MECHANICAL DEDUCTION OF FORMULAS OF DIFFERENTIAL EQUATIONS (I)

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ABSTRACT

Mechanical deduction of formulas of differential equations performed by a high-speed computer is a new field of mathematical research. It will modernize the means of research of differential equations.

This article describes the algorithm of the Differential Equation Program System (DEPS in abbreviated form). This system has been successfully applied to the research work of limit cycles, the criteria of focus and center, and the deduction of Ляпунов functions. As an example, this article gives the computer result of differential equations with polynomials of degree 2 and points out a critical error of sign in the famous result of Н. Н. Баутин.[6]

This article is the first part in a series of DEPS.

I. INTRODUCTION

In the study of ordinary differential equations, a great deal of labor is involved in formula deduction in many crucial steps of making assertions. For example, for Hilbert's 16th problem[1], in order to study the number of limit cycles, one has to investigate also the limit cycles sprung from singular points[2-3]. This problem is connected with the criteria of focus and center[4]. For the problem of stability of a motion, it is necessary to find out the corresponding Ляпунов functions. All these topics of primary importance lead to very complicate formula deduction.

In principle, in dealing with all these problems, we have well established algorithm to follow, nevertheless, so far only a few substantial results have been obtained, owing to the complexity met in actual computation and the large amount of work involved. In current literature, for instance, one can hardly find any article, which treats the general expression for the criteria of focus and center as well as the related Ляпунов functions, in case the equations dealt with have polynomials of degree not more than 2 on their right-hand side. For the special case of degree 2 such as the famous work of Н. Н. Баутин, there is an erroneous sign in his deduction, which has not been checked out ever since. This erroneous sign directly affects the proper discrimination between the cases of stability and instability, and is also connected with the probable number of limit cycles. It is only recently that in the course of investigating the number of limit cycles[3], this discrepancy is revealed by means of certain numerical calculation by Prof. L. K. Hua. Therefore, as a consequence of the complexity presented by deduction and the liability to error in manual computation, the more efficient way of research in this field, which promises to work theoretically, has long failed to be realized.

The emergence of modern high-speed large capacity electronic computers, however, paves the way to turn deduction from possibility into realization[7]. This paper together with a series of subsequent papers, will show some interesting results in this new direction, thus making many useful formulas available to research workers concerned.

II. The Algorithm of Mechanical Deduction

Given a system of literal equations

$$\begin{cases}\dfrac{dx}{dt} = y + X_2(x, y) = y + L_2x^2 + L_3xy + L_4y^2, \\ \dfrac{dy}{dt} = -x + Y_2(x, y) = -x + L_5x^2 + L_6xy + L_7y^2,\end{cases} \tag{2.1}$$

one is asked to find out its Ляпунов function

$$F \equiv F(x, y) = \sum_{j=2}^{N} F_j(x, y), \tag{2.2}$$

where

$$F_2(x, y) = \frac{1}{2}(x^2 + y^2),$$

$$F_j(x, y) = \sum_{k=0}^{j} f_{j,k}x^{j-k}y^k.$$

Differentiating $F(x, y)$ along the integral curve of (2.1) with respect to t, we have

$$\begin{aligned}\frac{dF(x, y)}{dt} &= \frac{\partial F(x, y)}{\partial x}\frac{dx}{dt} + \frac{\partial F(x, y)}{\partial y}\frac{dy}{dt} \\ &= \left(\sum_{j=2}^{N}\frac{\partial F_j(x, y)}{\partial x}\right)(y + X_2(x, y)) + \left(\sum_{j=2}^{N}\frac{\partial F_j(x, y)}{\partial y}\right)(-x + Y_2(x, y)) \\ &= (xy - yx) + \left(\frac{\partial F_3(x, y)}{\partial x}y - \frac{\partial F_3(x, y)}{\partial y}x + \frac{\partial F_2(x, y)}{\partial x}X_2(x, y)\right. \\ &\quad \left. + \frac{\partial F_2(x, y)}{\partial y}Y_2(x, y)\right) + \cdots \\ &\quad + \left(\frac{\partial F_j(x, y)}{\partial x}y - \frac{\partial F_j(x, y)}{\partial y}x + \frac{\partial F_{j-1}(x, y)}{\partial x}X_2(x, y)\right. \\ &\quad \left. + \frac{\partial F_{j-1}(x, y)}{\partial y}Y_2(x, y)\right) + \cdots \\ &\quad + \left(\frac{\partial F_N(x, y)}{\partial x}y - \frac{\partial F_N(x, y)}{\partial y}x + \frac{\partial F_{N-1}(x, y)}{\partial x}X_2(x, y)\right. \\ &\quad \left. + \frac{\partial F_{N-1}(x, y)}{\partial y}Y_2(x, y)\right).\end{aligned}$$

Now set each bracket in the above expression equal to zero and find out $F_j(x, y)$ in succession.

