GENERAL CONDITIONS FOR STABILITY OF OPTICAL RESONATORS

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ABSTRACT

The problem of stability of optical resonators in which x-direction and y-direction are coupled with each other is one of the new theoretical problems in laser development. In this paper, the general problem is considered and sufficient and necessary conditions for stability of optical resonators are obtained. Stability diagram is drawn. Thus the stability problem of optical resonators is solved in principle. All the previous cases in which x-direction and y-direction can be separately considered, are special cases of the present work.

I. INTRODUCTION

When an optical resonator possesses axial symmetry, the round trip matrix of rays in the resonator is a direct sum of two similar 2×2 matrices. Then the problem can be reduced to that of finding the effect of a 2×2 matrix on rays. Boyd and Kogelnik[1] and others have in different ways obtained the following stability condition for this kind of optical resonator

$$|T_r D_2| < 2,$$

where $T_r D_2$ denotes the trace D_2, D_2 being a round trip 2×2 matrix. This condition has been improved to the following form[2]:

$$|T_r D_2| < 2 \text{ or } D_2 = \pm 1. \tag{1.1}$$

When the cavity does not have axial symmetry but the round trip matrix of rays is the direct sum of round trip matrix of x-direction and that of y-direction, the stability conditions of the resonator is the simple combination of stability conditions of x-direction and those of y-direction. For unstable cavities, the theory of geometrical optics has been applied rather successfully[3]. Recently, one of the present authors, Zhu Ruzeng[4], has made a strict analysis of unstable cavities on the basis of geometrical optics. In this work, a new concept of the transfer matrices of beams was first introduced, quasi-modes and known stable modes were obtained, and Siegman's hypotheses[5] was proved in the framework of geometrical optics. Besides, he[6] has discussed the two-dimensional stability problem for modes of unstable cavity. However, in many cases, x-direction and y-direction couple with each other. In such circumstance the round trip matrix of rays in cavity is not the direct sum of two 2×2 matrices. For example, when cavity mirrors are not axially symmetric and there is astigmatism as a result of self-focusing[7,8], the round trip matrix of rays in cavity is not the direct sum of two 2×2 matrices. Astigmatic cavity[9] with 90° roof prism is just such an example. Recently, works on optical cavities represented by 4×4 matrix has been started. Fang Honglie[10] has given a canonical description of the cavity for which the matrix can be decomposed into direct sum of two 2×2 matrices. He has discussed[9] an astigmatic cavity with 90° roof prism in some special orientations. Wang Zhijiang and Fang Honglie[9] have further summarized the types of beams in those cavities for which matrix (or with coordinate axes rotated) can be decomposed into direct sum of two 2×2 matrices. In brief, the cases in which there is coupling between x-direction and y-direction are attracting more and more attention. Yet the stability problem of such cavities have not been properly solved. The aim of this paper is to solve completely the stability problem of cavities representable by 4×4 matrix, which can or cannot be decomposed into direct sum of two 2×2 matrices. We shall especially provide detailed and strict conclusions for every point on the boundary of stable region. It must be emphasized that the points on the boundary of stable region have important and practical significance. In section IV it will be shown that cavities of axial symmetry discussed before are just located on the boundary of the stable region given in this paper.

II. Sufficient and Necessary Conditions for Stability in Terms of Eigenvalues

In a cavity the relation between the parameters of a ray after n round trips and the initial parameters can be written as

$$\begin{bmatrix} x_n \\ \theta_n \\ y_n \\ \phi_n \end{bmatrix} = D^n \begin{bmatrix} x_0 \\ \theta_0 \\ y_0 \\ \phi_0 \end{bmatrix}, \tag{2.1}$$

where D is a 4×4 matrix, called round trip matrix of the ray in a cavity. The elements of D are all real numbers. To render discussion strict and complete, it is necessary to express the physical content of stability of a resonator based on strict mathematical definitions.

Definition. *A cavity is said to be stable, if to every positive number ε, no matter how small, a positive number δ can be found such that if only*

$$|x_0| < \delta,\ |\theta_0| < \delta,\ |y_0| < \delta \text{ and } |\phi_0| < \delta,$$

we have

$$|x_n| < \varepsilon,\ |\theta_n| < \varepsilon,\ |y_n| < \varepsilon \text{ and } |\phi_n| < \varepsilon,$$

for all values of positive integer n. Otherwise, the cavity is said to be unstable.

