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ON THE STABILITY OF ROTATING LIQUID STARS

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ABSTRACT

In this paper, it is proved by using the method in [1] that the stability criteria of rotating liquid stars, which depend on all of the normal modes of oscillation, are consistent with its secular stability criteria, if liquid stars are assumed to consist of incompressible, self-gravitating masses of viscous liquid. Moreover, the experimental evidence of the conclusion is provided by the Columbus problem[2]. This conclusion was predicted without proof by Thomson and Tait in 1883. Thus, it is reaffirmed that the evolution process of the rotating liquid stars in quasi-stationary state should be judged by the secular stability criteria. Jeans-Darwin's fission theory of binary stars, which has been denied during the last three decades, ought to be evaluated anew.

The problem of equilibrium and stability of the rotating liquid stars is classical. Early works on this subject were reviewed by Jeans[3] and Lamb[4]. Afterwards, the survey of it was made in succession by Lyttleton[5], Ledoux[6, 7], Chandrasekhar[8] and Araki Shunme[9].

Before the fifties of this century, researchers of the older generation believed that "In the practical applications we shall be concerned only with secular stability" and that "For problems of cosmogony it is secular instability alone which is of interest" ([3] § 182, [4] § 205). During those days Jeans-Darwin's fission theory of binary stars based on the above ideas was generally accepted in astronomical circles[10].

In 1953, Lyttleton asserted that there were serious errors in Jeans's exposition about the relation between secular stability and dynamic stability[5]. Henceforth, some authors attempted to seek for another fission theory of binary stars according to dynamic stability instead. Now, opinions vary and no unanimous conclusion can be drawn[9-14]. One of the main points at issue consists in that which stability criteria should be adopted in order to judge the evolution process of the rotating liquid stars.

Early in 1883, Thomson and Tait predicted without proof that the stability of the rotating liquid systems depends on whether total potential energy attains its minimun when viscosity is taken into account[15]. Owing to lack of proof, people have every reason to question whether it is a foregone conclusion that the secular instability means exponential increase in amplitude with time. If viscosity is small enough, the conclusion of stability for Maclaurin's spheroid is positive[16]. But a general proof has not yet been given. In this paper applying the variational method of non-adjoint operator, we proved the above conclusion, and showed how secular instability depends on initial infinitesimal amplitude increasing exponentially with time.

The dispute over the relation between secular stability and dynamic stability is very similar to the one over "D'Alenbert paradox". Without taking account of viscosity, the stability effect of Coriolis force corresponding to dynamic stability is untrue. Only by taking account of viscosity, it can conform to reality. The experimental evidence of this conclusion is provided by the Columbus problem. Applying "stability concept of vortex-lines", we can explain the essence of interaction between the viscosity and Coriolis force. Finally, the application of the above mentioned conclusion to cosmogony has been discussed.

I. Stability Theory of Rotating Liquid Stars

1. *Mathematical model of the rotating liquid star*

Suppose that liquid star consisting of incompressible, self-gravitating masses of viscous liquid uniformly rotate about a fixed axis with angular velocity $\boldsymbol{\Omega}_0$, the density of the liquid stars be non-homogeneous and becomes zero out of them, V denotes volume, and S boundary surface. We shall study the stability of its equilibrium only under its own gravitation.

In an equilibrium case, stars satisfy:

$$\rho_0 \nabla \left\{ \mathscr{B}_0 + \frac{1}{2} |\boldsymbol{\Omega}_0 \times \boldsymbol{x}|^2 \right\} = \nabla P_0, \tag{1}$$

$$\mathscr{B}_0 = G \iiint_V \frac{\rho_0(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}', \tag{2}$$

$$(P_0 = \text{on the surface } S), \tag{3}$$

where ρ_0, P_0, $\mathscr{B}_0$are density, pressure and gravitating potential respectively, x radial vector, G gravitating constant, and $U = \mathscr{B}_0 + 1/2\, |\boldsymbol{\Omega} \times \boldsymbol{x}|^2$. As we know, the equilibrium conditions require that the equipressure surface P_0 equals constant, both the equipotential surface U and the equidensity surface ρ_0 should be constants and the three surfaces should coincide with each other.

The figures of rotating liquid stars in equilibrium case are obtained by solving Eqs. (1)—(3). There have been a lot of articles reviewing this subject[3, 4, 8, 17-23], and recent references were given in [23].

2. *Equations of small disturbance and corresponding eigenvalue problems*

Now, we may deal with the stability of rotating liquid stars according to the stability theory of viscous fluid dynamics[24]. We take a system rotating uniformly with angular velocity $\boldsymbol{\Omega}_0$ relative to the fixed system as the reference one.