Two cases occur: j odd and j even. These two cases have substantial difference. When j is even, $F_j(x, y)$ will have solution only when the coefficients $L_2, \cdots, L_7$ of (2.1) satisfy a certain condition. The aim of our formula deduction is to find out such kind of condition and the corresponding $F_j(x, y)$. When j is odd, $F_j(x, y)$ always exists, no condition is needed. Given $F_2(x, y) = \frac{1}{2}(x^2 + y^2)$, we may find solutions $F_3, F_4, \cdots$ successively. In general, given $F_{j-1}(x, y)$ $(j \geqslant 3)$, we want to find $F_j(x, y)$, i.e. to find

$$\frac{\partial F_j(x,y)}{\partial x} y - \frac{\partial F_j(x,y)}{\partial y} x = -\frac{\partial F_{j-1}}{\partial x} X_2 - \frac{\partial F_{j-1}}{\partial y} Y_2$$

or

$$\sum_{k=0}^{j-1} (j-k) f_{j,k} x^{j-k-1} y^{k+1} - \sum_{k=1}^{j} k f_{j,k} x^{j-k+1} y^{k-1} = \sum_{k=0}^{j} h_{j,k} x^{j-k} y^k,$$

where $h_{j,k}$ on the right-hand side is known, and $f_{j,k}$ on the left waits to be found. When j is odd, there are $j+1$ expressions $f_{j,k}$ $(k = 0, 1, 2, \cdots, j)$. They naturally fall into two groups, i. e.

$$f_{j,1},\ f_{j,3},\ f_{j,5},\ \cdots,\ f_{j,j},$$

and

$$f_{j,0},\ f_{j,2},\ f_{j,4},\ \cdots,\ f_{j,j-1}.$$

The members of each group are related to one another by the formulas

$$\begin{aligned} -f_{j,1} \qquad\qquad\qquad\qquad &= h_{j,0}, \\ (j-1)f_{j,1} \qquad -3f_{j,3} \qquad &= h_{j,2}, \\ \cdots\cdots \qquad & \\ 2f_{j,j-2} - jf_{j,j} &= h_{j,j-1}, \end{aligned}$$

and

$$\begin{aligned} f_{j,j-1} \qquad\qquad\qquad\qquad &= h_{j,j}, \\ -(j-1)f_{j,j-1} + 3f_{j,j-3} \qquad &= h_{j,j-2}, \\ \cdots\cdots \qquad & \\ -2f_{j,2} + jf_{j,0} &= h_{j,1}. \end{aligned}$$

Both of the above two groups of equations are triangular matrices and we can find out in succession the following expressions:

$$f_{j,1},\ f_{j,3},\ \cdots,\ f_{j,j},$$

and

$$f_{j,j-1}, f_{j,j-3}, \cdots, f_{j,0}.$$

When j is even, the $j+1$ expressions $f_{j,k}$ $(k = 0, 1, \cdots, j)$ again fall naturally into two groups

$$f_{j,0},\ f_{j,2}, \cdots, f_{j,j}$$

and

$$f_{j,1},\ f_{j,3},\ \cdots,\ f_{j,j-1}.$$

In the first group, there are $\frac{j}{2}+1$ quantities satisfying $\frac{j}{2}$ equations, therefore solutions certainly exist. For example, take $f_{j,j}=0$ and we can find $f_{j,j-2}, f_{j,j-4}, \cdots, f_{j,2}, f_{j,0}$ in succession. However, the second group presents some difficulty: it has $\frac{j}{2}$ quantities which have to satisfy $\frac{j}{2}+1$ equations. Therefore, if solution exists, i. e. if equations are made compatible, we must impose a certain condition on $h_{j,k}$, i. e. on $L_m(m=2,3,\cdots,7)$. This condition is the very criterion we seek for. In particular, say, if from the first $\frac{j}{2}$ equations, we find in succession $f_{j,1}, f_{j,3}, \cdots, f_{j,j-1}$, then the last equation

$$f_{j,j-1}=h_{j,j}$$

will be the condition sought for. Let $V_{2n-1}=f_{2n,2n-1}-h_{2n,2n}$.