This definition ensures that there is such a 4-dimensional region A around the origin in the 4-dimensional space of initial conditions (x_0, θ_0, y_0, ϕ_0) that a ray with initial parameters bounded in the region will not walk out of the cavity. Thus a lot of rays can remain around the optical axis of the cavity, corresponding to the physical stability of optical cavity. Here we represent "a lot of" in physics by "4-dimension" of A in mathematics. Conversely, it is easy to prove that there will not be "a lot of" rays remaining around the optical axis when the conditions for definition of stability are not satisfied. From this we can prove

Theorem 1. *The necessary and sufficient condition for stability of a cavity is that the elements of D^n are bounded.*

The proof will be given in Appendix I. We shall refer to a system with elements of D^n bounded as a system of matrix bounded.

Theorem 2. *Let characteristic roots of the round trip matrix D of the ray in a cavity be λ_1, λ_2, λ_3, λ_4. If they satisfy any one of the following conditions, then the cavity is said to be stable, otherwise it is unstable:*

(i) $|\lambda_i| < 1, \quad i = 1, 2, 3, 4.$

(ii) *The modules of some characteristic roots are* 1, *but they are simple roots, and those of the rest are less than* 1.

(iii) *Modules of some characteristic roots* $\lambda_R (R = 1, 2, \ldots, t)$ *are* 1 *and the multiplicity of them are* μ_R, *but for them the rank of the characteristic matrix is* $(4 - \mu_R)$; *furthermore, the rest of the roots have the property* (ii).

Proof. According to an algebraic theorem, to any matrix D, there must exist a non-singular matrix V in complex domain which can transform D into Jordanian canonical form:

$$H = V^{-1}DV = H_{k_1} \dot{+} H_{k_2} \dot{+} \cdots + H_{k_m}, \quad (m \leqslant 4) \tag{2.2}$$

where every sub-block is a Jordan block, k_i being the number of order. Thus

$$H^n = H^n_{k_1} \dot{+} H^n_{k_2} \dot{+} \cdots + H^n_{k_m}. \tag{2.3}$$

First, let us prove the sufficiency of those conditions.

(i) The modules of all characteristic roots are less than one. According to (2.3) and Lemma 1 of Appendix II, H^n is bounded. In addition $D^n = VH^nV^{-1}$, therefore D^n is bounded. By Theorem 1, the cavity is stable.

(ii) From (i) and (ii) of Lemma 1 in Appendix II, we see that H^n is bounded, hence D^n is bounded, therefore the cavity is stable.

(iii) For characteristic root λ_R with multiplicity $\mu_R > 1$, because the rank of the characteristic matrix is $(4 - \mu_R)$, the corresponding eigenvectors constitute a μ_R-dimensional subspace. So the corresponding Jordan blocks are of order 1. In addition, $|\lambda_R| = 1$, thus the n power of Jordan block is bounded. The property of the rest of roots belongs to (ii), so the nth power of the corresponding Jordan block is bounded too. Hence, H^n is bounded. Therefore, D^n is bounded, and so the cavity is stable.

We shall now prove the necessity of these conditions. Suppose that the cavity is stable. According to Theorem 1, D^n is bounded, so H^n is bounded, i.e. each term on the right side of (2.3) is bounded. According to Lemma 1 in Appendix II, we know that there are only two possibilities:

(a) $|\lambda| < 1$,

(b) $|\lambda| = 1$, but $k_i = 1$.

If (b) does occur, the Jordan block(s) corresponding to a characteristic root of module one is (are) 1×1 Jordan block(s). Therefore either this root is a simple root or the rank of the corresponding characteristic matrix is $(N - \mu_R)$, where μ_R is the multiplicity of λ_R. Hence, D satisfies the condition (ii) or (iii) of this theorem.

If (b) does not occur, we know all Jordan blocks belong to case (a). Then (i) in this theorem is satisfied. Hence, the proof is complete.

Theorem 2 enables us to determine whether a cavity is stable or not, based on the eigenvalues of its round trip matrix of rays. However, when the eigenvalue is multiple, the rank of the corresponding characteristic matrix must be computed. This is troublesome. The following theorem will provide us with an equivalent but convenient method to treat this case.