Suppose that disturbing displacement vector is $\boldsymbol{\xi}(\boldsymbol{x}, t)$ so the euler variation of physical quantities of the liquid stars may be written as,

$$\left.\begin{aligned} &\boldsymbol{V}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{\xi}(\boldsymbol{x}, t)}{\partial t}, \quad \rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x}) + \tilde{\rho}(\boldsymbol{x}, t), \\ &P(\boldsymbol{x}, t) = P_0(\boldsymbol{x}) + \tilde{P}(\boldsymbol{x}, t), \quad \mathscr{B}(\boldsymbol{x}, t) = \mathscr{B}_0(\boldsymbol{x}) + \tilde{\mathscr{B}}(\boldsymbol{x}, t), \end{aligned}\right\}$$

where qualities with symbol "0", " ~ " indicate those in equilibrium and those disturbed respectively.

The equations of small disturbance of the rotating liquid stars are:

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + 2\rho_0 \boldsymbol{\Omega}_0 \times \frac{\partial \boldsymbol{\xi}}{\partial t} - \mu \nabla^2 \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \right) = -\nabla \widetilde{P} - \nabla \cdot (\rho_0 \boldsymbol{\xi}) \frac{\nabla P_0}{\rho_0} \tag{4}$$

$$+ \rho_0 \nabla_{\boldsymbol{x}} \left\{ \iiint_V G\rho_0(\boldsymbol{x}')(\boldsymbol{\xi} \cdot \nabla)_{\boldsymbol{x}'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x} \right\}, \tag{5}$$

$$\nabla \cdot \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \right) = 0. \tag{6}$$

Disturbing boundary condition: On surface S, it satisfies:

$$[\widetilde{P} + (\boldsymbol{\xi} \cdot \nabla) P_0] \cdot n_i - \mu \left[\frac{\partial^2 \xi_i}{\partial x_j \partial t} + \frac{\partial^2 \xi_j}{\partial x_i \partial t} \right] \cdot n_j = 0. \tag{7}$$

Suppose that

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{\xi}(\boldsymbol{x}) e^{\sigma t}, \tag{8}$$

other disturbing quantities are represented by $\boldsymbol{\xi}$. Thus, the eigenvalue problem for σ is derived as follows:

$$\rho_0 \sigma^2 \boldsymbol{\xi} + \sigma [2\rho_0 \boldsymbol{\Omega}_0 \times \boldsymbol{\xi} - \mu \nabla^2 \boldsymbol{\xi}] = -\nabla \widetilde{P}(\boldsymbol{\xi}) - \frac{(\boldsymbol{\xi} \cdot \nabla)\rho_0}{\rho_0} \nabla P_0$$

$$+ \rho_0 \nabla_{\boldsymbol{x}} \left\{ \iiint_V G\rho_0(\boldsymbol{x}')(\boldsymbol{\xi} \cdot \nabla)_{\boldsymbol{x}'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}' \right\}, \tag{9}$$

$$\nabla \cdot \boldsymbol{\xi} = 0. \tag{10}$$

On surface S,

$$[\widetilde{P}(\boldsymbol{\xi}) + (\boldsymbol{\xi} \cdot \nabla) P_0] \cdot n_i - \sigma\mu \left[\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right] \cdot n_j = 0. \tag{11}$$

The eigenvalue problem given by (9)—(11) are denoted as the problem A. If σ, $\boldsymbol{\xi}$, $\widetilde{P}(\boldsymbol{\xi})$ are the solutions of the problem A, then σ^*, $\boldsymbol{\xi}^*$, $\widetilde{P}^*(\boldsymbol{\xi})$ are its solutions too. Afterwards, according to the variational method of non-adjoint operator given by [1] we shall further treat this problem.

First, we derive adjoint eigenvalue problem (denoted as problem A^+)

$$\rho_0 \omega^2 \boldsymbol{\eta} - \omega [2\rho_0 \boldsymbol{\Omega}_0 \times \boldsymbol{\eta} + \mu \nabla^2 \boldsymbol{\eta}] = -\nabla \widetilde{P}(\boldsymbol{\eta}) - \frac{(\boldsymbol{\eta} \cdot \nabla)\rho_0}{\rho_0} \nabla P_0$$

$$+ \rho_0 \nabla_{\boldsymbol{x}} \left\{ \iiint_V G\rho_0(\boldsymbol{x}')(\boldsymbol{\eta} \cdot \nabla)_{\boldsymbol{x}'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}' \right\}, \tag{12}$$

$$\nabla \cdot \boldsymbol{\eta} = 0. \tag{13}$$

On surface S,

$$[\widetilde{P}(\boldsymbol{\eta}) + (\boldsymbol{\eta} \cdot \nabla) P_0] \cdot n_i - \omega\mu \left[\frac{\partial \eta_i}{\partial x_j} + \frac{\partial \eta_j}{\partial x_i} \right] \cdot n_j = 0. \tag{14}$$

If $\omega, \boldsymbol{\eta}, P(\boldsymbol{\eta})$ are the solutions of the problem A^+, then $\omega^*, \boldsymbol{\eta}^*, P^*(\boldsymbol{\eta})$ are also its solutions. The above mentioned two eigenvalue problems have their clear physical meaning. If A refers to the case of the right-handed rotation, then A^+ to the left-handed rotation.