Having done this, for even integer $2n\geqslant 4$, we have

$$F(x,y)=\sum_{j=2}^{2n}F_j(x,y),\ F_2(x,y)=\frac{1}{2}(x^2+y^2),$$

$$\frac{dF}{dt}=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}=V_3y^4+V_5y^6+\cdots+V_{2n-1}y^{2n}+\cdots.$$

Here all V_j are homogeneous polynomials in $L_m(m=2,3,\cdots,7)$, and it is not difficult to find out in principle, but is difficult to deduce by hand. Therefore it has to be deduced by mechanical deduction. It can be shown by mathematical deduction that $f_{2j,k}$ is a polynomial of degree $2(j-1)$ in L_m. Therefore V_{2k-1} is also a polynomial of degree $2k-2$ in L_m.

Here $F(x,y)=$ constant. Near the origin it is a family of closed curves. They can be used as Ляпунов functions to decide the stability property of a motion.

The specific criteria are:

$V_3>0$, unstable;

$V_3<0$, asymptotically stable;

$V_3=0, V_5>0$, unstable;

$V_5<0$, asymptotically stable;

$V_3=V_5=0, V_7>0$, unstable;

$V_7<0$, asymptotically stable;

$V_3=V_5=V_7=0$, after H. H. Баутин[6], this is stable of the center type, but is not asymptotically stable.

III. The Result for a Particular Case

Without loss of generality, we may apply a rotation transformation

$$\xi = x\cos\theta - y\sin\theta,$$
$$\eta = x\sin\theta + y\cos\theta,$$

to reduce Eqs. (2.1) to those of (ξ, η). It suffices to take θ to satisfy

$$(L_2 + L_4)\sin\theta + (L_5 + L_7)\cos\theta = 0,$$

in order to reduce the equation in (2.1) to a special form, in which the two coefficients in (2.1) $L_5 + L_7 = 0$. Thus, without loss of generality, we may eliminate one parameter, reducing the number of parameters to five, i.e. L_2, L_3, L_4, L_5 and L_6. Now we write these special equations as follows:

$$\begin{cases} \dfrac{dx}{dt} = y - L_3x^2 - (2L_2 + L_5)xy + L_6y^2 = X, \\ \dfrac{dy}{dt} = -x - L_2x^2 + (2L_3 + L_4)xy + L_2y^2 = Y. \end{cases} \tag{3.1}$$

The advantage of putting them in this way lies in the fact that

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = -2L_3x - 2(L_2 + L_5)y + (2L_3 + L_4)x + 2L_2y = L_4x - L_5y;$$

if $L_4 = L_5 = 0$, then $\dfrac{\partial X}{\partial x} + \dfrac{\partial Y}{\partial y} = 0$, i. e. we have centers, thus simplifying verification.

By means of machine computation (see appendix for details) we obtain the following result:

$$V_3 = -\frac{1}{3}L_3L_5 + \frac{1}{3}L_5L_6 = -\frac{1}{3}L_5(L_3 - L_6).$$

V_5 has 17 terms. When they are arranged in the ascending power of L_5, it reads:

$$\begin{aligned} V_5 = &\left\{\frac{1}{3}L_2L_4L_6^2 - \frac{2}{3}L_2L_4L_3L_6 + \frac{1}{3}L_2L_4L_3^2 + \frac{1}{15}L_2L_4^2L_3 - \frac{1}{15}L_2L_4^2L_6\right\} \\ &+ L_5\left\{\frac{14}{9}L_3L_6^2 - \frac{7}{9}L_3^2L_6 - \frac{11}{45}L_3^2L_4 + \frac{8}{15}L_3L_4^2\right. \\ &\left. + \frac{64}{45}L_3L_4L_6 - \frac{8}{15}L_4^2L_6 - \frac{53}{45}L_4L_6^2 - \frac{7}{9}L_6^3\right\} \\ &+ L_5^2\left\{-\frac{1}{3}L_2L_3 + \frac{1}{3}L_2L_6\right\} + L_5^3\left\{-\frac{1}{15}L_3 + \frac{1}{15}L_6\right\} \\ = &\frac{1}{15}L_2L_4(L_3 - L_6)(L_4 + 5L_3 - 5L_6) \\ &+ \frac{L_5(L_3 - L_6)}{3}\left\{-\frac{1}{5}L_5^2 - L_5L_2 - \frac{7}{3}(L_3 - L_6)L_6\right. \\ &\left. + \frac{8}{5}L_4^2 + \frac{1}{15}L_4(-11L_3 + 53L_6)\right\}, \end{aligned}$$

therefore

$$V_5 = \frac{1}{15}L_2L_4(L_3 - L_6)(L_4 + 5L_3 - 5L_6) + V_3(*),$$

where $(*)$ is a homogeneous quadratic form in L_m. Set

$$V_3 = \tilde{v}_3 = -\frac{1}{3} L_5(L_3 - L_6),$$

$$V_5 = \tilde{v}_5 + \tilde{v}_3(*),$$

where

$$\tilde{v}_5 = \frac{1}{15} L_2L_4(L_3 - L_6)(L_4 + 5L_3 - 5L_6).$$

Next comes V_7 having 74 terms of which 55 terms contain L_5 and 19 terms do not.

