsilon} \left(\frac{\partial v}{\partial n} S^2 - v \frac{\partial S^2}{\partial n} \right) ds + \int_{\Gamma'_\varepsilon \cup S'_\varepsilon} \left(\frac{\partial v}{\partial n} S^2 - v \frac{\partial S^2}{\partial n} \right) ds$$
$$+ \int_{\Omega_\varepsilon} \nabla u \nabla u \, dx - \int_{\Gamma_{N \setminus \gamma} \cap \partial\Omega_\varepsilon} \frac{\partial S^2}{\partial n} v \, ds - \int_{\Gamma'_\varepsilon \cup S'_\varepsilon \cup \Gamma_\varepsilon \cup S_\varepsilon} \frac{\partial v}{\partial n} u \, ds.$$

Let ε tend to zero. From Lemma 2.24, it is seen that

$$\int_\Omega w \Delta v \, dx = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} w \Delta v \, dx = - \int_\gamma \psi \cdot v \, ds \quad \text{for every } v \in E_0(\Omega).$$

Using this proposition, we can easily obtain the following corollary:

COROLLARY 2.26. *Let u be the very weak solution corresponding to*

$$\mathscr{F}(v) = -F(v) - \sum_i P_i v(A_i) - \sum_\alpha \int_{\gamma_\alpha} \varphi_\alpha v \, ds + \langle Nv, u_0 \rangle$$

with $F \in (H^1_D(\Omega))'$ and $u_0 \in H^1(\Omega)$. Moreover, A_i is a point in $\Omega \cup \Gamma_N$ and satisfies Hypothesis H_{A_i}, γ_α is a segment in $\Omega \cup \Gamma_N$ and satisfies Hypothesis H_{γ_α}, $\varphi_\alpha \in L^p(\gamma_\alpha)$ with p such that

(1) *$p = 1$ if the ends of γ_α do not join the vertices of Ω, and $\phi_\alpha < 2\pi$ or $\gamma \subset \Omega$;*
(2) *$p > 1$ if the ends of γ_α do not joint the vertices of Ω, and $\phi_\alpha = 2\pi$ with $\gamma \subset \partial\Omega$;*
(3) *$p > 2$ if the ends of γ_α join the vertices of Ω,*

where ϕ_α is the measure of the angle of the edge on which γ_α is lying (for γ_α in Ω we have $\phi_\alpha = 2\pi$, for γ_α in a face of Ω we have $\phi_\alpha = \pi$). Let σ_i be the measure of the cubic angle of Ω at point A_i; then we have

$$u = u_R + \sum_i \frac{P_i}{\sigma_i} \frac{1}{|X - A_i|} + \sum_\alpha \frac{1}{2\phi_\alpha} \int_{\gamma_\alpha} \frac{\varphi_\alpha(Y)}{|X - Y|} \, ds_\alpha, \tag{2.16}$$

where $u_R \in H^1(\Omega)$, and $|X - Y| = ((x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2)^{\frac{1}{2}}$.

3. The case of the Lamé system

Now we turn our attention to the Lamé system, i.e. (P2) mentioned in the Introduction. In this case the very weak formulation is analogous to that of the Laplace operator. But for the continuity of variational solutions we have no complete result. We are only able to prove the continuity of the solutions near the edges of a polyhedral domain. We shall present the results in the same order as in the previous section.

The very weak solution

Let us use the notation in Section 2 and introduce:

$$V_0(\Omega) = \{v \in H^1(\Omega)^3 \mid Lv \in L^2(\Omega)^3,\ Tv|_{\Gamma_N} = 0\},$$

where L is the Lamé operator, i.e.

$$Lv = \sigma_{ij,j}(v) \tag{3.1}$$

with

$$\sigma_{ij}(v) = 2\mu\varepsilon_{ij}(v) + \lambda\delta_{ij}\varepsilon_{kk}(v), \tag{3.2}$$

$$\varepsilon_{ij}(v) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.3}$$

and Tv is defined as follows:

DEFINITION 3.1. For $v \in V(\Omega) = \{v \in H^1(\Omega)^3 \mid Lv \in L^2(\Omega)^3\}$ we define $Tv \in (H^1(\Omega)^3)'$ if

$$\langle Tv, u\rangle = \int_\Omega [Lv \cdot u + \sigma_{ij}(v)\varepsilon_{ij}(u)]\, dx \quad \forall\, u \in H^1(\Omega)^3. \tag{3.4}$$

We have formally (see [4] or [5])

$$\langle Tv, u\rangle = \text{“}\int_{\partial\Omega} \sigma_{ij}(v) n_j u_i \, d\sigma\text{”},$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal vector of $\partial\Omega$. $Tv|_{\Gamma_N}$ is defined as the restriction of Tv on $(H_D^1(\Omega)^3)'$.