3. *Variational equation and the proof of variational principle*

(1) In the same way as is done by [1], we can prove that A and A^+ have the same series of eigenvalues, i.e. $\sigma_i = \omega_i$. Multiplying both sides of (9) by the solution $\boldsymbol{\eta}$ of A^+, integrating over the volume V, and using (10), (11), (13), we get the variational equation:

$$\sigma(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{I(\boldsymbol{\xi}, \boldsymbol{\eta})}\left\{-\left[\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) + \frac{\Phi(\boldsymbol{\xi}, \boldsymbol{\eta})}{2}\right] \pm\sqrt{\left[\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) + \frac{\Phi(\boldsymbol{\xi}, \boldsymbol{\eta})}{2}\right]^2 - I(\boldsymbol{\xi}, \boldsymbol{\eta})\delta^2 U(\boldsymbol{\xi}, \boldsymbol{\eta})}\right\}, \tag{15}$$

where

$$\left.\begin{aligned}
I(\boldsymbol{\xi}, \boldsymbol{\eta}) &= \iiint_V \rho_0 \boldsymbol{\xi} \cdot \boldsymbol{\eta} d\boldsymbol{x}, \\
\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) &= \iiint_V \rho_0(\boldsymbol{\Omega}_0 \times \boldsymbol{\xi}) \cdot \boldsymbol{\eta} d\boldsymbol{x}, \\
\Phi(\boldsymbol{\xi}, \boldsymbol{\eta}) &= \iiint_V \frac{\mu}{2}\left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i}\right) \cdot \left(\frac{\partial \eta_i}{\partial x_j} + \frac{\partial \eta_j}{\partial x_i}\right) d\boldsymbol{x}, \\
\delta^2 U(\boldsymbol{\xi}, \boldsymbol{\eta}) &= \iiint_V \boldsymbol{\eta} T \boldsymbol{\xi} d\boldsymbol{x} \\
&= -\iint_s \frac{dP_0}{dn}\eta_n \xi_n ds + \iiint_V \frac{\frac{d\rho_0}{dU}\frac{dP_0}{dU}(\boldsymbol{\xi} \cdot \nabla)U(\boldsymbol{\eta} \cdot \nabla)U}{\rho_0} d\boldsymbol{x} \\
&\quad - \iiint_V \iiint_V G\rho_0(\boldsymbol{x})\rho_0(\boldsymbol{x}')(\boldsymbol{\eta} \cdot \nabla)_{\boldsymbol{x}}(\boldsymbol{\xi} \cdot \nabla)_{\boldsymbol{x}'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}' d\boldsymbol{x}.
\end{aligned}\right\} \tag{16}$$

Then, the problems A and A^+ are reduced to a variation problem of functional $\sigma(\boldsymbol{\xi}, \boldsymbol{\eta})$. If $\boldsymbol{\xi}, \boldsymbol{\eta}$ are continuous and differentiable functions defined on τ and satisfying the additional conditions $\nabla \cdot \boldsymbol{\xi} = 0$, $\nabla \boldsymbol{\eta} = 0$, then the totality of $\boldsymbol{\xi}, \boldsymbol{\eta}$ forms a functional space M and $\sigma(\boldsymbol{\xi}, \boldsymbol{\eta})$is a functional defined on M.

(2) Let $\delta\boldsymbol{\xi}, \delta\boldsymbol{\eta}$ be the variations of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, and satisfy $\boldsymbol{\xi} + \delta\boldsymbol{\xi} \in M$, $\boldsymbol{\eta} + \delta\boldsymbol{\eta} \in M$, it can be shown that when the functional $\sigma(\boldsymbol{\xi}, \boldsymbol{\eta})$ reaches its extreme values, the following relations are obtained:

$$\begin{aligned}
&-\{2\sigma I(\boldsymbol{\xi}, \boldsymbol{\eta}) + 2\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) + \Phi(\boldsymbol{\xi}, \boldsymbol{\eta})\}\delta\sigma \\
&= \iiint_V \Big\{\rho_0\sigma^2\boldsymbol{\xi} + \sigma[2\rho_0\boldsymbol{\Omega} \times \boldsymbol{\xi} - \mu\nabla^2\boldsymbol{\xi}] + \nabla\tilde{P}(\boldsymbol{\xi}) + \frac{(\boldsymbol{\xi} \cdot \nabla)\rho_0}{\rho_0}\nabla P_0
\end{aligned}$$