When we make use of the condition $V_3 = 0$ or $L_3L_5 = L_5L_6$, the 55 terms containing L_5 all vanish (N. B. $L_5L_3^m = (L_5L_6)L_3^{m-1} = L_6(L_5L_3^{m-1}) = \cdots = L_5L_6^m$).

Therefore it is only necessary to study those terms which do not contain L_5. Now, by making use of $\tilde{v}_5 = 0$, we have

$$L_2L_4^2(L_3 - L_6) = -5L_2L_4(L_3 - L_6)^2.$$

Using this expression, we may reduce L_4 to the lowest degree, i. e. $L_2L_4^m(L_3 - L_6) = (-5)^{m-1}L_2L_4(L_3 - L_6)^m$, $m \geqslant 2$.

We arrange the 19 terms not containing L_5 in V_7 according to the descending power of L_4 and make use of $\tilde{v}_5 = 0$ to lower down the power of L_4. There are 2 terms containing L_4^4:

$$\frac{1}{315} L_2L_4^4(47L_6 - 47L_3) = -\frac{47}{315} L_2L_4^4(L_3 - L_6) = \frac{-47}{315}(-5)^3L_2L_4(L_3 - L_6)^4$$

$$= \frac{1}{315} L_2L_4(L_3 - L_6)^2[5875L_3^2 - 11750L_3L_6 + 5875L_6^2].$$

3 terms containing L_4^3:

$$\frac{1}{315} L_2L_4^3(-96L_6^2 - 166L_3^2 + 262L_3L_6)$$

$$= \frac{1}{315} L_2L_4^3(L_3 - L_6)(-166L_3 + 96L_6)$$

$$= \frac{1}{315}(-5)^2L_2L_4(L_3 - L_6)^3(-166L_3 + 96L_6)$$

$$= \frac{1}{315} L_2L_4(L_3 - L_6)^2(-4150L_3^2 + 6550L_3L_6 - 2400L_6^2).$$

6 terms containing L_4^2:

$$\frac{1}{315} L_2L_4^2(18L_2^2L_3 - 18L_2^2L_6 + 408L_3^3 - 1388L_3^2L_6 + 1562L_3L_6^2 - 582L_6^3)$$

$$= \frac{18}{315} L_2^3L_4^2(L_3 - L_6) + \frac{1}{315} L_2L_4^2(L_3 - L_6)(408L_3^2 - 980L_3L_6 + 582L_6^2)$$

$$= \frac{18 \times (-5)}{315} L_2^3L_4(L_3 - L_6)^2 + \frac{(-5)}{315} L_2L_4(L_3 - L_6)^2(408L_3^2 -$$

$$- 980L_3L_6 + 582L_6^2) = \frac{1}{315} L_2L_4(L_3 - L_6)(-90L_2^2$$

$$- 2040L_3^2 + 4900L_3L_6 - 2910L_6^2).$$

8 terms containing L_4^1:

$$\frac{1}{315} L_2L_4\{L_2^2(180L_3^2 - 360L_3L_6 + 180L_6^2)$$

$$+ 315L_3^4 - 420L_3^3L_6 - 490L_3^2L_6^2 + 980L_3L_6^3 - 385L_6^4\}$$

$$= \frac{1}{315} L_2L_4(L_3 - L_6)^2\{180L_2^2 + 315L_3^2 + 210L_3L_6 - 385L_6^2\}.$$

Combining the 19 terms above, we have, under the condition $\tilde{v}_3 = \tilde{v}_5 = 0$,

$$\tilde{v}_7 = \frac{1}{315} L_2L_4(L_3 - L_6)^2(90L_2^2 - 90L_3L_6 + 180L_6^2)$$

$$= -\frac{2}{7} L_2L_4(L_3 - L_6)^2(L_3L_6 - L_2^2 - 2L_6^2).$$

But

$$V_7 = \tilde{v}_7 + (*)\tilde{v}_5 + (**)\tilde{v}_3.$$

Here (*) is the homogeneous quadratic form in L_m; (**) the homogeneous form of degree four in L_m.