Theorem 3. *The condition* (iii) *in theorem* 2 *is equivalent to*

$$\frac{f(\lambda)}{\prod\limits_{l}^{t} (D - \lambda_l I)^{\mu_l - 1}} = 0, \tag{2.4}$$

where $f(\lambda)$ is the characteristic polynomial and t denotes the number of various multiple characteristic roots of module one.

Proof: According to condition (iii), the Jordan blocks corresponding to the multiple characteristic root λ_l of module one are of order one. The sufficient and necessary condition for this is that the minimum polynomial of D divides

$$f(\lambda)\Big/\prod_{l=1}^{t} (\lambda - \lambda_l I)^{\mu_l - 1}.$$

This, in turn, is equivalent to Eq. (2.4). Thus the proof is complete.

The important advantage of Theorem 2 and Theorem 3 above is that they are sufficient and necessary to constitute a complete set of criteria. In some other stability theories, when some eigenvalues are degenerate, i.e. the so-called "critical case", it is often impossible to make a thorough determination, and so only sufficient condition is given. As far as optical cavity is concerned, however, "the critical case" is particularly important. It will be seen later that all the axially symmetric optical cavities considered before belong to "critical case," for their characteristic roots are at least double. To use the above criteria, the characteristic roots must be found out beforehand, but this is not convenient. To save this trouble, we shall give in the next section sufficient and necessary conditions for stability in terms of elements of the matrix.

III. Sufficient and Necessary Condition for Cavity Stability in Terms of Elements of the Matrix

The characteristic equation of the round trip matrix of rays in a cavity is

$$f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0. \tag{3.1}$$

In Appendix III, by virtue of the reversibility of light paths, it is shown that Eq. (3.1) is an inverse number equation, i.e., if some number is a root, then its inverse is also a root.

Hence

$$\boldsymbol{a} = c, \quad d = 1. \tag{3.2}$$

Theorem 4. *The sufficient and necessary condition for stability of an optical cavity is that any one of the following six conditions be satisfied:*

$$\text{(i)} \quad \frac{\boldsymbol{a}^2}{4} + 2 > b > 2|\boldsymbol{a}| - 2; \tag{3.3}$$

$$\text{(ii)} \begin{cases} -4 < \boldsymbol{a} < 0, \\ b = -2(\boldsymbol{a} + 1), \\ D^3 + (\boldsymbol{a} + 1)D^2 - (\boldsymbol{a} + 1)D - 1 = 0; \end{cases} \tag{3.4}$$

$$\text{(iii)} \begin{cases} 0 < \boldsymbol{a} < 4, \\ b = 2(\boldsymbol{a} - 1) \\ D^3 + (\boldsymbol{a} - 1)D^2 + (\boldsymbol{a} - 1)D + 1 = 1; \end{cases} \tag{3.5}$$

$$\text{(iv)} \begin{cases} \boldsymbol{a} = 0, \\ b = -2, \\ D^2, = I; \end{cases} \tag{3.6}$$

$$\text{(v)} \begin{cases} |\boldsymbol{a}| < 4, \\ b = \dfrac{\boldsymbol{a}^2}{4} + 2, \\ D^2 + \dfrac{a}{2} D + I = 0; \end{cases} \tag{3.7}$$

$$\text{(vi)} \quad D = \pm 1; \ (\textit{in this case, we have } \boldsymbol{a} = \mp 4, b = 6), \tag{3.8}$$

where I is the 4×4 unity matrix,

$$\boldsymbol{a} = -T_r D, \tag{3.9}$$

$$b = \textit{the sume of all the principal minors of order two}, \tag{3.10}$$

$$-c = \textit{the sum of all the principal minors of order three.}$$

Proof. Now we are going to find out the domain of values to be taken by a and b such that the cavity will be stable, i.e., the roots of the characteristic equation of D satisfy theorem 2 and 3. This domain is the stable region, and the domain outside is the unstable region.

As the constant term of characteristic equation

$$f(\lambda) = \lambda^4 + \boldsymbol{a}\lambda^3 + b\lambda^2 + \boldsymbol{a}\lambda + 1 = 0 \tag{3.11}$$

is unity, so either all modules of characteristic roots are unity or else some of them are greater than unity. According to Theorm 2, the second case is unstable, so we need only discuss the case in which all modules of characteristic roots are unity.