$$- \rho_0 \nabla_x \left[\iiint_V G\rho_0(\boldsymbol{x}')(\boldsymbol{\xi} \cdot \nabla)_{x'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}' \right] \Big\} \delta\boldsymbol{\eta} d\boldsymbol{x}$$

$$+ \iint_s \left\{ - [\tilde{P}(\boldsymbol{\xi}) + (\boldsymbol{\xi} \cdot \nabla) P_0] \cdot n_i + \sigma\mu \left(\frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right) \cdot n_j \right\} \delta\eta_i ds$$

$$+ \iiint_V \Big\{ \rho_0 \sigma^2 \boldsymbol{\eta} - \sigma[2\rho_0 \boldsymbol{\Omega}_0 \times \boldsymbol{\eta} + \mu \nabla^2 \boldsymbol{\eta}] + \nabla \tilde{P}(\boldsymbol{\eta}) + \frac{(\boldsymbol{\eta} \cdot \nabla)\rho_0}{\rho_0} \nabla P_0$$

$$- \rho_0 G \nabla_{x'} \left[\iiint_V \rho_0(\boldsymbol{x}')(\boldsymbol{\eta} \cdot \nabla)_{x'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}' \right] \Big\} \delta\boldsymbol{\xi} d\boldsymbol{x}$$

$$+ \iint_s \left\{ - [\tilde{P}(\boldsymbol{\eta}) + (\boldsymbol{\eta} \cdot \nabla) P_0] \cdot n_i + \sigma\mu \left(\frac{\partial \eta_i}{\partial x_j} + \frac{\partial \eta_j}{\partial x_i} \right) \cdot n_j \right\} d\xi_i ds = 0. \quad (17)$$

From the above equation, by using the fundamental lemma of variational method ([25], Vol. 1, Chap. 4, §3) it is concluded that when the functional $\sigma(\boldsymbol{\xi}, \boldsymbol{\eta})$ defined on the region M takes its extreme value, the solution of problems A and A+ are derived. On the other hand, the solutions of problems A and A+ cause the functional $\sigma(\boldsymbol{\xi}, \boldsymbol{\eta})$ to take its extreme values. This is a variational problem of nonlinear functional under the natural boundary conditions. Hence, the variational equation (15) provides the variational basis of the approximate method in the treatment of stability and is also the fundamental starting point in the determination of the signs of the real parts of the eigenvalues.

4. *Stability criteria of the rotating liquid stars*

In the variational equation (15), with $\boldsymbol{\eta}$ substituted by $\boldsymbol{\xi}^*$, the integral relation of the eigenvalue may be obtained as follows:

$$\sigma = \frac{1}{I} \left\{ - \left(\Psi + \frac{\Phi}{2} \right) \pm \sqrt{\left(\Psi + \frac{\Phi}{2} \right)^2 - I \cdot \delta^2 U} \right\}, \quad (18)$$

where

$$I = I(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \iiint_V \rho_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}^* d\boldsymbol{x}, \quad (19)$$

$$\Psi = \Psi(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \iiint_V \rho_0 (\boldsymbol{\Omega}_0 \times \boldsymbol{\xi}) \cdot \boldsymbol{\xi}^* d\boldsymbol{x}, \quad (20)$$

$$\Phi = \Phi(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \iiint_V \frac{\mu}{2} \left| \frac{\partial \xi_i}{\partial x_j} + \frac{\partial \xi_j}{\partial x_i} \right|^2 d\boldsymbol{x}, \quad (21)$$

$$\delta^2 U = \delta^2 U(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \iiint_V \boldsymbol{\xi}^* T \boldsymbol{\xi} d\boldsymbol{x} = - \iint_s \frac{\partial P_0}{\partial n} (\boldsymbol{n} \cdot \boldsymbol{\xi})^2 ds$$

$$+ \iiint_V \frac{(\boldsymbol{\xi} \cdot \nabla) U (\boldsymbol{\xi}^* \cdot \nabla) U \frac{dP_0}{dU} \frac{d\rho_0}{dU}}{\rho_0} d\boldsymbol{x}$$

$$- \iiint_V \iiint_V G\rho_0(\boldsymbol{x}) \rho_0(\boldsymbol{x}') (\boldsymbol{\xi}^* \cdot \nabla)_x (\boldsymbol{\xi} \cdot \nabla)_{x'} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}'. \quad (22)$$

On the analogy of [1], it can be proved that each eigenvalue determined by the variational equation (15) corresponds to each one determined by (18).

The various terms in the integral relation of eigenvalue (18), as expressed by Eqs. (19)—(22), have their clear physical meaning: I represents the perturbed kinetic energy, 2Ψ corresponds formally to the work done by Coriolis' force ($\Psi^2 \leqslant 0$), $\Phi > 0$ represents the dissipative work done by the viscous force, $\delta^2 U$ corresponds to perturbed potential energy of the rotating liquid stars as will be proved in the next section.

