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THE STATIC EQUILIBRIUM OF AXIAL-SYMMETRIC MAGNETIC FLUX TUBE

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Abstract

The axial-symmetric magnetic flux tube may be described as the slender configuration. The series solutions of the exact nonlinear equations are obtained by expanding the polar angle in the spheric coordinate system for the boundary value problem of the static equilibrium configuration. By using these solutions, the features of the solar magnetic flux tube are discussed. The results show that the magnetic field will be strenghened in the flux tube in case that the transversal temperature has nonuniform distribution; the magnetic flux tube has the tendency to contract at the lower photosphere and expand at both sides; and the magnetic force lines are twisted in the magnetic surface in general. The feature of the magnetic field varies from the forced one at the lower level to the force-free one at the upper level in a flux tube, and the much twisting of the flux tube in the lower level will make larger the magnetic energy of the transversal field on the upper level to supply the energy needed for solar flare. Finally, some typical models of the flux tube are discussed in detail.

I. Introduction

Recent observations show that the elementary structure of solar magnetic field is solitary and discrete flux tubes. The typical radius of the flux tube is only 200—400 km, and the average magnetic field is 500—2000 gauss. The flux tubes emerge from convective region and stretch through the photosphere into the chromosphere and corona, thus giving complicated characteristic configurations. It is important to study the equilibrium configuration of the flux tube and the influence of thermodynamic parameters on the configuration for farther comprehension of the solar magnetic field and the dynamic processes in the active region[1]. The magnetic field in the active region is often described as a force-free field[2]. This assumption holds approximately only in the chromosphere and the lower corona. The pressure gradient and the gravitation are important in the convective region and the photosphere, and the general relations of static magnetohydrodynamics have to be dealt with. In addition, the quiet prominences could also be described as a magnetic flux tube[3]. It is also important to study the influence of plasma parameters on the equilibrium relations of magnetic field. A general discussion about the static equilibrium relations is given for the axial-symmetric magnetic flux tube, and the plasma temperature may be considered as two-dimensional non-uniform.

Lüst and Schüter had obtained the solution for the uniform cross-section of the flux tube[4]. Later Parker extended the one-dimensional result of [4] to a two-dimensional one[5,6], though his approach is not so rigorous in mathematics. Then, Wilson obtained a kind of special asymptotic solutions of the static magnetohydrodynamic equations by the expansion method in cylindric coordinate[7]. Recently, Low discussed generally the properties of two-dimensional forced field using the magnetic potential[8,9], restudied the equilibrium configuration of Schlüter-Temesvary (1954) solar spot model and obtained the Wilson solution as one of his special examples. On the other hand, Comfort et al. found some special solutions of the nonlinear equations using separate variables, and made use of the results obtained to discuss some solar spot configurations in detail[10].

The difficulties of solving the static magnetohydrodynamic problems stem from the nonlinear properties of the differential equations. The problem will be more difficult if some initial and boundary conditions are given. And what is more, the thermodynamic parameters vary rapidly with the height, as a result, the computations are not easy to carry on for special problems. By considering the slender feature of the magnetic flux tube, the parameters may be expanded for small polar angle θ in the spheric coordinate (r, θ, φ), and the solutions of two-dimensional problem may be obtained for the static magnetohydrodynamic equations in general. The terse form of the expanding solution are convenient to apply for various problems.

II. Basic Equations and Their Solutions

The equations of static magnetohydrodynamic equilibrium include the conservative relations of mass, momentum and energy, the condition of solenoidal magnetic field and the state equation. In the case of axial-symmetry, these equations may be written in the spheric coordinates as follow:

$$B_\theta\left(\frac{\partial B_r}{\partial\theta}-\frac{\partial rB}{\partial r}\right)-B_\varphi\frac{\partial rB_\varphi}{\partial r}=4\pi r\left(\frac{\partial p}{\partial r}+\frac{GM_0}{r^2}\rho\right), \tag{2.1}$$

$$B_r\left(\frac{\partial rB_\theta}{\partial r}-\frac{\partial B_r}{\partial\theta}\right)-\frac{B_\varphi}{\sin\theta}\frac{\partial B_\varphi\sin\theta}{\partial\theta}=4\pi\frac{\partial p}{\partial\theta}, \tag{2.2}$$

$$B_r\frac{\partial rB_\varphi}{\partial r}+\frac{B_\theta}{\sin\theta}\frac{\partial B_\varphi\sin\theta}{\partial\theta}=0, \tag{2.3}$$

$$\frac{1}{r}\frac{\partial r^2B_r}{\partial r}+\frac{1}{\sin\theta}\frac{\partial B_\theta\sin\theta}{\partial\theta}=0, \tag{2.4}$$

$$p=\rho\mathscr{R}T, \tag{2.5}$$

where B_r, B_θ, B_φ, are the magnetic field intensities; ρ, p, T are the plasma density, pressure and temperature respectively; R is the gaseous constant; $M_\odot$, the solar mass; and G, the gravitational constant. The energy equation is not included here and the temperature of plasma $T(r, \theta)$ is thought to be given or from observations. By introducing the following non-dimensional parameters, we have

$$\begin{cases} R=\dfrac{r}{r_0}, \quad \rho^*=\dfrac{\rho}{\rho_0}, \quad p^*=\dfrac{p}{p_0}, \quad \boldsymbol{B}^*=\dfrac{\boldsymbol{B}}{B_0}, \\ \sigma=\dfrac{GM_\odot\rho_0/r_0}{B_0^2/4\pi}, \quad \beta=\dfrac{p_0}{B_0^2/4\pi}, \quad \delta=\dfrac{\sigma}{\beta}, \end{cases} \tag{2.6}$$

where the subscript 0 denotes the corresponding typical value of the parameter. Omitting the superscript $*$ which denotes the non-dimensional quantity, we get the non-dimensional equations as follows:

$$B_\theta\left(\frac{\partial B_r}{\partial\theta}-\frac{\partial RB_\theta}{\partial R}\right)-B_\varphi\frac{\partial RB_\varphi}{\partial R}=R\left(\beta\frac{\partial p}{\partial R}+\frac{\sigma}{R^2}\rho\right), \tag{2.7}$$