From the variational resolution, we have [7, 8]:

THEOREM 3.2. *The Lamé operator L is an isomorphism from $V_0(\Omega)$ onto $L^2(\Omega)^3$, if $\Gamma_D \neq \varnothing$.*

REMARK 3.3. For $\Gamma_D = \varnothing$, we may replace $V_0(\Omega)$ by the quotient space of $V_0(\Omega)$: $V_0(\Omega)/u_d$, where u_d represents the rigid displacement:

$$u_d = MX + B$$

with $B \in \mathbb{R}^3$ and $M \in \mathbb{R}^3 \times \mathbb{R}^3$, $M^T = -M$, and replace $L^2(\Omega)^3$ by a subspace of $L^2(\Omega)^3$ of which the functions take the zero mean value and the zero 1-order moment.

By the transposition method, we can introduce the very weak solution:

DEFINITION 3.4. Let $\mathscr{F} \in (V_0(\Omega))'$, we say $\boldsymbol{u}$ is the *very weak solution for the Lamé system corresponding to* $\mathscr{F}$ if and only if

$$\begin{cases} \boldsymbol{u} \in L^2(\Omega)^3 \\ \text{such that } \displaystyle\int_\Omega \boldsymbol{u} \cdot L\boldsymbol{v}\, dx = \mathscr{F}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V_0(\Omega). \end{cases} \tag{3.5}$$

Similar to the first subsection of Section 2, we can show:

THEOREM 3.5. *For every* $F \in (V_0(\Omega))'$ (P2) *posssesses a unique (very weak) solution* $\boldsymbol{u}$ *in the sense of* (3.5).

We also have a similar statement to Proposition 2.9.

The continuity of variational solutions

THEOREM 3.6. *Let* Ω *be a bounded polyhedral domain, and* $\mathscr{S} = \{S_i\}$ *be the set of its vertices. If* $\boldsymbol{u} \in V_0(\Omega)$, *then for all* $\varepsilon > 0$ *we have*

$$\boldsymbol{u} \in C_b\Big(\Omega - \bigcup_{\mathscr{S}} B(S_i, \varepsilon)\Big)^3,$$

where $B(S_i, \varepsilon)$ *is the ball centred at point* S_i *with radius* ε.

The proof follows.

Partial regularity. Let G_1 be a (two-dimensional) bounded polygon, and γ_N, $\gamma_D \subset \partial G_1$ with $\gamma_N \cap \gamma_D = \varnothing$, $\bar{\gamma}_N \cup \bar{\gamma}_D = \partial G_1$. We impose $\Omega_1 = G_1 \times \mathbb{R}$, $\Gamma_N = \gamma_N \times \mathbb{R}$, $\Gamma_D = \gamma_D \times \mathbb{R}$; then $\boldsymbol{u} \in V_0(\Omega_1)$ implies that $\boldsymbol{u} \in H^1_D(\Omega_1)^3$ and $L\boldsymbol{u} \in L^2(\Omega_1)^3$. According to Theorem 3.2 this is equivalent to saying that there exists $\boldsymbol{f} \in L^2(\Omega_1)^3$ and that $\boldsymbol{u}$ is the unique solution of the following variational problem ($\Gamma_D \neq \varnothing$, for $\Gamma_D = \varnothing$ see Remark 3.3):

$$\begin{cases} \boldsymbol{u} \in H^1_D(\Omega_1)^3, \\ \displaystyle\int_{\Omega_1} \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, dx = \int_{\Omega_1} \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall\, \boldsymbol{v} \in H^1_D(\Omega_1)^3. \end{cases} \tag{3.6}$$

Then we have (see [**4, 9, 17**] etc.)

LEMMA 3.7. *Let* $\boldsymbol{f} \in L^2(\Omega_1)^3$ *and* $\boldsymbol{u}$ *be the variational solution defined by* (3.6); *then*

$$\frac{\partial \boldsymbol{u}}{\partial z} \in H^1(\Omega_1)^3,$$

where $z \in \mathbb{R}$, *with* $(m, z) \in \Omega_1$, $m \in G_1$.