According to Eq. (18), we may use the following lemma:

Lemma 1. *Let $Z=e+if$ $(e>o)$ and a be any real number, then*

$$(1)\ \operatorname{Re}\{-z \pm \sqrt{z^2 - a^2}\} < 0, \tag{23}$$

$$(2)\ 0 < \max\{\operatorname{Re}[-z \pm \sqrt{z^2 + a^2}]\} < a. \tag{24}$$

The stability criteria may then be obtained as follows:

Theorem 1. *If all of the eigenvectors $\boldsymbol{\xi}$ satisfy*

$$\delta^2 U = \iiint_V \boldsymbol{\xi}^* T \boldsymbol{\xi} d\boldsymbol{x} > 0, \tag{25}$$

then the rotating liquid stars must be stable.

Proof. According to the theory of stability of viscous fluid motion, when the real parts of all eigenvalues are negative, the motion is stable. Now we may prove the theorem by the deduction to absurdity. If there existed a certain eigenfunction $\boldsymbol{\xi}_k$ satisfying

$$\delta^2 U(\boldsymbol{\xi}, \boldsymbol{\xi}^*) < 0,$$

and in Eq. (18), we set

$$e = \frac{\Phi(\boldsymbol{\xi}_k, \boldsymbol{\xi}_k^*)}{2} > 0,$$

$$\text{if} = \Psi(\boldsymbol{\xi}_k^*, \boldsymbol{\xi}_k), \quad a^2 = -I(\boldsymbol{\xi}_k, \boldsymbol{\xi}_k^*), \quad \delta^2 U(\boldsymbol{\xi}_k, \boldsymbol{\xi}_k^*) > 0,$$

then according to (24), the corresponding eigenvalue λ_k would satisfy $\operatorname{Re}\sigma > 0$. This is obviously contrary to the assumption for stability.

Theorem 2. *If there exists at least an eigenfunction $\boldsymbol{\xi}$ satisfying*

$$\delta^2 U(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \iiint_V \boldsymbol{\xi}^* T \boldsymbol{\xi} d\boldsymbol{x} < 0, \tag{26}$$

then the rotating liquid stars must be unstable.

Synthesizing Theorem 1 and 2, we obtain:

Theorem 3. *Suppose that the liquid stars rotating uniformly consist of incompressible self-gravitating masses of viscous liquid, then its critical stability conditions are*

$$\delta^2 U(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \iiint_V \boldsymbol{\xi}^* T \boldsymbol{\xi} d\boldsymbol{x} = 0. \tag{27}$$

II. Secular Stability and Dynamic Stability

1. *Proof of $\delta^2 U$ to be the second order variation of the total potential energy*

In order to derive the second order variation of the total potential energy of the rotating liquid stars, we have examined the stability problem of the equilibrium of the following system: Apart from viscous force and Coriolis force, its potential force including gravitational force and centrifugal force is identical with that of the rotating liquid stars. From Eqs. (9)—(11), an eigenvalue problem determining stability of this system may be obtained as follows:

$$\left.\begin{aligned} &\rho_0\sigma^2\boldsymbol{\xi} = -\nabla\tilde{P} - \frac{(\boldsymbol{\xi}\cdot\nabla)\rho_0}{\rho_0}\nabla P_0 + \rho_0\nabla_x\left\{\iiint\limits_V G\rho_0(\boldsymbol{x}')(\boldsymbol{\xi}\cdot\nabla)_{x'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}d\boldsymbol{x}\right\} \\ &\nabla\cdot\boldsymbol{\xi} = 0 \\ &\text{On surface } S,\ \tilde{P}(\boldsymbol{\xi}) + (\boldsymbol{\xi}\cdot\nabla)P_0 = 0. \end{aligned}\right\} \quad (28)$$

It is easy to prove that this is a self-adjoint eigenvalue problem and its corresponding variational equation satisfies the Rayleigh principle:

$$\sigma^2 = -\frac{\delta^2 U(\boldsymbol{\xi},\boldsymbol{\xi}^*)}{I(\boldsymbol{\xi},\boldsymbol{\xi}^*)} = -\frac{\iiint\limits_V \boldsymbol{\xi}^* T\boldsymbol{\xi}d\boldsymbol{x}}{\iiint\limits_V \rho_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}^* d\boldsymbol{x}}. \quad (29)$$

Thus, the sufficient and necessary conditions of stability of the system become

$$\delta^2 U(\boldsymbol{\xi},\boldsymbol{\xi}^*) = \iiint\limits_V \boldsymbol{\xi}^* T\boldsymbol{\xi}d\boldsymbol{x} > 0. \quad (30)$$

It is well known that with the effect of potential force, the sufficient and necessary conditions of stability of conservative system are that the potential energy attains its minimum. Thus, it can be seen that $\delta^2 U$ is the second order variation of the potential energy. According to Theorem 3, we immediately obtain:

Theorem 4. *If the liquid stars rotating uniformly are incompressible, self-gravitating masses of viscous liquid, then its stability criteria depending on all normal modes of oscillation are consistent with secular stability criteria derived from minimizing potential energy.*

2. *The relations between secular stability and dynamic stability*

It is the existence of Coriolis force that leads to the difference between secular stability and dynamic stability for rotating systems. They are identical for system in static equilibrium.