$$B_r\left(\frac{\partial RB_\theta}{\partial R}-\frac{\partial B_r}{\partial\theta}\right)-\frac{B_\varphi}{\sin\theta}\frac{\partial B_\varphi\sin\theta}{\partial\theta}=\beta\frac{\partial p}{\partial\theta}, \tag{2.8}$$

$$B_r\frac{\partial RB_\varphi}{\partial R}+\frac{B_\theta}{\sin\theta}\frac{\partial B_\varphi\sin\theta}{\partial\theta}=0, \tag{2.9}$$

$$\frac{1}{R}\frac{\partial R^2B_r}{\partial R}+\frac{1}{\sin\theta}\frac{\partial B_\theta\sin\theta}{\partial\theta}=0, \tag{2.10}$$

$$p=\rho T. \tag{2.11}$$

As the magnetic flux tube is generally of a slender configuration, the polar angle θ is small in the tube if the origin of the spheric coordinate is placed at some point in the convective region. Hence, the two-dimensional parameters may be expanded for θ as follows:

$$\begin{aligned} p&=\sum_{m=0}^{\infty}p^{(m)}\theta^m, \quad \rho=\sum_{m=0}^{\infty}\rho^{(m)}\theta^m,\\ T&=\sum_{m=0}^{\infty}T^{(m)}\theta^m, \quad \boldsymbol{B}=\sum_{m=0}^{\infty}\boldsymbol{B}^{(m)}\theta^m. \end{aligned} \tag{2.12}$$

By substituting the expanding relations into the basic equations (2.7)—(2.11), and using the formulae

$$\cos\theta=\sum_{n=0}^{\infty}\frac{(-1)^m}{(2m)!}\theta^{2m}, \quad \sin\theta=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\theta^{2m-1},$$

then we obtain the differential equations for every order of θ^m.

The relations of the zeroth order are

$$\frac{dp^{(0)}}{dR}+\frac{\delta}{R^2}\rho^{(0)}=0, \tag{2.13}$$

$$B_\theta^{(0)}=0, \quad B_\varphi^{(0)}=0, \tag{2.14}$$

$$p^{(0)}=\rho^{(0)}T^{(0)}. \tag{2.15}$$

By using the relations (2.13) and (2.15), the pressure distribution can be written as

$$p^{(0)}(R)=p_0^{(0)}\exp\left(-\int_1^R\frac{\delta}{R^2T^{(0)}}dR\right), \tag{2.16}$$

where the boundary value $p_0^{(0)}=p^{(0)}(1)=1$, which is the same value of pressure $p(1, 0)$. This relation is reasonable. The zeroth relations correspond to the relations at the symmetric axis $\theta=0$ which must be a magnetic force line, and the pressure gradient is balanced with the gravitation along the magnetic force line[8]. $B_r^{(0)}$ is arbitrary in the equations of the zeroth relations. The higher order quantities can be determined if the distribution of $B_r^{(0)}$ is given as any special type.

The equations of the first order are given as:

$$B_\theta^{(1)}B_r^{(1)} = \beta R\,\frac{dp^{(1)}}{dR} + \frac{\sigma}{R}\,\rho^{(1)}, \tag{2.17}$$

$$-B_r^{(0)}B_r^{(1)} = \beta p^{(1)}, \tag{2.18}$$

$$B_\theta^{(1)} = -\frac{1}{2R}\,\frac{dR^2B_r^{(0)}}{dR}, \tag{2.19}$$

$$p^{(1)} = \rho^{(0)}T^{(1)} + \rho^{(1)}T^{(0)}. \tag{2.20}$$

Substituting (2.20) and (2.18) into Eq. (2.17), the equation of pressure can be written as,

$$\frac{dp^{(1)}}{dR} + \left(\frac{\delta}{R^2T^{(0)}} - \frac{1}{2RB_r^{(0)}}\,\frac{dR^2B_r^{(0)}}{dR}\right)p^{(1)} = \delta\,\frac{p^{(0)}}{R^2T^{(0)}}\,\frac{T^{(1)}}{T^{(0)}}.$$

By using the relation (2.16), the solution of the above equation is

$$p^{(1)}(R) = R\sqrt{B_r^{(0)}(R)}\,p^{(0)}(R)\left(p_0^{(1)} + \int_1^R \frac{\delta}{R^3\sqrt{B_r^{(0)}}}\,\frac{T^{(1)}}{T^{(0)2}}\,dR\right), \tag{2.21}$$

where $p_0^{(1)} = p^{(1)}(1)$. Substituting (2.21) into (2.18), we get $B_r^{(1)}$ as

$$B_r^{(1)}(R) = -\frac{p^{(0)}R}{\sqrt{B_r^{(0)}(R)}}\left[p_0^{(1)} + \int_1^R \frac{\delta}{R^3\sqrt{B_r^{(0)}}}\,\frac{T^{(1)}}{T^{(0)2}}\,dR\right]. \tag{2.22}$$

The relation of $B_\theta^{(1)}$ is given in Eq. (2.19). In the one-dimensional cases, there is $B_\theta^{(1)}=0$, then $R^2B_r = \text{constant}$. The non-zero $B_\theta^{(1)}$ just reflects the two-dimensional features of the magnetic field. In integrating (2.21), the condition $B_r^{(0)}(1) = 1$ is taken.