Proof. It can be proved by using the invariant property of the domain Ω_1 and of the Lebesgue's measure with respect to the transition in the z direction.

Let us take $\varepsilon \in \mathbb{R}$ and define

$$w_\varepsilon = \frac{w(m, z+\varepsilon) - w(m, z)}{\varepsilon}$$

for $(m, z) \in \Omega_1 = G_1 \times \mathbb{R}$. Then from the invariant properties we have

$$\int_{\Omega_1} \sigma_{ij}(\boldsymbol{u}_\varepsilon)\varepsilon_{ij}(\boldsymbol{v})\,dx = \int_{\Omega_1} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{v}\,dx.$$

So

$$\|\boldsymbol{u}_\varepsilon\|_{H^1(\Omega_1)^3} \leqq C\|\boldsymbol{f}_\varepsilon\|_{(H_D^1(\Omega_1)^3)'}.$$

If we have

$$\|\boldsymbol{f}_\varepsilon\|_{(H_D^1(\Omega_1)^3)'} \leqq C\|\boldsymbol{f}\|_{L^2(\Omega_1)^3}, \tag{3.7}$$

we could conclude that

$$\frac{\partial \boldsymbol{u}}{\partial z} \in H^1(\Omega_1)^3.$$

In fact from (3.7) one can deduce that

$$\|\boldsymbol{u}_\varepsilon\|_{H^1(\Omega_1)^3} \leqq C\|\boldsymbol{f}\|_{L^2(\Omega_1)^3}.$$

Therefore, $\{u_\varepsilon \mid \varepsilon \in \mathbb{R}\}$ is bounded in $H^1(\Omega_1)^3$. Hence it is weakly compact, i.e. there exists a sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$ such that

$$\boldsymbol{u}_{\varepsilon_k} \xrightarrow[\text{weakly}]{} \boldsymbol{w} \in H^1(\Omega_1)^3.$$

But we have

$$\boldsymbol{u}_{\varepsilon_k} \xrightarrow[\text{in the sense of distribution}]{} \frac{\partial \boldsymbol{u}}{\partial z},$$

so that

$$\frac{\partial \boldsymbol{u}}{\partial z} = \boldsymbol{w} \in H^1(\Omega_1)^3$$

(one can obtain an even better result: $\partial \boldsymbol{u}/\partial z \in H_D^1(\Omega_1)^3$). For (3.7), we take $v \in H_D^1(\Omega_1)^3$, then

$$\boldsymbol{v}_{-\varepsilon} = -\frac{1}{\varepsilon}\int_{z-\varepsilon}^{z} \frac{\partial \boldsymbol{v}}{\partial z}\,dz = -\int_0^1 \frac{\partial \boldsymbol{v}}{\partial z}(m, z - t\varepsilon)\,dt.$$

Using the above relation, we have

$$\begin{aligned}\left|\int_{\Omega_1} \boldsymbol{f}_\varepsilon \cdot \boldsymbol{v}\,dx\right| &= \left|-\int_{\Omega_1} \boldsymbol{f} \cdot \boldsymbol{v}_{-\varepsilon}\,dx\right| \\ &= \left|\int_{\Omega_1}\int_0^1 \boldsymbol{f}\frac{\partial \boldsymbol{v}}{\partial z}(m, z - t\varepsilon)\,dt\,dm\right| \\ &\leqq \int_0^1 \|\boldsymbol{f}\|_{L^2(\Omega_1)^3}\left\|\frac{\partial \boldsymbol{v}}{\partial z}(m, z - t\varepsilon)\right\|_{L^2(\Omega_1)^3} dt\end{aligned}$$

$$= \|f\|_{L^2(\Omega_1)^2} \left\| \frac{\partial v}{\partial z} \right\|_{L^2(\Omega_1)^3}$$

$$\leqq C \|f\|_{L^2(\Omega_1)^3} \|v\|_{H^1_D(\Omega_1)^3}$$

This implies (3.7). □

Continuity. THEOREM 3.8. *Let Ω_1 be defined as previously, and $f \in L^2(\Omega_1)^3$. If u is the variational solution defined by* (3.6) *then*