Finally we obtain

$$\tilde{v}_3 = -\frac{1}{3} L_5(L_3 - L_6),$$

$$\tilde{v}_5 = \frac{1}{15} L_2L_4(L_3 - L_6)(L_4 + 5L_3 - 5L_6),$$

$$\tilde{v}_7 = -\frac{2}{7} L_2L_4(L_3 - L_6)^2(L_3L_6 - L_2^2 - 2L_6^2).$$

$$\tilde{v}_3 = V_3,$$

$$\tilde{v}_5 = V_5|_{\tilde{v}_3=0},$$

$$\tilde{v}_7 = V_7|_{\tilde{v}_3=\tilde{v}_5=0}.$$

Now we are in a position to compare our results with that of Н. Н. Баутин[6]. Н. Н. Баутин equations in the reduced form read as follows:

$$\begin{cases} \dfrac{dx}{dt} = \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dfrac{dy}{dt} = x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{cases} \tag{3.2}$$

For $\lambda_1 = 0$, we have to decide whether there is a center or focus. Inserting $-y$ for y gives (3.1), where the stability property remains unaltered.

We apply the polar coordinates transformation $x = \rho\cos\varphi, y = \rho\sin\varphi$, to obtaining

$$\frac{d\rho}{d\varphi} = \frac{\rho^2 H_3(\varphi)}{1+\rho K_3(\varphi)},$$

$$H_3(\varphi) = -\lambda_3 \cos^3\varphi + (3\lambda_2 + \lambda_5)\cos^2\varphi \sin\varphi + (2\lambda_3 + \lambda_4 + \lambda_6)\cos\varphi \sin^2\varphi - \lambda_2 \sin^3\varphi,$$

$$K_3(\varphi) = \lambda_2 \cos^3\varphi + (3\lambda_3 + \lambda_4)\cos^2\varphi \sin\varphi + (-3\lambda_2 - \lambda_5)\cos\varphi \sin^2\varphi - \lambda_6 \sin^3\varphi.$$

We want to find, for initial conditions $\varphi = 0, \rho = \rho_0$, the solution expressed in series

$$\rho = \rho(\varphi; \lambda_j; \rho_0) = \sum_{n=1}^{\infty} \rho_0^n v_n(\varphi, \lambda_j),$$

where

$$\rho_0 = \rho(0;\ \lambda_j;\ \rho_0).$$

When $\lambda_1 = 0$, Н. Н. Баутин gave:

$$v_1(2\pi;\lambda_j) = 1,$$
$$v_2(2\pi;\lambda_j) = 0,$$
$$v_3(2\pi;\lambda_j) = \bar{v}_3,$$
$$v_4(2\pi;\lambda_j) = \bar{v}_3\theta_4^{(3)},$$
$$v_5(2\pi;\lambda_j) = \bar{v}_5 + \bar{v}_3\theta_5^{(3)},\ v_6(2\pi;\ \lambda_j) = \bar{v}_5\theta_6^{(5)} + \bar{v}_3\theta_6^{(3)},$$
$$v_7(2\pi;\lambda_j) = \bar{v}_7 + \bar{v}_5\theta_7^{(5)} + \bar{v}_3\theta_7^{(3)},$$

$$\bar{v}_3 = -\frac{\pi}{4}\lambda_5(\lambda_3 - \lambda_6),$$

$$\bar{v}_5 = \frac{\pi}{24}\lambda_2\lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5\lambda_3 - 5\lambda_6),$$

$$\bar{v}_7 = \frac{25\pi}{32}\lambda_2\lambda_4(\lambda_3 - \lambda_6)^2(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2).$$

Stability is just the behaviour for increasing t or for increasing θ. Therefore the criteria are:

$\bar{v}_3 > 0$, unstable;
$\bar{v}_3 < 0$, asymptotically stable;
$\bar{v}_3 = 0, \bar{v}_5 > 0$, unstable;
$\bar{v}_5 < 0$, asymptotically stable;
$\bar{v}_3 = 0, \bar{v}_5 = 0, \bar{v}_7 > 0$, unstable;
$\bar{v}_7 < 0$, asymptotically stable;
$\bar{v}_3 = \bar{v}_5 = \bar{v}_7 = 0$, center, stable but not asymptotically stable.

Comparing the above two results, we have Table 1.