Similarly, the second order equations may be written as:

$$R\left(\beta\,\frac{dp^{(2)}}{dR} + \frac{\sigma}{R^2}\rho^{(2)}\right) = B_\theta^{(1)}\left(2B_r^{(2)} - \frac{dRB_\theta^{(1)}}{dR}\right) + B_\theta^{(2)}B_r^{(1)} - B_\varphi^{(1)}\,\frac{dRB_\varphi^{(1)}}{dR}, \tag{2.23}$$

$$2B_r^{(0)}B_r^{(2)} - B_r^{(0)}\,\frac{dRB_\theta^{(1)}}{dR} + B_r^{(1)2} - 2B_\varphi^{(1)2} = 2\beta p^{(2)}, \tag{2.24}$$

$$B_r^{(0)}\,\frac{dRB_\varphi^{(1)}}{dR} + 2B_\varphi^{(1)}B_\theta^{(1)} = 0, \tag{2.25}$$

$$B_\theta^{(2)} = -\frac{1}{3R}\,\frac{dR^2B_r^{(1)}}{dR}, \tag{2.26}$$

$$p^{(2)} = \rho^{(0)}T^{(2)} + \rho^{(1)}T^{(1)} + \rho^{(2)}T^{(0)}. \tag{2.27}$$

By using Eqs. (2.24)—(2.27), Eq. (2.23) becomes

$$\frac{dp^{(2)}}{dR} + \left(\frac{\delta}{RT^{(0)}} + \frac{2}{R}\,\frac{B_\theta^{(1)}}{B_r^{(0)}}\right)p^{(2)} = S_2(R), \tag{2.28}$$

where the term $S_2(R)$ in the right-hand side is

$$S_2(R) = \frac{\delta}{R^2T^{(0)}}\left\{p^{(1)}\,\frac{T^{(1)}}{T^{(0)}} + p^{(0)}\left[\frac{T^{(2)}}{T^{(0)}} - \left(\frac{T^{(1)}}{T^{(0)}}\right)^2\right]\right\}$$
$$+ \frac{1}{\beta R}\left[\frac{B_\theta^{(1)}}{B_r^{(0)}}\,(2B_\varphi^{(1)2} - B_r^{(1)2}) + B_\theta^{(2)}B_r^{(1)} - B_\varphi^{(1)}\,\frac{dRB_\varphi^{(1)}}{dR}\right]. \tag{2.29}$$

Considering the relation (2.19), we have the solution of $p^{(2)}$ as :

$$p^{(2)}(R) = R^2 B_r^{(0)}(R) p^{(0)}(R) \left\{ p_0^{(2)} + \int_1^R \frac{S_2(R)}{R^2 B_r^{(0)} p^{(0)}} dR \right\}. \tag{2.30}$$

Using the relation (2.19), we obtain, by integrating Eq. (2.25), the following result:

$$B_\varphi^{(1)}(R) = B_{\varphi 0}^{(1)} R B_r^{(0)}(R), \tag{2.31}$$

where $B_{\varphi 0}^{(1)} = B_\varphi^{(1)}(1)$. From Eqs. (2.22) and (2.26), we get

$$B_\theta^{(2)}(R) = \frac{1}{3R} \frac{d}{dR} \left\{ \frac{\beta R^3 p^{(0)}(R)}{\sqrt{B_r^{(0)}(R)}} \left[p_0^{(1)} + \int_1^R \frac{\delta}{R^3 \sqrt{B_r^{(0)}}} \frac{T^{(1)}}{T^{(0)}} dR \right] \right\}. \tag{2.32}$$

The source term $S_2(R)$ is determined completely using Eqs. (2.31) and (2.32), and furthermore, the pressure is determined by Eq. (2.30). Thus $B_r^{(2)}$ can be obtained from (2.30) and (2.32) as:

$$B_r^{(2)}(R) = \frac{1}{2} \frac{dRB_\theta^{(1)}}{dR} + \frac{2B_\varphi^{(1)2} - B_r^{(1)2}}{2B_r^{(0)}} + \frac{\beta}{2} R^2 p^{(0)} \left\{ p_0^{(2)} + \int_1^R \frac{S_2(R) dR}{R^2 B_r^{(0)} p^{(0)}} \right\}. \tag{2.33}$$

The equations of the third order may be given as:

$$B_\theta^{(1)} \left(3B_r^{(3)} - \frac{dRB_\theta^{(2)}}{dR} \right) + B_\theta^{(2)} \left(2B_r^{(2)} - \frac{dRB_\theta^{(1)}}{dR} \right) + B_r^{(1)} B_\theta^{(3)} - \left(B_\varphi^{(1)} \frac{dRB_\varphi^{(2)}}{dR} + B_\varphi^{(2)} \frac{dRB_\varphi^{(1)}}{dR} \right) = R \left(\beta \frac{dp^{(3)}}{dR} + \frac{\sigma}{R} \rho^{(3)} \right), \tag{2.34}$$

$$B_r^{(0)} \left(\frac{dRB_\theta^{(2)}}{dR} - 3B_r^{(3)} \right) + B_r^{(1)} \left(\frac{dRB_\theta^{(1)}}{dR} - 2B_r^{(2)} \right) - B_r^{(2)} B_r^{(1)} - 5B_\varphi^{(1)} B_\varphi^{(2)} + \frac{1}{6} B_r^{(0)} B_r^{(1)} = \beta \left(3p^{(3)} - \frac{1}{6} p^{(1)} \right), \tag{2.35}$$

$$B_r^{(0)} \frac{dRB_\varphi^{(2)}}{dR} + B_r^{(1)} \frac{dRB_\varphi^{(1)}}{dR} + 3B_\theta^{(1)} B_\varphi^{(2)} + 2B_\theta^{(2)} B_\varphi^{(1)} = 0, \tag{2.36}$$

$$\frac{1}{R} \frac{dR^2 B_r^{(2)}}{dR} - \frac{1}{6R} \frac{dR^2 B_r^{(0)}}{dR} + 4B_\theta^{(3)} - \frac{2}{3} B_\theta^{(1)} = 0, \tag{2.37}$$

$$p^{(3)} = \rho^{(0)} T^{(3)} + \rho^{(1)} T^{(2)} + \rho^{(2)} T^{(1)} + \rho^{(3)} T^{(0)}. \tag{2.38}$$