$$u \in C_b(\Omega_1).$$

Proof. We put

$$u = \begin{Bmatrix} v \\ u_z \end{Bmatrix}.$$

Using Lemma 3.7, we arrive at

$$\begin{cases} \dfrac{\partial^2 v}{\partial z^2} + \underline{L}v = F_u \in L^2(\Omega_1)^2, & \\ v = 0 & \text{on } \Gamma_D, \\ -\underline{\sigma}_{ij}(v)\underline{n}_j = H_u \in H^{\frac{1}{2}}(\Gamma_N)^2, & \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2 u_z}{\partial z^2} + \underline{\Delta} u_z = \tilde{f}_u \in L^2(\Omega_1), & \\ u_z = 0 & \text{on } \Gamma_D, \\ \dfrac{\partial u_z}{\partial \underline{\underline{n}}} = h_u \in H^{\frac{1}{2}}(\Gamma_N), & \end{cases}$$

where $\underline{\Delta}, \underline{L}$ are the Laplace operator and the Lamé operator in two-dimensional space, respectively. Also $\underline{\sigma}$ is the two-dimensional stress tensor, $\underline{n}$ is the outward normal vector of γ_N in $\mathbb{R}^2$. We have

$$\underline{\sigma}(v) \cdot \underline{n} = \left\{ \sum_{\beta=1}^{2} \sigma_{\alpha\beta}(v) \cdot n_\beta \right\} = \{\sigma_{\alpha\beta} \cdot n_\beta\}, \quad \alpha = 1, 2.$$

Using the results about traces (see e.g. [4, Appendix III]) one obtains: there exists a $u^1 \in H^2(\Omega_1)^3$ such that

$$\begin{cases} u^1 = 0 & \text{on } \Gamma_D, \\ \underline{\sigma}(v^1)\underline{n} = H_u & \text{on } \Gamma_N, \\ \dfrac{\partial u_z^1}{\partial n} = h_u & \text{on } \Gamma_N. \end{cases}$$

If we put

$$u^2 = u - u^1,$$

then

$$\begin{cases} \dfrac{\partial^2 u^2}{\partial z^2} + \begin{pmatrix} \underline{L} & 0 \\ 0 & \underline{\Delta} \end{pmatrix} u^2 = \mathscr{F} \in L^2(\Omega_1)^3, & \\ u^2 = 0 & \text{on } \Gamma_D, \\ \underline{\sigma}(v^2)\underline{n} = 0 & \text{on } \Gamma_N, \\ \dfrac{\partial u_z^1}{\partial n} = 0 & \text{on } \Gamma_N. \end{cases}$$

We define

$$\mathbb{D} = \begin{pmatrix} D_{\underline{L}} \\ D_{\underline{\Delta}} \end{pmatrix} = \left\{ u \in H_D^1(G_1)^3 \;\middle|\; \begin{pmatrix} \underline{L} & 0 \\ 0 & \underline{\Delta} \end{pmatrix} u \in L^2(G_1), \underline{\sigma}(v)\underline{n} = 0 \quad \text{and} \quad \frac{\partial u_z}{\partial \underline{n}} = 0 \quad \text{on } \gamma_N \right\}$$

($\underline{\sigma}(v)\underline{n}$ and $\partial u_z/\partial \underline{n}$ are defined in a weak sense similar to the definition of Tv); then

$$\begin{pmatrix} \underline{L} & 0 \\ 0 & \underline{\Delta} \end{pmatrix}$$

is a positive selfadjoint invertible ($\Gamma_D \neq \varnothing$) operator with domain $\mathbb{D}$. Similarly to the proofs of Theorem 2.13 and Corollary 2.15, we could obtain

$$u^2 \in H^2(\mathbb{R}, L^2(\Omega_1)^3) \cap L^2(\mathbb{R}, \mathbb{D})^3 \hookrightarrow C_b(\Omega_1)^3.$$

So, from the Sobolev embedding theorem, we have

$$u = u^2 + u^1 \in C_b(\Omega_1)^3.$$

Theorem 3.6. can finally be obtained from the properties of traces and partition of unity.

REMARK 3.9. From the recent result of Kozlov and Maz'ya [12], one can conclude that the functions of $V_0(\Omega)$ are continuous up to the boundary if the domain Ω contains only isolated singular points, e.g. if Ω is a cone with regular intersection with the unit sphere. For such domains Grisvard (personal communication) proved that the functions of $V_0(\Omega)$ ($\Gamma_N = \partial\Omega$, or $\Gamma_D = \partial\Omega$) belong to $H^{3/2+\varepsilon}(\Omega)^3$ for a certain $\varepsilon > 0$. From Sobolev's embedding theorem we have the continuity.

