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SELF-CONSISTENT SOLUTION FOR GASEOUS SHOCK AND STELLAR DENSITY WAVE*

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ABSTRACT In this paper, the self-consistent density wave theory containing both a gaseous shock and a linear stellar density wave is studied, and a quasi-stable, tightly-wound, two-arm solution is obtained. The solution is convergent if the incomplete, linearized hydrodynamic equations are used, and the solution then gives the same dispersion relation as the local, asympotic solution, but the density and field profiles will be non-sinusoidal. The stellar wave will be unstable if the complete, linearized hydrodynamic equations are used.

1. INTRODUCTION

Since the 60's, on the basis of Lindblad's density wave theory, C.C. Lin and his co-workers carried out a large research program and found a self-consistent, local, asympotic solution for linear stellar and gaseous density waves [1]. Large differences in gas density are observed inside and outside the spiral arms. Theoretically, as the velocity dispersion of the gas is much smaller than that of the stars, when the stellar component can be treated by a linear theory, the gaseous component should be treated by a non-linear theory. A more complete theory should therefore aim at a self-consistent totality of linear stellar behaviour and non-linear gaseous behaviour. As the mass of the gas is always a small fraction of the total mass of the system, the first step is to find a linear density response of the stars while neglecting all effects of the gas, the second step is to find a non-linear density response of the gas, then we may consider the effect of the non-linear behaviour of the gas on the stellar density wave and so on. If the results are convergent, then a self-consistent solution can be said to have been obtained.

In the theory of interstellar gas, after the idea of galactic shock was proposed by Fujimoto [2], Roberts [3] and Shu et al. [4] have used density wave theory to find local shock wave solutions and HU Wen-rui [5] has studied self-consistent solutions of the self-gravitation of gas and shock. But the coupled problem of a gaseous shock and a stellar density wave has not been dealt with so far.

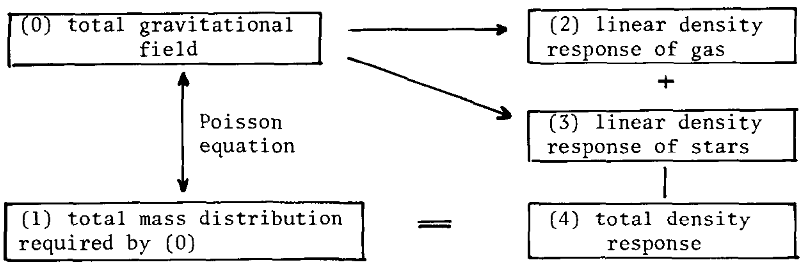
In this paper, I shall give a solution of this coupled problem. When finding the linear stellar density response and the non-linear gas density response, I shall adopt the usual assumptions of the quasi-steady, tightly-wound, two-arm model. The peculiar difficulty of the present problem lies in finding an analytical relation between the local perturbing potential and the local perturbing density, hence I shall depart from the usual procedure and

*A condensed translation

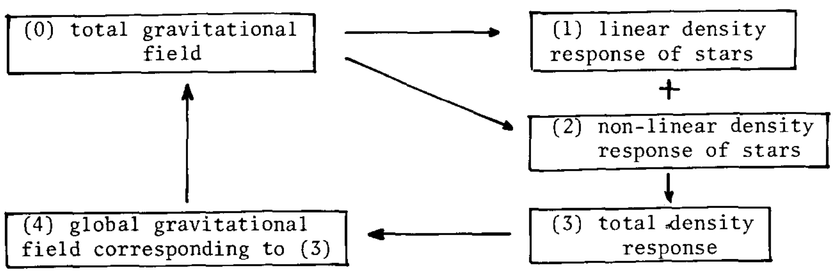
iterate between the total, global, gravitational field and the stellar and gaseous responses.

2. BASIC ASSUMPTIONS, METHOD AND SOLUTION

In a density wave theory, the stellar disk is taken to be so thin that the problem is reduced to two-dimensions, and the sprials are taken to be tightly wound. In the linear theory, the spiral structure is identified with a wave pattern, which is stationary in the coordinates rotating with the pattern speed. The following block diagram summarizes the usual procedure, in which the Poisson equation and the hydrodymanic equations are transformed into a relation between the local perturbing field and density.



The method I shall use is summarized in the following block diagram.



Initially, a specified spiral field (=5% of the basic state) is introduced in (0); the density responses (1), (2), (3) having been found, the total gravitational field corresponding to (3) is calculated, using the global form of the gravitation formula. This is then introduced in (0) to start the next iteration.