Substituting (2.35) and (2.37) into (2.34), we derive the equation of pressure as:

$$\frac{dp^{(3)}}{dR} + \left(\frac{\delta}{RT^{(0)}} - \frac{3}{2R^2 B_r^{(0)}} \frac{dR^2 B_r^{(0)}}{dR} \right) p^{(3)} = S_3(R), \tag{2.39}$$

and

$$S_3(R) = \frac{B_\theta^{(1)}}{6RB_r^{(0)}} p^{(1)} + \frac{\delta}{RT^{(0)}} \left\{ p^{(2)} \frac{T^{(1)}}{T^{(0)}} + p^{(1)} \left[\frac{T^{(2)}}{T^{(0)}} - \left(\frac{T^{(1)}}{T^{(0)}} \right)^2 \right] + p^{(0)} \left[\frac{T^{(3)}}{T^{(0)}} - 2 \frac{T^{(1)} T^{(2)}}{T^{(0)2}} + \left(\frac{T^{(1)}}{T^{(0)}} \right)^3 \right] \right\} + \frac{B_\theta^{(1)}}{\beta R B_r^{(0)}} \left[B_r^{(1)} \frac{dRB_\theta^{(1)}}{dR} - 3B_r^{(1)} B_r^{(2)} - 5B_\varphi^{(1)} B_\varphi^{(2)} \right] + \frac{1}{\beta R} \left[B_r^{(1)} B_\theta^{(3)} + 2B_r^{(2)} B_\theta^{(2)} + \frac{1}{6} B_r^{(1)} B_\theta^{(1)} \right.$$

$$- B_\theta^2 \frac{dRB_\theta^{(1)}}{dR} - B_\varphi^{(1)} \frac{dRB_\varphi^{(2)}}{dR} - B_\varphi^{(2)} \frac{dRB_\varphi^{(1)}}{dR}\Big]. \tag{2.40}$$

The pressure $p^{(3)}$ can be derived from Eq. (2.39) as:

$$p^{(3)}(R) = R^3(B_r^{(0)})^{\frac{3}{2}} p^{(0)}(R) \left\{p_0^{(3)} + \int_1^R \frac{S_3(R)dR}{R^3(B_r^{(0)})^{3/2} p^{(0)}}\right\}, \tag{2.41}$$

where the boundary condition $p_0^{(3)} = p^{(3)}(1)$. The linear differential equation of $B_\varphi^{(2)}$ can be given using (2.36):

$$\frac{dRB_\varphi^{(2)}}{dR} - \left(\frac{3}{2RB_r^{(0)}} \frac{dR^2B_r^{(0)}}{dR}\right) B_\varphi^{(2)} + \frac{B_r^{(1)}(R)}{B_r^{(0)}(R)} \frac{dRB_\varphi^{(1)}}{dR} + 2\frac{B_\varphi^{(1)}B_\theta^{(2)}}{B_r^{(0)}} = 0. \tag{2.42}$$

The solution of the above equation is

$$B_\varphi^{(2)}(R) = R^2(B_r^{(0)})^{\frac{3}{2}} \left[B_{\varphi 0}^{(2)} - \int_1^R \frac{2B_\varphi^{(1)}B_\theta^{(2)} + B_r^{(1)} \dfrac{dRB_\varphi^{(1)}}{dR}}{R^3(B_r^{(0)})^{5/2}} dR\right], \tag{2.43}$$

where $B_\theta^{(2)}$, $B_\varphi^{(1)}$, $B_r^{(1)}$ are given respectively by Eqs. (2.32), (2.31) and (2.22). Eq. (2.37) gives

$$B_\theta^{(3)}(R) = \frac{B_\theta^{(1)}}{6} + \frac{1}{24R} \frac{dR^2B_r^{(0)}}{dR} - \frac{1}{4R} \frac{dR^2B_r^{(2)}}{dR}. \tag{2.44}$$

The solution of $B_r^{(3)}$ is obtained from Eq. (2.35) as:

$$B_r^{(3)}(R) = \frac{1}{3} \frac{dRB_\theta^{(2)}}{dR} + \frac{B_r^{(1)}}{3B_r^{(0)}}\left(\frac{dRB_\theta^{(1)}}{dR} - 3B_r^{(2)}\right) - \frac{5B_\varphi^{(1)}B_\varphi^{(2)}}{3B_r^{(0)}} + \frac{B_r^{(1)}}{18}$$
$$+ \frac{\beta p^{(1)}}{18B_r^{(0)}} - \beta R^3\sqrt{B_r^{(0)}}\, p^{(0)}\left\{p_0^{(3)} + \int_1^R \frac{S_3(R)dR}{R^3(B_r^{(0)})^{3/2}p^{(0)}}\right\}. \tag{2.45}$$

On the analogy of this, the equations and solutions of higher orders may be demonstrated. There are only contained some boundary value of parameters at $R=1$ and some simple differential and integral operations in these solutions. Hence, the solutions of expanding series can be obtained formally for the initial problem. By applying these solutions, we may begin to discuss the general features of the magnetic flux tube.

III. General Features of Solar Magnetic Flux Tube

1. *The influence of non-uniform temperature distribution*

The total pressures between the inside and outside of the magnetic flux tube must be kept in balance as,

$$p + \frac{B^2}{8\pi} = p_e, \tag{3.1}$$

where p_e is the pressure just outside the tube. Hence, the magnetic field magnitude is determined from the pressure difference between both sides $p_e - p$. Observations of solar photosphere show that the magnitude of the magnetic field is not larger than 1500 gauss if the observed value $p_e - p$ is used. So some mechanisms are required so as to increase the magnetic field[1]. Parker has pointed out that the temperature difference between both sides of the magnetic flux tube can increase the magnetic field in the tube. Without losing generality, let temperature field be described as,

$$T(R,\theta)=T^{(0)}(R)(1+\alpha_1\theta+\alpha_2\theta^2+\cdots), \tag{3.2}$$

and the relation between the functions $\alpha_i(R)$ and $T^{(i)}(R)$ be

$$\alpha_i(R)=\frac{T^{(i)}(R)}{T^{(0)}(R)}. \tag{3.3}$$

Substituting (3.2) into the formulas (2.21), (2.28) and (2.39), we have the expanding expression of pressure as

$$p(R,\theta)=\exp\left(-\delta\int_1^R\frac{dR}{R^2T^{(0)}}\right)\left\{1+R\sqrt{B_r^{(0)}}\left[p_0^{(1)}+\delta\int_1^R\frac{\alpha_1(R)dR}{R^3\sqrt{B_r^{(0)}}}\right]\theta\right.$$

$$\left.+\sum_{i=2}^{\infty}R^i(B_r^{(0)})^{i/2}\left[p_0^{(i)}+\int_1^R\frac{\alpha_iS_idR}{R^i(B_r^{(0)})^{i/2}}\right]\theta^i\right\}. \tag{3.4}$$