For the rotating liquid systems, the properties of Coriolis force are as follows:

(1) For the inviscid case, the Coriolis force may possess stabilizing effect. In fact, set $\phi = 0$ in Eq. (18), we obtain

$$\sigma = \frac{1}{I}\{-\Psi \pm \sqrt{\Psi^2 - I\cdot\delta^2 U}\}, \quad (31)$$

then, the sufficient and necessary conditions of the stability are $\Phi^2 - I\cdot\delta^2U<0$. Because $\Psi^2\leqslant 0$, $\delta^2U>0$ only is sufficient condition of the stability. In the region $\Psi^2/I<\delta^2U<0$, Coriolis force may possess stabilizing effect. It is a region of secular instability and one of dynamic stability.

(2) For the viscid case, the Coriolis force cannot possess stabilizing effect. Critical conditions depend on whether the total potential energy of the system attains its minimum which is directly deduced from Theorem 4.

(3) When the liquid system rotates uniformly around its symmetric axis, then, whether we consider viscosity or not, the Coriolis force will never achieve stabilizing effect for the axisymmetric disturbance. In fact, for the axisymmetric disturbance, $\boldsymbol{\xi}(r, z) = \boldsymbol{\xi}_r(r, z), i\zeta_r(r, z), \zeta_j(r, z)$, it is easy to prove that $\Psi = \iiint_V \rho_0(\boldsymbol{\Omega}_0\times\boldsymbol{\xi})\cdot \boldsymbol{\xi}^* d\boldsymbol{x} = 0$, i. e. Coriolis force does not achieve stabilizing effect.

According to the above properties of Coriolis force, we can easily get four relations between secular stability and dynamic stability as listed by Lyttleton ([5] p. 22). The cases emphasized by him in which two kinds of stability disappear simultaneously correspond to those in which Coriolis force does not acquire stabilizing effect. These relations are not essential. In fact, for inviscid case, the dynamic stability conditions corresponding to the stabilizing effect of Coriolis force, are untrue. For viscid case, two stability concepts which were obtained by use of the normal mode method, are identical.

3. *Properties of the secular instability*

Making use of minimizing total potential energy, we could obtain only secular stability conditions, but could not calculate the growth rate of the instability. Thus, somebody raises a question whether secular instability should have been unstable with initial infinitesimal amplitude increasing exponentially with time[16]. Some hold that only the initial infinitesimal amplitude for dynamic instability increases exponentially with time, but the amplitude for secular instability linearly vary with time, the growth rate depending on the magnitude of viscosity, and when viscosity disappears, so does the growth rate[5].

According to Eq. (18), we may calculate the growth rate of secular instability, making use of Lemma 2.

Lemma. 2 *Suppose that $Z=e+if$ $(e>0)$ and a is any real number, then*

(1) *when* $f=0$, $$-z+\sqrt{z^2+a^2} = \frac{a}{\sqrt{a^2+e^2}+e}, \tag{32}$$

(2) *when* $e \ll 1$, $$-z+\sqrt{z^2+a^2} = \begin{cases} \dfrac{a^2}{f^2-a^2}e + i(-f+\sqrt{f^2-a^2}), & \text{if } a^2<f^2, \quad (33) \\ \sqrt{a^2-f^2}-if, & \text{if } a^2>f^2, \quad (34) \\ \sqrt{\dfrac{e^2}{2}+\dfrac{1}{2}\sqrt{e^4+4e^2f^2}}-e-if, & \text{if } a^2=f^2, \quad (35) \end{cases}$$

where

$$e = \frac{\Phi}{2I}, \quad if = \frac{\Psi}{I}, \quad a^2 = -\frac{\delta^2 U}{I}.$$

At first, when $\Psi = 0$, i.e. in the cases when dynamic and secular stabilities disappear simultaneously, making use of Eq. (32), we get

$$\sigma = \sigma_r = \frac{-\dfrac{\delta^2 U}{I}}{\sqrt{-\dfrac{d^2 U}{I} + \dfrac{\Phi}{4I^2}} + \dfrac{\Phi}{2I}} > 0. \tag{36}$$

At this moment, it will not be overstable any longer, even if viscosity exists; in this case small disturbance will increase exponentially with time, and viscosity effect only reduces the growth rate of instability. Such a view that secular instability would also cause small disturbance to increase exponentially with time is of great difference from earlier concept. It seems to be the most dangerous instability, for if only $\delta^2 U < 0$, small disturbance would increase exponentially with time:

$$\exp \sigma_r t = \exp \left\{-\frac{\delta^2 U}{I} t \Big/ \left[\sqrt{-\frac{\delta^2 U}{I} + \frac{\Phi^2}{4I^2}} + \frac{\Phi}{2I}\right]\right\}.$$

For small viscid case, the growth rates of instability would hardly depend on viscosity. Approximately we have

$$\exp \sigma_r t = \exp \left\{t \sqrt{-\frac{\delta^2 U}{I}}\right\},$$

It will not vanish when viscosity is equal to zero.