Thus, my calculation differs from the local asymptotic solution in that I allow for the non-linear gas density response and, more importantly, I use the global gravitational formula.

In my calculations, I took Schmidt's model [6] for the axisymmetric basic state of the star component and Mezger's model [7] for the gas basic state. A pattern of 2 trailing arms rotating with a speed of $\Omega_p = 11$ km sec^{-1} Mpc^{-1} was used. Velocity dispersion of gas was taken to be 8 km sec^{-1} and that of the stars taken such that $\kappa a_* = \pi G \sigma_0$ (definitions below).

1. Linear Density Response of Stars

Using cylindrical coordinates $\tilde{\omega}$, θ, in which the spiral pattern is at rest, the complete, linearized hydrodynamic equations are

$$\begin{cases} \dfrac{\partial u_*}{\partial \varpi} + \dfrac{(\Omega-\Omega_p)}{\sigma_{*0}} \dfrac{\partial \sigma_*}{\partial \theta} + \dfrac{1}{\varpi} \dfrac{\partial v_*}{\partial \theta} + \left(\dfrac{1}{\varpi} + \dfrac{1}{\sigma_{*0}} \dfrac{\partial \sigma_{*0}}{\partial \varpi}\right) u_* = 0, & (1) \\ \dfrac{\partial \sigma_*}{\partial \varpi} + \dfrac{\sigma_{*0}(\Omega-\Omega_p)}{a_*^2} \dfrac{\partial u_*}{\partial \theta} - \dfrac{2\Omega\sigma_{*0}}{a_*^2} v_* = - \dfrac{\sigma_{*0}}{a_*^2} \dfrac{\partial \mathcal{V}}{\partial \varpi}, & (2) \\ (\Omega-\Omega_p) \dfrac{\partial v_*}{\partial \theta} + \dfrac{\kappa^2}{2\Omega} u_* = - \dfrac{1}{\varpi} \dfrac{a_*^2}{\sigma_{*0}} \dfrac{\partial \sigma_*}{\partial \theta} - \dfrac{1}{\varpi} \dfrac{\partial \mathcal{V}}{\partial \theta}, & (3) \end{cases}$$

where $\kappa^2 = (2\Omega)^2 \left[1 + \dfrac{\varpi}{2\Omega} \dfrac{d\Omega}{d\varpi}\right]$ All the symbols have their usual meanings, in particular σ, u, v are the perturbed density and velocity in the radial and transverse directions, a is the velocity dispersion, and $\mathcal{V}$ the total gravitational potential. Suffix asterisk and suffix 'g' refer respectively to stars and gas and suffices 0 and 1 refer to the basic state and perturbed quantities.

In the usual asymptotic treatment, we omit from (1) the last two terms and from (3) the 2 terms on the right and obtain

$$\begin{cases} \dfrac{\partial u_*}{\partial \varpi} + \dfrac{(\Omega-\Omega_p)}{\sigma_{*0}} \dfrac{\partial \sigma_*}{\partial \theta} = 0, & (4) \\ \dfrac{\partial \sigma_*}{\partial \varpi} + \dfrac{\sigma_{*0}(\Omega-\Omega_p)}{a_*^2} \dfrac{\partial u_*}{\partial \theta} - \dfrac{2\Omega\sigma_{*0}}{a_*^2} v_* = - \dfrac{\sigma_{*2}}{a_*^2} \dfrac{\partial \mathcal{V}}{\partial \varpi}, & (5) \\ (\Omega-\Omega_p) \dfrac{\partial v_*}{\partial \theta} + \dfrac{\kappa^2}{2\Omega} u_* = 0. & (6) \end{cases}$$

We shall call eqns. (4)-(6), the incomplete linearized equations. If the terms omitted are unimportant, then these equations will give about the same solution as the complete equations (1)-(3).

The range 3-25 kpc was divided into 2; in the outer range 15.5-25 kpc where the effect of the shock is small, calculation was made according to the local asymptotic theory [1]; in the inner part, 3-15.5 kpc, according to either (1)-(3) or to (4)-(6). Boundary conditions were taken to be

$$u(\varpi_{\rm I}, \theta) = 0, \quad \varpi_{\rm I} = 3\ \text{kpc},$$
$$\sigma(\varpi_{\rm II}, \theta) = \sigma_A(\varpi_{\rm II}, \theta), \quad \varpi_{\rm II} = 15.5\ \text{kpc},$$

suffix A refers to values in the asymptotic theory.