The pressure increases in the flux tube at the solar surface with the increase of θ. Generally, there are relations $p_0^{(1)}>0$ and $\alpha_1(R)>0$, then $p^{(1)}(R)>0$. Hence, formula (2.22) gives

$$B_r^{(1)}(R)=-\frac{\beta R}{\sqrt{B_r^{(0)}}}\cdot\exp\left(-\delta\int_1^R\frac{dR}{R^2T^{(0)}}\right)\left[p_0^{(1)}+\int_1^R\frac{\alpha_1\delta dR}{R^3T^{(0)}\sqrt{B_r^{(0)}}}\right]<0. \tag{3.5}$$

The magnetic energy density may be written as:

$$\frac{B^2}{8\pi}=\frac{1}{8\pi}[B_r^{(0)2}+2B_r^{(0)}B_r^{(1)}\theta+(B_r^{(1)2}+B_\theta^{(1)2}+B_\varphi^{(1)2}+2B_r^{(0)}B_r^{(2)})\theta^2+\cdots], \tag{3.6}$$

where the coefficient of term θ is negative. This means that the magnetic energy distribution has marked nonuniformity if the temperature is nonuniform across the flux tube. The magnetic energy of nonuniform part is a small quantity of higher order if the transversal state is uniform.

For the linear problem, the nonuniform component of pressure is,

$$\frac{p^{(1)}(R)}{p^{(0)}(R)}\theta=R\sqrt{B_r^{(0)}}\left[p_0^{(1)}+\delta\int_1^R\frac{\alpha_1(R)dR}{R^3\sqrt{B_r^{(0)}}}\right]\theta, \tag{3.7}$$

and the nonuniformity of magnetic energy may be described as,

$$\frac{2B_r^{(1)}}{B_r^{(0)}}\theta=-\frac{2\beta p^{(0)}}{B_r^{(0)2}}\frac{p^{(1)}\theta}{p^{(0)}}. \tag{3.8}$$

Outside the solar convective region, both $p^{(0)}$ and $B_r^{(0)}$ decrease as the height increases. The magnitude of $p^{(0)}$ decreases nearly 5 magnitude orders from the lower photosphere to the transition region, while $B_r^{(0)}$ decreases only 1—2 magnitude order. Therefore, the nonuniformity of the pressure and the magnetic pressure caused by the nonuniformity of the temperatures are larger in the photosphere or in the region under the photosphere. On the other hand, the nonuniformity of magnetic pressure is weakened more rapidly and much less than the nonuniformity of pressure in the region above the photosphere. So, the increase of magnetic pressure caused by the nonuniform temperature occurs mainly in the lower solar atmosphere.

Let us discuss once more the case where the transversal temperature gradient is absent. Then, there are

$$\alpha_i = 0, \quad i = 1, 2, 3, \cdots$$

If the initial pressure is uniform at $R = 1$, we have:

$$p_0^{(i)} = 0, \quad i = 1, 2, 3, \cdots$$

Using (2.21) and (2.22), we get:

$$p^{(1)}(R) = 0, \quad B_r^{(1)}(R) = 0. \tag{3.9}$$

In Eq. (2.29), the source term caused by the variation of magnetic field is not zero, that is,

$$\begin{aligned} S_2(R) &= \frac{1}{\beta R}\left[\frac{2B_\theta^{(1)}B_\varphi^{(1)2}}{B_r^{(0)}} - B_\varphi^{(1)}\frac{dRB_\varphi^{(1)}}{dR}\right] \\ &= -\frac{(B_{\varphi 0}^{(1)})^2}{\beta} B_r^{(0)}\frac{d}{dR}[R(R+1)B_r^{(0)}]. \end{aligned} \tag{3.10}$$

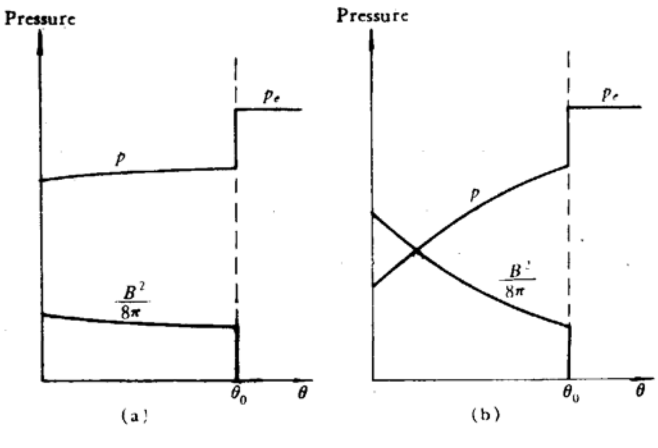
The distribution of $p^{(2)}(R)$ is obtained correspondingly:

$$p^{(2)}(R) = -\frac{(B_{\varphi 0}^{(1)})^2}{\beta} R^2 B_r^{(0)} p^{(0)} \int_1^R \frac{1}{R^2 p^{(0)}}\frac{d}{dR}[R(R+1)B_r^{(0)}]dR. \tag{3.11}$$

The above formula shows that $p^{(2)}(R) = 0$ if the initial magnetic flux tube is not twisted at the base, that is, $B_{\varphi 0}^{(1)} = 0$; otherwise, $p^{(2)}$ is positive, when $B_r^{(0)}$ decays more rapidly than $1/R(R+1)$, and *vice versa*. Correspondingly, the variation of the magnetic pressure is:

$$\begin{aligned} \frac{B^2}{8\pi} = \frac{(B_r^{(0)})^2}{8\pi} + \frac{\theta^2}{8\pi}\Bigg\{\frac{1}{4R^2}\left(\frac{dR^2B_r^{(0)}}{dR}\right)^2 - \frac{B_r^{(0)}}{2}\frac{d^2}{dR^2}(R^2B_r^{(0)}) + (B_{\varphi 0}^{(1)})^2\Bigg[R^2(B_r^{(0)})^2 \\ + 2R^2B_r^{(0)} - R^2B_r^{(0)}p^{(0)}\int_1^R \frac{dR(R+1)B_r^{(0)}}{dR}\frac{dR}{R^2p^{(0)}}\Bigg]\Bigg\} + O(\theta^3). \end{aligned} \tag{3.12}$$

Therefore, the magnetic pressure and the thermodynamic pressure may be nonuniform



$\alpha_1 = 0$ $\alpha_1 \neq 0$

Fig. 1. The influence of temperature distribution on the magnetic pressure.

even if the transversal temperature is uniform, but these nonuniformities appear in the terms θ^2. The magnetic pressure may be increased if the decay of $B_r^{(0)}$ is faster.