When $\Psi \neq 0$ and in small viscid case, i. e. $\Phi \ll 1$, from (18), by using Lemma 2, we get

$$\sigma = \sigma_r + i\sigma_i = \begin{cases} \dfrac{-\dfrac{1}{2}\Phi \cdot \delta^2 U}{-\Psi^2 + I \cdot \delta^2 U} - \dfrac{\Psi}{I} + i\dfrac{\sqrt{-\Psi^2 + I \cdot \delta^2 U}}{I}, \\ \qquad \text{if } -\Psi^2 + I \cdot \delta^2 U > 0, & (37) \\ \dfrac{1}{I}\sqrt{\Psi^2 - I \cdot \delta^2 U} - \dfrac{\Psi}{I} \text{ if } -\Psi^2 + I \cdot \delta^2 U < 0, & (38) \end{cases}$$

When $0 > \delta^2 U > \Psi^2/I$, this is a region of secular instability but of dynamic stability. Right now, secular instability is a kind of overstability, the growth rate of instability

$$\sigma_r = \frac{-\dfrac{1}{2}\phi \cdot \delta^2 U}{-\Psi^2 + I \cdot \delta^2 U}$$

depends on the magnitude of viscosity. When $\delta^2 U < \Psi^2/I < 0$, this is a region of dynamic instability as well as of secular instability, and it is also a kind of overstability. In the approximate expression (38) corresponding to small viscid case, σ does not depend on Φ, thus, the growth rate of instability σ_r does not depend on the magnitude of viscosity. It will not vanish when viscosity is equal to zero.

Here, we have not only given the expressions for calculating the growth rate of instability in various cases, but also cleared some erroneous points of view about secular instability.

4. *Physical essence of the viscous effect on instability*

In the problem of motion stability, the viscosity possesses double effects: first, it possesses an ordinary effect of decreasing the disturbance; second, it indirectly gives rise to instability by cancelling factors of motion stability. Owing to viscosity, the stability effect of Coroiolis force disappears in the system of rotating liquid stars, which is known as the viscous instability effect. Now, we may explain the essence of interaction between the viscosity and Coriolis force by applying the concept of vortex-lines stability."

According to the Taylor-Prondman theorem, we know that[26] (i) the liquid system rotating with angular velocity $\boldsymbol{\Omega}_0$ uniformly possesses vorticity $2\boldsymbol{\Omega}_0$; (ii) in the rotating frame, the rotating system of barotropic fluid (containing incompressible fluid) possesses a total vorticity rot $\boldsymbol{v}+2\boldsymbol{\omega}$ and in the process of motion the strength of vortex tube $\iint_S (\text{rot}\, \boldsymbol{v}+2\boldsymbol{\Omega}_0)\, d\boldsymbol{s}$ remains constant (where S is an arbitrary curved surface enclosed by a simple closed curve C). It is easy from the proof of the theorem to see that vorticity $2\boldsymbol{\Omega}_0$ exactly corresponds to Coriolis force.

Before disturbance, the rotating liquid stars possesses uniformly vorticity $2\boldsymbol{\Omega}_0$. In proceeding of disturbance, its vorticity $2\boldsymbol{\Omega}_0$ remains constant. At this moment, we may envisage that the fluid elements resemble small spheres adhering to vortex-line, and rotating with angular velocity $2\boldsymbol{\Omega}_0$around it. In such case the total system could be regarded as a set of the above-mentioned clusters of vortex-lines, which attach to the fluid permanently. The original elements on the vortex-lines before disturbance will still form the same vortex-lines during the period of disturbance, i.e. all of fluid elements forming the vortex-lines are disturbed as a whole. Here the vorext-lines just like frozen-in magnetic lines would produce stabilizing effect to decrease the disturbance. Thus, the stabilizing effect of Coriolis force resembles exactly that of the frozen-in vortex-lines. As soon as viscous effects are taken into account, the frozen-in vorticity and the stabilizing effect of vortex-lines would disappear. In this way, the Coriolis force would not actually arouse any stabilizing effect.