2. Non-Linear Density Response of the Gas

Under the above assumptions, and in terms of spiral coordinates,

$$\eta = \ln\left(\frac{\varpi}{\varpi_0}\right)\cos i + (\theta - \Omega_p t)\sin i, \quad \xi = -\ln\left(\frac{\varpi}{\varpi_0}\right)\sin i + (\theta - \Omega_p t)\cos i,$$

the equations for a gaseous shock are

$$(\sigma_{g0} + \sigma_{g1})(u_{\eta 0} + u_{\eta 1}) = \sigma_{g0} u_{\eta 0}, \tag{7}$$

$$\frac{du_\eta}{d\eta} = \frac{(u_{\eta 0} + u_\eta)\left(2\Omega\varpi u_\xi - \left.\dfrac{\partial \mathcal{V}}{\partial \eta}\right|_{z=0}\right)}{(u_{\eta 0} + u_\eta)^2 - a_g^2}, \tag{8}$$

$$\frac{du_\xi}{d\eta} = -\frac{\kappa^2 \varpi}{2\Omega(u_{\eta 0} + u_\eta)} u_\eta, \tag{9}$$

in which u_η, u_ξ are the perturbed velocity components in the directions specified.

Our work differs from a solution of the uncoupled shock wave problem in this: we find the density response of the shock from the given total gravity, add it to the linear density response of the stars, then use the global form of gravitation to calculate a new field, to start the next iteration.

The eqns (7)-(9) are solved for a fixed $\bar{\omega}$, $\bar{\omega}_i$. The position of the shock front is determined by the spiral pattern calculated in the local asymptotic solution. The initial field is taken such that the minimum potential is 10° behind the shock in the η-direction. Calculation begins at the "acoustic point" (coordinate η_a) where the equations are

$$\frac{du_\eta}{d\eta}\Big|_{u_{\eta 0}+u_\eta=a} = \sqrt{(\kappa\bar{\omega})^2\frac{(u_{\eta 0}-a_g)}{2a_g} - \frac{1}{2}\bar{\omega}_i\frac{d(F_\eta)}{d\eta}\Big|_{u_{\eta 0}+u_\eta=a_g}}. \tag{10}$$

$$u_\eta(\bar{\omega}_i, \eta_a) = a_g - u_{\eta 0}, \tag{11}$$

$$u_\xi(\bar{\omega}_i, \eta_a) = \frac{-1}{2\Omega}F_\eta, \tag{12}$$

where F_η is the gravity in the η-direction. Following [5], calculation of the shock is made in the range $\frac{du_\eta}{d\eta}\Big|_{u_{\eta 0}+u_\eta=0} > 0$. A final check of the local shock solution is made using the equations of the flow lines,

$$\frac{d\xi}{d\eta} = \frac{u_{\xi 0}+u_\xi}{u_{\eta 0}+u_\eta}\Big|_{\bar{\omega}=\bar{\omega}_i} \tag{13}$$

3. Calculation of Gravitational Field and Iteration

The sum of the linear density response of stars and the non-linear density response of gas satisfies the Poisson equation,

$$\Delta\mathscr{V} = 4\pi G(\sigma_* + \sigma_{g1})\delta(z), \tag{14}$$

However, instead of (14), I shall use Newton's formula to find directly the new radial and transverse field components $F_{\bar{\omega}}$, F_θ. The new field components are then substituted into (5) and (8), and eqns. (4)-(6) then give the next σ_*, and eqns. (7)-(9) give the next σ_{g1}. These in turn give the next field components. And so on.

3. RESULTS OF CALCULATION

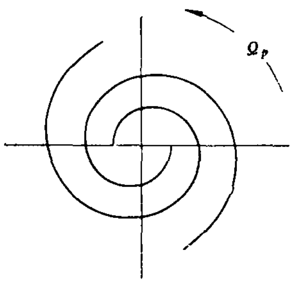


Fig.1 The Spiral pattern (Ω_p=11 km/s/Mpc)

The spiral pattern calculated according to the local asymptotic theory is shown in Fig. 1. This pattern did not change during the iteration.

The incomplete equations (4)-(6) were used in the results shown in Figs. 2,3,4,5, and the complete equations (1)-(3) in Fig. 6. In all these figures, results from the 2nd and 3rd iterations are shown together with the results in the asymptotic solution. Fig. 2 shows the density and velocity profiles of the stars at a constant ϖ and θ. Because of the effect of the shock on the field, the iterated solutions differ