Because the coefficients of Eq. (3.11) is real, complex roots must appear in pairs. In addition, the constant term being $+1$, real roots $+1$ must appear in pairs, and -1 too. Therefore the four roots can be written as

$$\lambda_1 = e^{i\theta_1}, \ \lambda_2 = e^{-i\theta_1}, \ \lambda_3 = e^{i\theta_2}, \ \lambda_4 = e^{-i\theta_2}, \tag{3.12}$$

where θ_1 and θ_2 are real numbers.

Substituting (3.12) into (3.11), we obtain

$$a = -2(\cos\theta_1 + \cos\theta_2), \tag{3.13}$$

$$b = 2 + 4\cos\theta_1\cos\theta_2. \tag{3.14}$$

So

$$\begin{aligned} \cos\theta_1 &= \frac{1}{4}\left[-a + \sqrt{a^2 - 4b + 8}\,\right], \\ \cos\theta_2 &= \frac{1}{4}\left[-a - \sqrt{a^2 - 4b + 8}\,\right]. \end{aligned} \tag{3.15}$$

Analysing Eq. (3.15), we can see that (i)—(vi) in the present theorem correspond to the following cases respectively:

(i) All the four characteristic roots are of module 1 and unequal;

(ii) two distinct roots are of module 1, the other two are double roots $\lambda = 1$, but

$$f(D)/(D - I) = 0;$$

(iii) two distinct roots are of module 1 and a double root $\lambda = -1$, but

$$f(D)/(D + I) = 0;$$

(iv) two distinct double roots $\lambda = \pm 1$, but

$$f(D)/(D^2 - I) = 0;$$

(v) two distinct double roots $\lambda = e^{\pm i\theta}(\theta \neq 0, \pi)$, but

$$f(D)/(D - e^{i\theta}I)\,(D - e^{-i\theta}I) = 0;$$

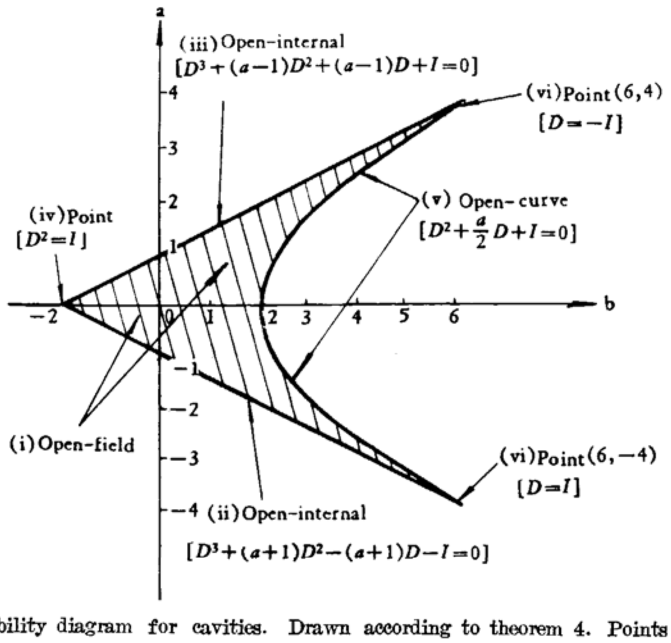

Fig 1. Stability diagram for cavities. Drawn according to theorem 4. Points inside the shaded area represent stable cavities. A point on the boundary represents stable cavity only if it satisfies the corresponding condition indicated in the square bracket. The area outside is unstable.

(vi) a four-fold roots $\lambda = \pm 1$, but

$$f(D)/(D\mp I)^3 = 0.$$

According to Theorems 2 and 3, all the six cases mentioned above are stable. When a and b satisfy none of the six cases, the solution of θ_1 and θ_2 will not be real, so the cavity must be unstable.

The stability diagram for cavities is shown in Fig. 1. Points inside the shaded area in the figure represent stable cavities, while points on the boundary represent stable cavities, only if they satisfy corresponding conditions indicated in square brackets. Outside of the shaded area is the unstable region.