5. *Comment on the dispute over relation between secular stability and dynamic stability*

The historical dispute over the question which lasted nearly about one hundred years, might be attributed to the following two factors:

First of all, the stability criterion obtained for the dynamic stability without considering viscosity is false. In the theory of hydrodynamic stability, a lot of examples have shown that the neglect of viscosity would bring about false conclusions. For example, as to the stability of plane-parallel flow, the flow is always stable for inviscid case, and the instability of laminar motion would never appear; the critical Reynold number could only be obtained for viscid case. This is also a long-standing dispute[24,27], very similar to the one over "D'Alembert paradox". Only for the viscid cases, can such a conclusion conform to reality.

On the other hand, when the secular stability based on the minimum of total potential energy is taken, only stability criteria may be obtained, but the growth rate of secular instability could not be calculated. As a result, it is only under the excessive condition that the stability is required everywhere that the evolution process of the rotating liquid stars in quasi-stationary state would come true.

Taking account of viscosity, we may deal with the stability of the rotating liquid stars by the method of small disturbance. It might be concluded that the two stability concepts are identical. Moreover, to give growth rate of secular instability would bring about dynamic property of quasi-stationary evolution process. Thus, a real process need not demand stability everywhere.

VI. Discussion and Conclusion

Experimental evidence proving truthfulness of secular stability

Although direct simulation of rotating liquid stars is impossible, after altering self-gravitating field to uniform force field, it would be possible to carry out experiment of simulation. The possibility for doing this is provided by the Columbus problem. Kelvin's experiment had already proved that the stabilizing effect of Coriolis force is false[27], which is fairly clear by comparing Maclaulin's ellipsoid with Kelvin's experiment in their theory modes, eigenequations and stability conditions (see Table 1). The consistency between Kelvin's experiment and theory is exactly the proof of the reality of secular stability condition of Maclaulin's ellipsoid.

Table 1

Comparison Between Rotating Liquid Stars and Columbus Problem in Stability Conditions

		Inviscid Cases		Viscid Cases	
		Dynamic Stability Conditions	Dynamic Instability Conditions	Secular Stability Conditions	Seucular Instability Conditions
Rotating liquid stars	generl cases	$\Psi^2 - I \cdot \delta^2 U<0$	$\Psi^2 - I \cdot \delta^2 U>0$	$\delta^2 U>0$	$\delta^2 U<0$
	Maclaulin's ellipsoid	$0<e<0.8127$ $0.8127<e<0.9125$	$0.9125<e<1$	$0<e<0.8127$	$0.8127<e<1$
Columbus problem	general cases	$\Psi^2 - I \cdot \delta^2 W_2<0$	$\Psi^2 - I \cdot \delta^2 W_2>0$	$\delta^2 W_2>0$	$\delta^2 W^2<0$
	ellipsoid-shape fluid rotor gyros	$a>c$ $3a<c$	$a<c<3a$	$a>c$	$a<c$
	Kelvin's experiment	–	–	$a>c$	$a<c$

In Table 1, I, Ψ are given by Eqs. (19) and (20). $\delta^2 U$, $\delta^2 W_2$ represent the potential energy variation of the second order in two cases respectively, a being equator radius, c polar radius and e ellipticity.

2. *Quasi-stationary evolution process of the rotating liquid stars must be judged by secular stability criteria*

This paper has proved that stability criteria of rotating liquid stars, which depend on all the normal modes of oscillation, are consistent with its secular stability criteria. This conclusion had proved theoretically the falsehood of stabilizing effect of Coriolis force. By applying "concept of vortex-lines stability", we have further explained physically the essence of interaction between the viscosity and the Coriolis force, and made use of Kelvin's experiment to prove this conclusion. Thus, it may be affirmed that quasi-stationary evolution process of the rotating liquid stars must be judged by secular sability criteria.

3. *Jeans-Darwin's fission theory of binary stars must be evaluated anew.*

Previously, the secular stability criteria could only be obtained by minimizing total potential energy, whereas the growth rate of secular instability still remains unknown, and this leads to such an excessive condition that the evolution process of rotating liquid stars in quasi-stationary state would come true only by demanding stability everywhere. It is just due to the above-mentioned weak point that Jeans-Darwin's fission theory of binary stars was rejected.

On the condition that the dependence of secular instability on small disturbance increasing exponentially with time is established, that would bring about dynamic property of quasi-stationary evolution process, and that the characteristic time of the secular instability is larger than that of the process evolution, such process will be realized. On such basis, we can discuss the reality of Jeans-Darwin's fission process of binary stars and some new light may be thrown upon the problem.

4. *The conclusion of this paper is suitable for rotating magnetic stars as well*

As for the case of rotating MHD system, our conclusion is identical to that of the present paper, i.e. the stabilizing effect of Coriolis force is false[28]. The application of the method in the present paper and the conclusion on rotating magnetic stars will be described elsewhere.

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