According to the above analyses, the influence of temperature distribution on the magnetic pressure is shown in Fig. 1, where the surface $\theta = \theta_0$ is the boundary of the magnetic flux tube.

2. *Configuration features of magnetic surface*

The equation of magnetic force line can be written as,

$$\frac{dR}{B_r} = \frac{Rd\theta}{B_\theta} = \frac{R\sin\theta d\varphi}{B_\varphi}. \tag{3.13}$$

In the axial-symmetric problem, every magnetic force line in the meridional surface rotates around the symmetric axis and thus forms a magnetic surface which may be written as:

$$\frac{d\theta}{dR} = \frac{1}{R}\frac{B_\theta^{(1)}\theta + B_\theta^{(2)}\theta^2 + \cdots}{B_r^{(0)} + B_r^{(1)}\theta + B_r^{(2)}\theta^2 + \cdots}. \tag{3.14}$$

Expanding the equation in series of θ, the asymptotic equation is given as:

$$\frac{d\theta}{dR} = \frac{1}{R}\frac{B_\theta^{(1)}}{B_r^{(0)}}\theta + \frac{1}{R}\left(\frac{B_\theta^{(2)}}{B_r^{(0)}} - \frac{B_\theta^{(1)}B_r^{(1)}}{B_r^{(0)2}}\right)\theta^2 + \frac{1}{R}\left[\frac{B_\theta^{(3)}}{B_r^{(0)}} - \frac{B_r^{(1)}B_\theta^{(2)}}{B_r^{(0)2}} + \frac{B_\theta^{(1)}}{B_r^{(0)}}\left(2\frac{B_r^{(1)2}}{B_r^{(0)2}} - \frac{B_r^{(2)}}{B_r^{(0)}}\right)\right]\theta^3 + \cdots \tag{3.15}$$

This is a complete nonlinear equation which is generally difficult to solve. If we confine ourselves only to the θ^2 terms, omit the higher order terms and let Eq. (3.15) be divided by θ^2, we then obtain a linear equation of $1/\theta$ as,

$$\frac{d}{dR}\left(\frac{1}{\theta}\right) + \left(\frac{1}{R}\frac{B_\theta^{(1)}}{B_r^{(0)}}\right)\frac{1}{\theta} = -\frac{1}{R}\left(\frac{B_\theta^{(2)}}{B_r^{(0)}} - \frac{B_\theta^{(1)}B_r^{(1)}}{B_r^{(0)2}}\right). \tag{3.16}$$

Eq. (3.16) gives

$$\frac{1}{\theta} = R\sqrt{B_r^{(0)}}\left\{\frac{1}{\theta_0} - \int_1^R\left(B_\theta^{(2)} - \frac{B_\theta^{(1)}B_r^{(1)}}{B_r^{(0)}}\right)\frac{dR}{(R^2\sqrt{B_r^{(0)}})^3}\right\},$$

or

$$\frac{\theta}{\theta_0} = \frac{1/R\sqrt{B_r^{(0)}}}{1 + \theta_0\int_1^R\left(B_\theta^{(2)} - \frac{B_\theta^{(1)}B_r^{(1)}}{B_r^{(0)}}\right)\frac{dR}{(R^2\sqrt{B_r^{(0)}})^3}}. \tag{3.17}$$

The above result can be expressed approximately as:

$$\frac{\theta}{\theta_0} \simeq \frac{1}{R\sqrt{B_r^{(0)}(R)}}\left\{1 + \theta_0\int_1^R\left(B_\theta^{(2)} - \frac{B_\theta^{(1)}B_r^{(1)}}{B_r^{(0)}}\right)\frac{dR}{(R^2\sqrt{B_r^{(0)}})^3}\right\} + O(\theta^2). \tag{3.18}$$

The result of (3.18) shows that the basic configuration of the magnetic surface may be written as:

$$\theta = \theta_0/R\sqrt{B_r^{(0)}(R)}.$$

The magnetic force line diverges with the increase of R if the decay of $B_r^{(0)}$ is faster than $1/R^2$, otherwise, the force line will converge. In application of the result to the magnetic flux tube near the solar surface, $B_r^{(0)}$ decays rapidly outward the photosphere, so the force lines diverge. But with some dynamo effects that increase the magnetic field in the convective region, the force line would concentrate outward to the photosphere. Therefore, the whole force lines concentrate at the base of the photosphere, and then diverge at both sides, as shown in Fig. 2. When the thermodynamic parameters are uniform in transversal cross section of the flux tube, there are

$$B_r^{(1)} = 0, \quad B_\theta^{(2)} = 0,$$

and the linear term in (3.18) disappear. Therefore, the nonlinear terms give expression about the influence of nonuniform temperature on the configuration of magnetic surface.

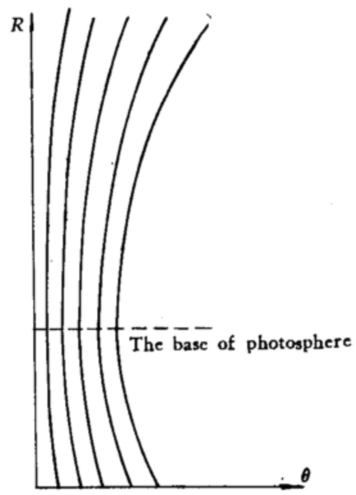


Fig. 2. Schematic diagram of the configuration of the magnetic surface.