By this table, $\bar{v}_j$ and $\tilde{v}_j$ should be of the same sign. However, in our table $\bar{v}_3$ and $\tilde{v}_3$ are of the same sign, so are $\bar{v}_5$ and $\tilde{v}_5$, but $\bar{v}_7$ and $\tilde{v}_7$ have opposite sign. Therefore one of the two cases must be wrong. Actually $\bar{v}_7$ has a wrong sign that we should add a negative sign to it. This fact was first found in the course of

Table 1

	Н. Н. Баутин Result	Our Result
3	$\bar{v}_3 = \frac{-\pi}{4}\lambda_5(\lambda_3 - \lambda_6)$	$\tilde{v}_3 = -\frac{1}{3}L_5(L_3 - L_6)$
5	$\bar{v}_5 = \frac{\pi}{24}\lambda_2\lambda_4(\lambda_3 - \lambda_6)(\lambda_4 + 5\lambda_3 - 5\lambda_6)$	$\tilde{v}_5 = \frac{1}{15}L_2L_4(L_3 - L_6)(L_4 + 5L_3 - 5L_6)$
7	$\bar{v}_7 = \frac{25\pi}{32}\lambda_2\lambda_4(\lambda_3 - \lambda_6)^2(\lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2)$	$\tilde{v}_7 = -\frac{2}{7}L_2L_4(L_3 - L_6)^2(L_3L_6 - 2L_6^2 - L_2^2)$

making a certain numerical computation for constructing limit cycles[2-3]. Now, we reveal definitely this fact using literal deduction in our mechanical deduction of formulas. This discrepancy in error is concerned with the condition of stability at infinity and may affect the number of existing limit cycles by one[3]. Consequently we have to make the necessary correction.

Other results concerning the mechanical deduction and the related Ляпунов functions will appear in a series of forthcoming articles.

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Appendix

List of expressions of V_3, V_5 and V_7 by mechanical deduction

```
                                        V3=
                                        +(
1/- 3  .L1**0  .L2**0  .L3**1  .L4**0  .L5**1  .L6**0  .L7**0
1/  3  .L1**0  .L2**0  .L3**0  .L4**0  .L5**1  .L6**1  .L7**0
                       ).x**0  .y**4
                                        V5=
                                        +(
-1/- 3  .L1**0  .L2**1  .L3**2  .L4**1  .L5**0  .L6**0  .L7**0
-1/-15  .L1**0  .L2**1  .L3**1  .L4**2  .L5**0  .L6**0  .L7**0
 1/-15  .L1**0  .L2**1  .L3**0  .L4**2  .L5**0  .L6**1  .L7**0
 1/  3  .L1**0  .L2**1  .L3**0  .L4**1  .L5**0  .L6**2  .L7**0
 2/- 3  .L1**0  .L2**1  .L3**1  .L4**1  .L5**0  .L6**1  .L7**0
-1/  3  .L1**0  .L2**1  .L3**1  .L4**0  .L5**2  .L6**0  .L7**0
 1/  3  .L1**0  .L2**1  .L3**0  .L4**0  .L5**2  .L6**1  .L7**0
-7/  9  .L1**0  .L2**0  .L3**2  .L4**0  .L5**1  .L6**1  .L7**0
-11/ 45  .L1**0  .L2**0  .L3**2  .L4**1  .L5**1  .L6**0  .L7**0
 8/ 15  .L1**0  .L2**0  .L3**1  .L4**2  .L5**1  .L6**0  .L7**0
-64/-45  .L1**0  .L2**0  .L3**1  .L4**1  .L5**1  .L6**1  .L7**0
```

−14/− 9	.L1**0	.L2**0	.L3**1	.L4**0	.L5**1	.L6**2	.L7**0
−1/ 15	.L1**0	.L2**0	.L3**1	.L4**0	.L5**3	.L6**0	.L7**0
8/−15	.L1**0	.L2**0	.L3**0	.L4**2	.L5**1	.L6**1	.L7**0
53/−45	.L1**0	.L2**0	.L3**0	.L4**1	.L5**1	.L6**2	.L7**0
−7/ 9	.L1**0	.L2**0	.L3**0	.L4**0	.L5**1	.L6**3	.L7**0
1/ 15	.L1**0	.L2**0	.L3**0	.L4**0	.L5**3	.L6**1	.L7**0