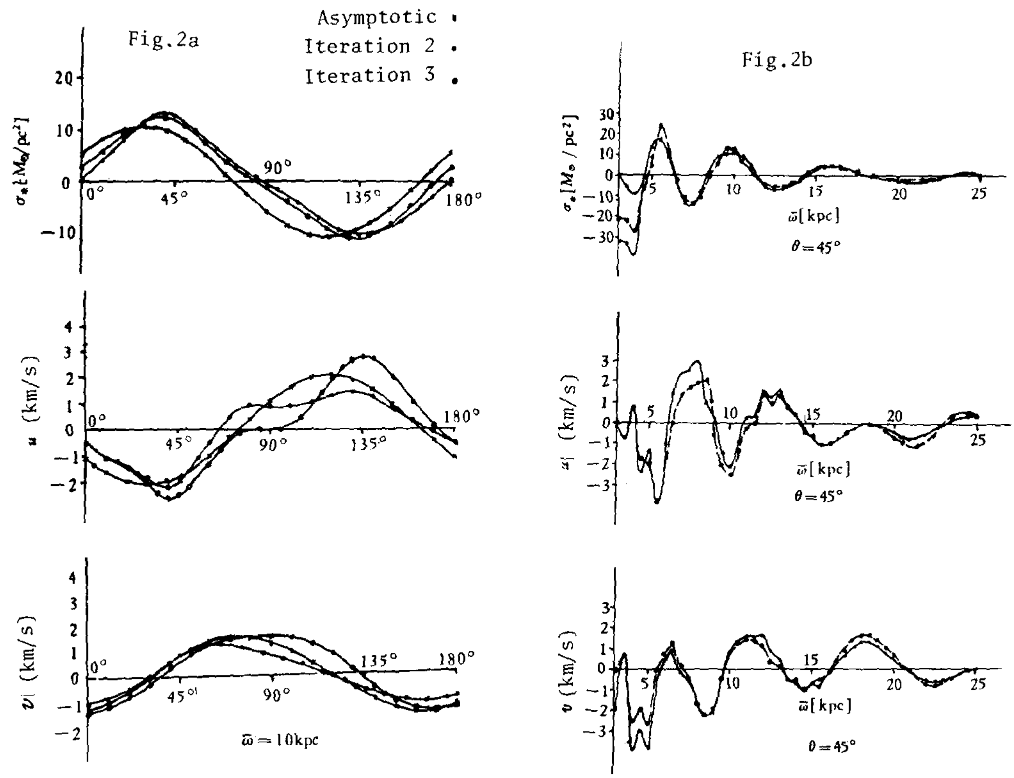


Fig.2 Stellar density wave profiles at constant $\bar{\omega}$ (2a) and constant θ (2b)

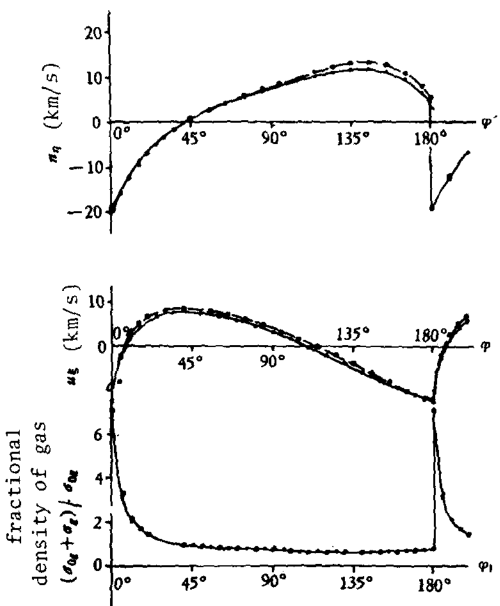


Fig.3 Gaseous shock profiles at $\bar{\omega}=10$kpc

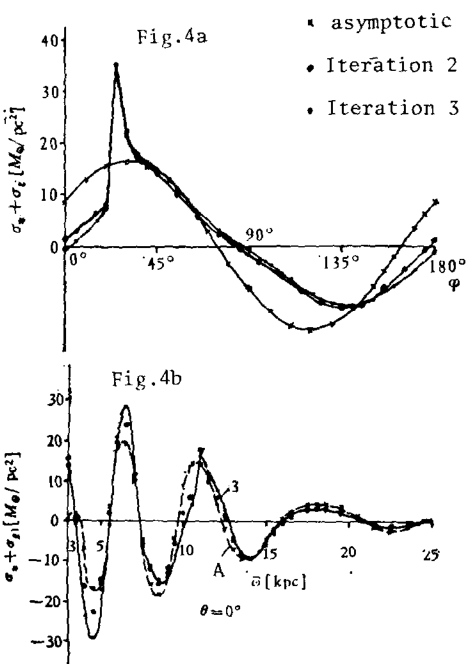


Fig.4 Total surface density profile from (4)-(6). (a) at constant $\bar{\omega}$. (b) at constant θ

somewhat from the asymptotic solution.