So far, we have shown the sufficient and necessary conditions for stability of cavities by coefficients a and b of the characteristic equation. Furthermore, a and b are directly connected with elements of matrix D by Eqs. (3.9) and (3.10), so we have actually expressed the sufficient and necessary conditions for stability of cavities by the elements of D. Hence, as soon as the round trip matrix of rays in a cavity is known, a and b can be calculated, and we can find out immediately from the stability diagram whether the cavity is stable or not. Thus, the stability problem of the cavities representable by 4×4 matrix has been solved in principle. What remains is only to apply the results of the present paper to practical cavities.

IV. Cavities of Axial Symmetry and Cavities of Separable Matrix

When an optical cavity is of axial symmetry, the round trip matrix of rays in the cavity can be represented by

$$D = D_2 \dot{+} D_2,$$

and[11]

$$|D_2| = 1,$$

where D_2 is a 2×2 matrix. The stability condition of such system is well known[2]. However, to include it in our formulation, we shall deduce it from the present theorems. Besides, we shall observe where in the general stability diagram given in this paper its stability regions are located.

Corollary 1. *For cavities with round trip matrices of rays representable by*

$$D = D_2 \dot{+} D_2,$$

and $|D_2| = 1$, *the sufficient and necessary condition for stability is*

$$|T_rD| < 2, \text{ or } D_2 = \pm I.$$

Proof. Because $D = D_2 \dot{+} D_2$, the multiplicity of characteristic roots must be even. Hence only when one of the three cases (iv), (v) and (vi) in Theorem 4 occurs, can the cavity be stabe. In the case (iv), characteristic roots must be $+1$ and -1, thus $|D_2| = -1$. But it is contrary to the condition $|D_2| = 1$. Therefore, this case is ruled out. Case (v) means

$$|T_rD_2| < 2,$$

and (vi) means

$$D_2 = \pm I.$$

The proof is complete.

Here, we can see that all points represented by (4.2) belong to the boundaries (v) and (vi) of the stable region given in our stability diagram. Thus it is clear that the boundaries of stable region correspond to very important and practical situations.

For those cavities for which the matrix are not of axial symmetry but for which the x-direction and y-direction can still be considered separately, we have

$$D = D_1 \dotplus D_2,$$

where D_1 and D_2 are both 2×2 matrices and

$$|D_1| = |D_2| = 1.$$

In such cases, obviously the sufficient and necessary condition is that the following be both satisfied:

$$|T_r D_1| < 2 \text{ (or } D_1 = \pm I),$$

and

$$|T_r D_2| < 2 \text{ (or } D_2 = \pm I).$$

Appendix I

Proof of Theorem 1

Theorem 1. *The sufficient and necessary condition for stability of a cavity is that the elements of D^n be bounded.*

Proof. For convenience, we introduce following symbols:

$$x_{n_1} = x_n, \quad x_{n_2} = \theta_n, \quad x_{n_3} = y_n, \quad x_{n_4} = \phi_n.$$

$$x_{01} = x_0, \quad x_{02} = \theta_0, \quad x_{03} = y_0, \quad x_{04} = \phi_0.$$

The elements of D^n will be denoted by $d_{nij}(i, j = 1, 2, 3, 4)$. From Eq. (2.1), we get

$$x_{ni} = \sum_{j=1}^{4} d_{nij} x_{0j} \quad (i = 1, 2, 3, 4). \tag{5.1}$$

We shall first show the condition to be sufficient. Let

$$|d_{nij}| < K, K > 0.$$

For an arbitrary positive number ε, take $\delta = \varepsilon/4K$, then when

$$|x_{0j}| < \delta = \frac{\varepsilon}{4K},$$

from (5.1), we have

$$|x_{ni}| \leqslant K \sum_{j=1}^{4} |x_{0j}| < 4K\delta = \varepsilon.$$

Thus, the definition of stability is satisfied.

We shall next show the condition to be necessary.