).x**0 .y**6

V7=

+(

−4/ 3	.L1**0	.L2**1	.L3**3	.L4**1	.L5**0	.L6**1	.L7**0
28/ −3	.L1**0	.L2**2	.L3**2	.L4**0	.L5**1	.L6**1	.L7**0
1388/−315	.L1**0	.L2**1	.L3**2	.L4**2	.L5**0	.L6**1	.L7**0
1562/−315	.L1**0	.L2**1	.L3**1	.L4**2	.L5**0	.L6**2	.L7**0
−8/ 7	.L1**0	.L2**3	.L3**1	.L4**1	.L5**0	.L6**1	.L7**0
20/ 21	.L1**0	.L2**2	.L3**1	.L4**1	.L5**1	.L6**1	.L7**0
262/ 315	.L1**0	.L2**1	.L3**1	.L4**3	.L5**0	.L6**1	.L7**0
−28/ −9	.L1**0	.L2**1	.L3**1	.L4**1	.L5**0	.L6**3	.L7**0
1/ 1	.L1**0	.L2**1	.L3**4	.L4**1	.L5**0	.L6**0	.L7**0
−8/ −3	.L1**0	.L2**2	.L3**3	.L4**0	.L5**1	.L6**0	.L7**0
136/ 105	.L1**0	.L2**1	.L3**3	.L4**2	.L5**0	.L6**0	.L7**0
−4/ −7	.L1**0	.L2**3	.L3**2	.L4**1	.L5**0	.L6**0	.L7**0
−8/ 35	.L1**0	.L2**2	.L3**2	.L4**1	.L5**1	.L6**0	.L7**0
−166/ 315	.L1**0	.L2**1	.L3**2	.L4**3	.L5**0	.L6**0	.L7**0
−14/ 9	.L1**0	.L2**1	.L3**2	.L4**1	.L5**0	.L6**2	.L7**0
2/ 35	.L1**0	.L2**3	.L3**1	.L4**2	.L5**0	.L6**0	.L7**0
2287/−315	.L1**0	.L2**1	.L3**2	.L4**0	.L5**2	.L6**1	.L7**0
1313/ 945	.L1**0	.L2**1	.L3**3	.L4**0	.L5**2	.L6**0	.L7**0
−179/ 105	.L1**0	.L2**1	.L3**2	.L4**1	.L5**2	.L6**0	.L7**0
47/−315	.L1**0	.L2**1	.L3**1	.L4**4	.L5**0	.L6**0	.L7**0
192/ 35	.L1**0	.L2**1	.L3**1	.L4**1	.L5**2	.L6**1	.L7**0
1097/ 105	.L1**0	.L2**1	.L3**1	.L4**0	.L5**2	.L6**2	.L7**0
−773/ 630	.L1**0	.L2**1	.L3**1	.L4**2	.L5**2	.L6**0	.L7**0
32/ 3	.L1**0	.L2**2	.L3**1	.L4**0	.L5**1	.L6**2	.L7**0
493/ 630	.L1**0	.L2**2	.L3**1	.L4**2	.L5**1	.L6**0	.L7**0
58/−315	.L1**0	.L2**3	.L3**1	.L4**0	.L5**2	.L6**0	.L7**0
−37/ 90	.L1**0	.L2**2	.L3**1	.L4**0	.L5**3	.L6**0	.L7**0
−11/ 90	.L1**0	.L2**1	.L3**1	.L4**0	.L5**4	.L6**0	.L7**0
−194/ 105	.L1**0	.L2**1	.L3**0	.L4**2	.L5**0	.L6**3	.L7**0
−4/ −7	.L1**0	.L2**3	.L3**0	.L4**1	.L5**0	.L6**2	.L7**0
−76/ 105	.L1**0	.L2**2	.L3**0	.L4**1	.L5**1	.L6**2	.L7**0
−32/ 105	.L1**0	.L2**1	.L3**0	.L4**3	.L5**0	.L6**2	.L7**0
−11/ 9	.L1**0	.L2**1	.L3**0	.L4**1	.L5**0	.L6**4	.L7**0
2/ −35	.L1**0	.L2**3	.L3**0	.L4**2	.L5**0	.L6**1	.L7**0
47/ 315	.L1**0	.L2**1	.L3**0	.L4**4	.L5**0	.L6**1	.L7**0
−397/ 105	.L1**0	.L2**1	.L3**0	.L4**1	.L5**2	.L6**2	.L7**0
865/−189	.L1**0	.L2**1	.L3**0	.L4**0	.L5**2	.L6**3	.L7**0
−773/ 630	.L1**0	.L2**1	.L3**0	.L4**2	.L5**2	.L6**1	.L7**0
−4/ 1	.L1**0	.L2**2	.L3**0	.L4**0	.L5**1	.L6**3	.L7**0
493/−630	.L1**0	.L2**2	.L3**0	.L4**2	.L5**1	.L6**1	.L7**0