Fig. 3 shows the solutions for the shock. When the stellar component was calculated from the incomplete equations (4)-(6), the results for the shock did not vary much from one iteration to the next. Within 12 kpc, self-consistent solutions exist containing both a linear stellar density wave and a gaseous shock wave. Our control calculation shows that the relative errors in the flow lines and radius vector are less than 15% in the region beyond 6 kpc.

Fig. 4 gives the total density response. It shows that the self-consistent solution will be stable, when eqns. (4)-(6) are used, and will give the same dispersion relation as the local asymptotic solution. At the shock front, the total density response presents a sharp maximum and its profile differs greatly from the sinusoidal shape of the asymptotic solution.

Fig. 5 shows the results for the corresponding total field. Again, the shock wave distorts the field from a sinusoidal shape.

Fig. 6 again shows the results for the total density response, but here the star component was calculated from the complete equation (1)-(3). Because the stellar linear density wave is unstable in this case, so is the total density wave.

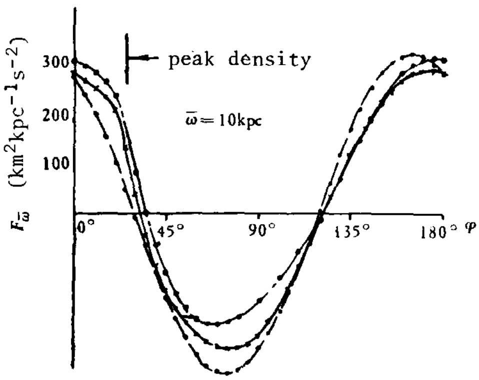


Fig.5 The gravity field profile

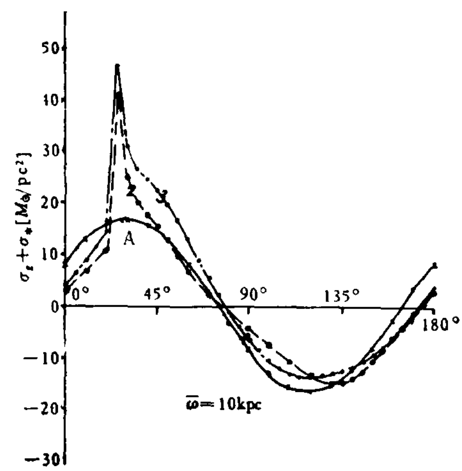


Fig.6 Total density from (1)-(3).

4. DISCUSSION

A self-consistent solution for a linear stellar density wave and a non-linear gaseous shock wave has been obtained in the quasi-stationary, tightly-wound, two-dimensional model. Because the gas is a small part of the total mass, iterative calculation of the coupled system gave results (cf. Figs. 2b,4b) which basically agree with the local asymptotic solution, namely, a stationary spiral density wave and the same dispersion relation.

However, certain variations are introduced by the non-linear behaviour of the gas. Under a spiral field that is 5% of the basic field, while the stellar density response remains linear (Fig. 2), large-scale shocks are formed in the gas (Fig. 3). At the shock front, the gas contributes as much as the stars to the perturbed density, and so becomes no longer negligible theoretically. Fig. 4 shows that the wave profile in the coupled solution is not sinusoidal, and the total density shows a discontinuity at the shock front. At other locations, differences between the coupled and asymptotic solutions are also present, due to the global influence of the shock.

The galactic shock exerts a similar effect on the total field, causing its profile to be non-sinusoidal. At the position of the density discontinuity, the first derivative of the radial field is discontinuous (cf Fig. 5), giving a minimum in the potential. As gas passes through the potential well, the density rapidly grows, heralding the birth of young stars and HII regions, which are precisely the spiral arm tracers.

Again the effect of the shock is felt also eleswhere through the global relation between density and field. Fig. 5 shows, for example, the field curve is steepened on the side containing the shock front, and flattened on the other side.

The incomplete equations (4)-(6) were obtained from the complete equations by the omission of 4 terms. In the asymptotic theory, where $m/k\bar{\omega}$ is regarded as a small quantity, these 4 terms are omitted. Our calculations, using always the global form of gravitation, have shown that equns. (4)-(6) led to stable solutions, while eqns (1)-(3) led to unstable ones. This suggests that $m/k\varpi$ may not be small everywhere throughout the range considered (3-15.5 kpc), and that retaining the 4 terms may lead to global instability. Our results point to the existence of linear instability, and we should mention that a recent study [8] has shown that a linear density wave has a very fast, non-linear rate of growth. This problem should be pursued further.

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