Suppose that the cavity is stable. Given a positive number ε, by definition of stability, we can find another positive number δ which has the following property:

If only

$$|x_{0i}| < \delta, \quad (i = 1, 2, 3, 4) \tag{5.2}$$

then

$$|x_{ni}| < \varepsilon \quad (i = 1, 2, 3, 4 \quad n = 1, 2, \cdots). \tag{5.3}$$

Now, choose the following x_{0i} which satisfy Eq. (5.2)

$$x_{01} = \frac{\delta}{2}, \quad x_{02} = x_{03} = x_{04} = 0.$$

Putting these into (5.1), we get

$$|x_{ni}| = \left|\sum_{j=1}^{4} d_{nij}x_{0j}\right| = \frac{\delta}{2}|d_{ni1}|. \tag{5.4}$$

Substituting (5.4) into (5.3) leads to

$$|d_{ni1}| < \frac{2\varepsilon}{\delta}.$$

Similarly, we can prove

$$|d_{ni2}| < \frac{2\varepsilon}{\delta}, \quad |d_{ni3}| < \frac{2\varepsilon}{\delta}, \quad |d_{ni4}| < \frac{2\delta}{\delta}.$$

Namely,

$$|d_{nij}| < \frac{2\varepsilon}{\delta}, \quad (i, j = 1, 2, 3, 4).$$

Therefore, $2\varepsilon/\delta$ is the bound of the elements of matrix D^n. The proof is complete.

Appendix II

Lemma 1. *Let the characteristic root of a Jordan block H_k of order k be λ, then when n takes all of the natural numbers, the elements of matrix H_k^n can be classified into the following four kinds:*

(i) *If $|\lambda| < 1$, then the elements are bounded;*

(ii) *if $|\lambda| > 1$, then the elements are unbounded;*

(iii) *if $|\lambda| = 1$ and $k = 1$, then the elements are bounded;*

(iv) *if $|\lambda| = 1$ and $k > 1$, then the elements are unbounded.*

Proof. Using method of mathematical induction, we can prove

$$H_k^n = \begin{bmatrix} \lambda^n & & & \\ C_n^1\lambda^{n-1} & \lambda^n & & \\ \vdots & \vdots & \ddots & \\ C_n^{k-1}\lambda^{n-k+1} & C_n^{k-2}\lambda^{n-k+2} & & \lambda^n \end{bmatrix}, \tag{5.5}$$

where

$$C_n^m = \begin{cases} \dfrac{n!}{m!(n-m)!}, & \text{when } m \leqslant n, \\ 0, & \text{when } m > n, \end{cases} \tag{5.6}$$

$m = 0, 1, 2, \ldots, k-1$. By making $n \to \infty$ in (5.5) and using (5.6), the four points to be proved can be obtained.

Appendix III

Theorem. *The characteristic equation of the round trip matrix of the rays in a cavity is an inverse number equation.*

Proof. There always exist reflecting surfaces. Taking any one of them, as shown in Fig. 2, with reflection matrix denoted by R_A, we can write the round trip matrix of rays in the form

$$M = R_A G.$$

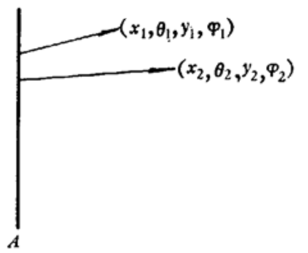

Fig. 2

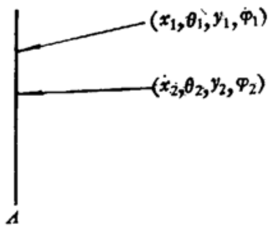

Fig. 3

Suppose

$$R_A G \begin{bmatrix} x_1 \\ \theta_1 \\ y_1 \\ \varphi_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ \theta_2 \\ y_2 \\ \varphi_2 \end{bmatrix}. \tag{5.7}$$

Now, we turn $(x_2, \theta_2, y_2, \varphi_2)$ into the opposite directions as shown in Fig. 3. By the symbolism of rays, its symbol is still $(x_2, \theta_2, y_2, \varphi_2)$. Because light paths are reversible, we get

$$G R_A \begin{bmatrix} x_2 \\ \theta_2 \\ y_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \theta_1 \\ y_1 \\ \varphi_1 \end{bmatrix}. \tag{5.8}$$

From Eqs. (5.7) and (5.8), we get

$$M = R_A G = (G R_A)^{-1}.$$

So

$$G^{-1} = R_A M.$$

Hence,

$$G^{-1}R_A^{-1} = R_A M R_A^{-1},$$

namely,

$$M^{-1} = R_A M R_A^{-1}.$$

Therefore, M has the same characteristic equation as M^{-1}, and so it is an inverse number equation, i.e. its roots consist of a set of inverse number pairs.

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