−58/−315	.L1** 0	.L2** 3	.L3** 0	.L4** 0	.L5** 2	.L6** 1	.L7** 0
37/ 90	.L1** 0	.L2** 2	.L3** 0	.L4** 0	.L5** 3	.L6** 1	.L7** 0
−11/ −90	.L1** 0	.L2** 1	.L3** 0	.L4** 0	.L5** 4	.L6** 1	.L7** 0
−2/ 3	.L1** 0	.L2** 0	.L3** 3	.L4** 0	.L5** 1	.L6** 2	.L7** 0
−79/ 105	.L1** 0	.L2** 0	.L3** 3	.L4** 1	.L5** 1	.L6** 1	.L7** 0
1247/ 315	.L1** 0	.L2** 0	.L3** 2	.L4** 2	.L5** 1	.L6** 1	.L7** 0
2507/ 315	.L1** 0	.L2** 0	.L3** 2	.L4** 1	.L5** 1	.L6** 2	.L7** 0
152/ 27	.L1** 0	.L2** 0	.L3** 2	.L4** 0	.L5** 1	.L6** 3	.L7** 0
−67/ 42	.L1** 0	.L2** 0	.L3** 2	.L4** 0	.L5** 3	.L6** 1	.L7** 0
−4/ 3	.L1** 0	.L2** 0	.L3** 4	.L4** 0	.L5** 1	.L6** 1	.L7** 0
1/ 45	.L1** 0	.L2** 0	.L3** 4	.L4** 1	.L5** 1	.L6** 0	.L7** 0
41/ 35	.L1** 0	.L2** 0	.L3** 3	.L4** 2	.L5** 1	.L6** 0	.L7** 0
−433/−1890	.L1** 0	.L2** 0	.L3** 3	.L4** 0	.L5** 3	.L6** 0	.L7** 0
3629/−1890	.L1** 0	.L2** 0	.L3** 2	.L4** 3	.L5** 1	.L6** 0	.L7** 0
218/−315	.L1** 0	.L2** 0	.L3** 2	.L4** 1	.L5** 3	.L6** 0	.L7** 0
1/ 3	.L1** 0	.L2** 0	.L3** 5	.L4** 0	.L5** 1	.L6** 0	.L7** 0
9469/−630	.L1** 0	.L2** 0	.L3** 1	.L4** 2	.L5** 1	.L6** 2	.L7** 0
−211/ 15	.L1** 0	.L2** 0	.L3** 1	.L4** 1	.L5** 1	.L6** 3	.L7** 0
14857/−1890	.L1** 0	.L2** 0	.L3** 1	.L4** 3	.L5** 1	.L6** 1	.L7** 0
−751/ 630	.L1** 0	.L2** 0	.L3** 1	.L4** 4	.L5** 1	.L6** 0	.L7** 0
1027/ 630	.L1** 0	.L2** 0	.L3** 1	.L4** 1	.L5** 3	.L6** 1	.L7** 0
−101/−630	.L1** 0	.L2** 0	.L3** 1	.L4** 2	.L5** 3	.L6** 0	.L7** 0
−151/ 27	.L1** 0	.L2** 0	.L3** 1	.L4** 0	.L5** 1	.L6** 4	.L7** 0
257/ 105	.L1** 0	.L2** 0	.L3** 1	.L4** 0	.L5** 3	.L6** 2	.L7** 0
1/−105	.L1** 0	.L2** 0	.L3** 1	.L4** 0	.L5** 5	.L6** 0	.L7** 0
99/ 10	.L1** 0	.L2** 0	.L3** 0	.L4** 2	.L5** 1	.L6** 3	.L7** 0
−718/−105	.L1** 0	.L2** 0	.L3** 0	.L4** 1	.L5** 1	.L6** 4	.L7** 0
−802/−135	.L1** 0	.L2** 0	.L3** 0	.L4** 3	.L5** 1	.L6** 2	.L7** 0
−751/−630	.L1** 0	.L2** 0	.L3** 0	.L4** 4	.L5** 1	.L6** 1	.L7** 0
−197/ 210	.L1** 0	.L2** 0	.L3** 0	.L4** 1	.L5** 3	.L6** 2	.L7** 0
−101/ 630	.L1** 0	.L2** 0	.L3** 0	.L4** 2	.L5** 3	.L6** 1	.L7** 0
44/ 27	.L1** 0	.L2** 0	.L3** 0	.L4** 0	.L5** 1	.L6** 5	.L7** 0
−146/ 135	.L1** 0	.L2** 0	.L3** 0	.L4** 0	.L5** 3	.L6** 3	.L7** 0
−1/−105	.L1** 0	.L2** 0	.L3** 0	.L4** 0	.L5** 5	.L6** 1	.L7** 0
			).x** 0 .	y** 8			

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