Solution for concentrated loads

Similarly to what is done in the third subsection of Section 2, we shall shall give some results for concentrated loads, i.e. under certain restrictions, f and h may contain Dirac's measures.

Let $P_i \in \mathbb{R}^3$ and $A_i \in \Omega \cup \Gamma_N$. We also assume that $\varphi_\alpha \in L^1(\gamma_\alpha)^3$, with γ_α being a rectifiable curve in $\Omega \cup \Gamma_N$. The following constraint would be imposed:

$$\{A_j\} \cap \{S_i\} = \left(\bigcup_\alpha \bar{\gamma}_\alpha\right) \cap \{S_i\} = \varnothing.$$

Then, from Theorem 3.6, we have the following lemma:

LEMMA 3.10. *With the above assumption, the functional*

$$F(\boldsymbol{v}) = -\sum_\alpha \int_{\gamma_\alpha} \varphi_\alpha \cdot \boldsymbol{v}\, ds - \sum_j \boldsymbol{P}_j \cdot \boldsymbol{v}(A_j) \tag{3.8}$$

is in $(V_0(\Omega))'$.

Take

$$\mathscr{F}(\boldsymbol{v}) = F(\boldsymbol{v}) + \langle T\boldsymbol{v}, \boldsymbol{u}_0 \rangle \tag{3.9}$$

with $\boldsymbol{u}_0 \in H^1(\Omega)^3$. Due to Theorem 3.5 we have

COROLLARY 3.11. *Let* $\mathscr{F} \in (V_0(\Omega))'$ *be defined by* (3.9). *Then there exists a unique* $\boldsymbol{u} \in L^2(\Omega)^3$ *solution of* (P2), *in the sense of* (3.5), *under the concentrated loads* $\boldsymbol{P}_i$ *and* φ_α *in* $\Omega \cup \Gamma_N$ *and the imposed displacement* $\boldsymbol{u}_0|_{\Gamma_D}$ *on* Γ_D.

REMARK 3.12. In Section 2, we discussed the decomposition of solutions corresponding to concentrated loads. The key point, that allows us to use the fundamental solution to construct the singular solutions on segments and to decompose the solutions, is the fact that the normal derivative of the fundamental solution on any plane passing through the singular point of the fundamental solution is zero except at the singular point. Unfortunately this is not the case when we deal with the Lamé system, and in general we do not have an explicit solution for a polyhedral cone or edge (even locally). So far as the author's knowledge goes, the explicit solutions corresponding to a point load exist only for the whole space and the particular case of a half space. Here is the solution in the whole space [12]: let $\boldsymbol{F} = (f_1, f_2, f_3)$ act at point $\boldsymbol{Y} = (y_1, y_2, y_3)$; then the solution of the Lamé system is

$$\boldsymbol{G}_F(\boldsymbol{X} - \boldsymbol{Y}) = \frac{1+\nu}{8\pi E(1-\nu)} \cdot \frac{(3-4\nu)\boldsymbol{F} + \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{F})}{|\boldsymbol{X} - \boldsymbol{Y}|}$$

where $\boldsymbol{n} = (\boldsymbol{X} - \boldsymbol{Y})/|\boldsymbol{X} - \boldsymbol{Y}|$ and the relations between the Lamé coefficients μ, λ and E, ν are $\mu = E/(1+\nu)/2$, $\lambda = \nu E/(1+\nu)/(1-2\nu)$. So, similarly to what is done in Section 2, the solution u of the Lamé system in Ω corresponding to concentrated loads $\boldsymbol{F}$ at a point $A \in \Omega$, $\boldsymbol{H}$ on a segment $\gamma \subset \Omega$ can be decomposed as

$$\boldsymbol{u} = \boldsymbol{u}_R + \boldsymbol{G}_F(\boldsymbol{X} - A) + \int_\gamma \boldsymbol{G}_H(\boldsymbol{X} - \boldsymbol{Y}) \cdot \boldsymbol{H}(\boldsymbol{Y})\, ds_Y,$$

with $\boldsymbol{u}_R \in H^1(\Omega)$.

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