3. *Twisting effect of magnetic force lines*

In the cases of force-free field, the force lines tend to twist. The force lines twist round in the magnetic surface as space lines in the case of axial-symmetry[2]. Initially, if $B_{\varphi 0}^{(i)} = 0$, with Eqs. (2.31) and (2.43), the following relations always hold:

$$B_\varphi^{(1)} = 0, \quad B_\varphi^{(2)} = 0.$$

On the contrary, the force lines will twist as they stretch outward, if they are twisted initially at the base. It seems that the twisting features exist not only in the force-free field but also in the forced magnetic field.

As the expansion is confined to the geometric coordinate θ, not the value of β, so these solutions can be applied to study the match region between the forced field and the force-free field. A lot of solar flare theories consider that the transversal component of magnetic field is increasing gradually in the chromosphere and corona as the magnetic field is twisted at the base of photosphere, so that the energy is stored. The solution (2.31) shows that the magnetic field will twist in the photosphere if it is twisted in the convective region, with $B_{\varphi 0}^{(1)} \neq 0$. In this case, the magnitude of the twisted field will increase or decrease according as the variation of $RB_r^{(0)}(R)$. As the decay of $RB_r^{(0)}(R)$ is not faster under the photosphere, the twisting degree does not differ greatly from the initial one, and then the twisting degree is weakened above the photosphere. The twisting degree will be changed in the upper level according to the initial one $B_{\varphi 0}^{(1)}$. The transversal component of magnetic field will be increased in the upper atmosphere if the magnetic field is twisted gradually in the convective region and as a result the magnetic energy is stored. Combining the result with the analyses about the force-free field, the process about storage energy in solar flare may be well explained. Of course, in a detailed discussion we need to study the dynamic process about the flux tube. Estimating roughly from (2.31), we get the ratio of a longitudinal magnetic energy to a transversal one as:

$$\frac{B_\varphi^2}{B_r^2} \simeq \left(\frac{B_\varphi^{(1)}\theta}{B_r^{(0)}}\right)^2 \simeq (B_{\varphi 0}^{(1)})^2 R^2 \theta^2. \tag{3.19}$$

If $B_{\varphi 0}^{(1)}$ is not small and the above ratio may reach 0.1, then the corresponding energy is large enough to supply a solar flare.

It must be pointed out that the expanding asymptotic solution is a local one and we need to pay attention to the effective region in application. The solar parameters in the atmosphere change rapidly; the local value $\beta = p/(B^2/8\pi)$ is larger than 1 in the convective region, near 1 in the photosphere, and much smaller than 1 in the outer region. When β is much smaller than 1, the equilibuium relation between the local quantities will change, and the magnetic force would play an increasing important role. The balance between the pressure gradient and the gravitation holds only in the region nearer to the symmetric axis.

4. *A kind of solution*

Let us discuss the following typical two-dimensional temperature field.

$$T(R, \theta) = R^n(1 + \alpha_1 R^{n_1}\theta + \alpha_2 R^{n_2}\theta^2 + \cdots), \tag{3.20}$$

where the constant coefficient α_i and the power index n_i are given. The temperature increases as R increases, when $n > 0$; the temperature is uniform along the symmetric axis when $n = 0$; and the temperature is constant in the cross section of the flux tube when $\alpha_i = 0$. Furthermore, we assumes that,

$$B_r^{(0)} = R^{-m}. \tag{3.21}$$

This is a decaying field when $m > 0$.

Substituting the above relations into Eqs. (2.16) and (2.21), we have respectively:

$$p^{(0)}(R) = \exp\left[-\frac{\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right], \quad n \neq -1; \tag{3.22}$$

$$p^{(1)}(R)=\begin{cases}R^{1-\frac{m}{2}}\exp\left[-\frac{\delta}{n+1}\left(1-\frac{1}{R^{n+1}}\right)\right]\left[p_0^{(1)}+\frac{\alpha_1\delta}{2+n-n_1-m/2}\right.\\ \qquad\left.\times\left(1-R^{(n_1-n-2+m/2)}\right)\right],\quad (n+2\neq n_1+m/2)\\ R^{1-\frac{m}{2}}\exp\left[-\frac{\delta}{n+1}\left(1-\frac{1}{R^{n+1}}\right)\right][p_0^{(1)}+\alpha_1\delta\ln R],\quad \left(n+2=n_1+\frac{m}{2}\right).\end{cases} \tag{3.23}$$

Using (2.19), (2.22) and (2.31), we obtain the first order relations as:

$$B_r^{(1)}(R)=$$

$$\begin{cases}-\beta R^{1+\frac{m}{2}}\exp\left[-\frac{\delta}{n+1}\left(1-\frac{1}{R^{1+n}}\right)\right]\left[p_0^{(1)}+\frac{\alpha_1\delta}{2+n-n_1-m/2}\right.\\ \qquad\left.\times\left(1-R^{(n_1-n-2+m/2)}\right)\right],\ \left(n+2\neq n_1+\frac{m}{2}\right),\\ -\beta R^{1+\frac{m}{2}}\exp\left[-\frac{\delta}{n+1}\left(1-\frac{1}{R^{n+1}}\right)\right][p_0^{(1)}+\alpha_1\delta\ln R],\ \left(n+2=n_1+\frac{m}{2}\right),\end{cases} \tag{3.24}$$

$$B_\theta^{(1)}(R)=\frac{m-2}{2}R^{-m}, \tag{3.25}$$

$$B_\varphi^{(1)}(R)=B_{\varphi0}^{(1)}R^{1-m}. \tag{3.26}$$

Eq. (2.32) gives:

$$\begin{aligned}B_\theta^{(2)}(R)=\frac{\beta}{3}\exp\left[-\frac{\delta}{n+1}\left(1-\frac{1}{R^{1+n}}\right)\right]&\left\{p_0^{(1)}R^{\frac{m}{2}}\left[\left(3+\frac{m}{2}\right)R-\frac{\delta}{R^n}\right]\right.\\ &+\alpha_1\delta\left[R^{m+n_1-n-1}+\frac{R^{m/2}}{2+n-n_1-m/2}\left[\left(3+\frac{m}{2}\right)R-\frac{\delta}{R^m}\right]\right.\\ &\left.\left.\times[1-R^{n_1-n-2+m/2}]\right]\right\},\quad\left(n+2\neq n_1+\frac{m}{2}\right)\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}B_\theta^{(2)}(R)=\frac{\beta}{3}\exp\left[-\frac{\delta}{n+1}\left(1-\frac{1}{R^{n+1}}\right)\right]&\left\{p_0^{(1)}R^{\frac{m}{2}}\left[\left(3+\frac{m}{2}\right)R-\frac{\delta}{R^n}\right]\right.\\ &\left.+\alpha_1\delta\left[R+\left[\left(3+\frac{m}{2}\right)R-\frac{\delta}{R^n}\right]\ln R\right]R^{m/2}\right\}.\quad\left(n+2=n_1+\frac{m}{2}\right)\end{aligned} \tag{3.28}$$

Other corresponding components may be further obtained.

In the case of solar surface flux tube, the rate of temperature increase is nearly equal to the rate of the decrease of magnetic field, so $n>m/2$. At the same time, the transversal temperature gradient decreases as the height increases, that is $n_1<0$. Therefore, the power index in the above equations satisfies $n+2>n_1+m/2$. So, for R that is several scales larger in height, the asymptotic relations of (3.23) and (3.24) are given as:

$$p^{(1)}(R)\simeq R^{-\left(\frac{m-2}{2}\right)}\exp\left(-\frac{\delta}{n+1}\right)\left[p_0^{(1)}+\frac{\alpha_1\delta}{2+n-n_1-m/2}\right], \tag{3.29}$$

$$B_r^{(1)}(R)\simeq-\beta R^{1+\frac{m}{2}}\exp\left(-\frac{\delta}{n+1}\right)\left[p_0^{(1)}+\frac{\alpha_1\delta}{2+n-n_1-m/2}\right]. \tag{3.30}$$

The above relations show that owing to conditions $p_0^{(1)} > 0$ and $a_1 > 0$ for the transversal nonuniformity of thermodynamic parameters as in the solar flux tube, $B_r^{(1)}$ is a larger negative quantity for several scales in height. The magnetic field is enormously amplituded by the nonuniform temperature in this model.

As a simple case, there is $a_i = 0$ if the nonuniformity of temperature is one-dimensional, and the solutions are obtained as below:

$$p = \exp\left[-\frac{\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right]\left\{1 + p_0^{(1)}R^{1-\frac{m}{2}}\theta + R^{2-m}\left[p_0^{(2)} + \frac{2(m-1)}{\beta}\right.\right.$$
$$\left.\left. \times B_{\varphi 0}^{(1)2}e^{\frac{\delta}{n+1}}\int_1^R \exp\left(-\frac{\delta}{n+1}\frac{1}{R^{n+1}}\right)dR\right]\theta^2 + \cdots\right\}, \tag{3.31}$$

$$B_r = R^{-m} - \beta p_0^{(1)}R^{1+\frac{m}{2}}\exp\left[-\frac{\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right]\theta + \left\{-\frac{(m-2)(m-1)}{4}\right.$$
$$- \frac{\beta^2}{2}R^{2+2m}\exp\left[-\frac{2\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right]p_0^{(1)2} + B_{\varphi 0}^{(1)2}R^{2-m}$$
$$+ \frac{\beta}{2}R^2\exp\left[-\frac{2\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right]\left[p_0^{(1)} + \frac{2(m-2)}{\beta}B_{\varphi 0}^{(1)2}e^{\frac{\delta}{n+1}}\right.$$
$$\left.\left. \times \int_1^R \exp\left(-\frac{\delta}{n+1}\frac{1}{R^{n+1}}\right)dR\right]\right\}\theta^2 + \cdots \tag{3.32}$$

$$B_\theta = \frac{m-2}{2}\frac{\theta}{R^m} + \frac{\beta}{3}p_0^{(1)}\exp\left[-\frac{\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right]R^{\frac{m}{2}}\left[\left(3 + \frac{m}{2}\right)R\right.$$
$$\left. - \frac{\delta}{R^n}\right]\theta^2 + \cdots \tag{3.33}$$

$$B_\varphi = B_{\varphi 0}^{(1)}R^{1-m}\theta + R^{2-3m/2}\left\{B_{\varphi 0}^{(2)} - \frac{2}{3}\beta p_0^{(1)}B_{\varphi 0}^{(1)}\int_1^R \exp\left[-\frac{\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right]\right.$$
$$\left. \cdot\left(2m - \frac{\delta}{R^{n+1}}\right)\frac{dR}{R^{1-2m}}\right\}\theta^2 + \cdots \tag{3.34}$$

In case of a uniform initial pressure at the entry section ($p_i^{(0)} = 0$) and without twisting of the magnetic flux tube ($B_{\varphi 0}^{(i)} = 0$), a very simple basic configuration of the magnetic field is given by:

$$\begin{cases} p = \exp\left[-\frac{\delta}{n+1}\left(1 - \frac{1}{R^{n+1}}\right)\right][1 + O(\theta^3)], \\ B_r = R^{-m} - \frac{(m-2)(m-1)}{4}\theta^2 + O(\theta^3), \\ B_\theta = \frac{m-2}{2}R^{-m}\theta + O(\theta^3), \\ B_\varphi = O(\theta^3). \end{cases} \tag{3.35}$$

This is an untwisting flux tube, which is one of the simplest configurations and corresponds to the case with the lowest density of magnetic energy.

Now, the axial-symmetric configuration of slender magnetic flux tube and the thermodynamic parameters in the tube are given. Applying this expanding solution, we have discussed the features of magnetic field configurations in the solar atmosphere and the influence about the nonuniform temperature distribution on the increase of the magnetic field. This method is efficient only if the angle θ is small. When the angle θ is not small, local expansion may be taken at θ_0, and the feature of the solutions can be discussed in the region near θ_0. The complete picture about the whole magnetic flux tube is finally presented by the depiction of the properties of solutions in several regions.

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