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FROZEN-IN AND RESISTIVE FORCE-FREE FIELDS IN MOVING FLUIDS

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ABSTRACT. Conclusion in [1] have been further verified. New results are: (1) If the electric potential $\nabla\phi$ obeys the gauge relation

$$\frac{\partial \bar{A}}{\partial t}\cdot \bar{B} = 0$$

then the necessary and sufficient condition for α to be constant is

$$\nabla \times (\bar{V}\times\bar{B}) = \beta\bar{B}, \qquad \frac{\partial \hat{A}}{\partial t} = 0.$$

(2) An equation governing the behaviour of α has been derived, which shows that the electrical resistance is the agency that causes a non-constant α to evolve into a constant one.

1. INTRODUCTION

However large the electrical conductivity in a conducting gas may be, there is always some resistance. When we deal with the evolution of an astrophysical plasmoid constrained by a magnetic field, we must take into account the dispersive effect of the finite resistance on the field. In this paper, we shall consider the case where electrical resistance and gas motion co-exist, for in actual cases, the gas need not be at rest.

The equations of the force-free field are

$$\nabla \times \bar{B} = \alpha\bar{B}, \tag{1}$$

$$\frac{4\pi}{c}\bar{J} = \nabla \times \bar{B}. \tag{2}$$

A solution of eqn (1) need not be stable. In [1], the present writer gave an answer to the question of stability in the presence of both resistance and a flow field and pointed out that the magnetic energy of the system is a potential energy with regard to the gas motion and it is only when the potential energy is at its minimum state that the gas will not be perturbed out of the force-free equilibrium state. Using a variational analysis, I found in [1] that, whether the field is resistive or frozen-in, and whether the gas is static or in motion, a constant value of α characterizes the state of minimum magnetic energy. This paper is a continuation of [1], and considers further the necessary and sufficient conditions for α to be a constant.

2. BASIC EQUATIONS

For the force-free field in magnetohydrodynamics, in addition to eqns. (1) and (2), we have also the Maxwell's equations,

$$\nabla \cdot \bar{B} = 0, \tag{3}$$

$$\bar{J}/\sigma = \bar{E} + \bar{V} \times \bar{B}/c, \tag{4}$$

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}. \tag{5}$$

$\bar{E}$ and σ are the electric field and conductivity of the gas and $\bar{V}$ is its velocity of flow.

Taking the divergence of (1) and noting (3), we have

$$\hat{B} \cdot \nabla\alpha = 0. \tag{6}$$

The circumflex or the double-bar will be used indiscriminately to denote *unit* vectors. To satisfy (3), we define

$$\bar{B} = \nabla \times \bar{A}. \tag{7}$$

Inserting this in (5) and integrating, we have

$$\bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t} - \nabla\phi. \tag{8}$$

The potential field $\Delta\phi$ permits some freedom of adjustment, usually we take the gauge that gives vanishing $\nabla.\bar{V}$, then the source of ϕ is the charge density. We shall be making some other adjustment below. The above set of equations are not closed and the $\bar{E}$ can be given.

3. GEOMETRICAL MEANING OF α AND ITS GOVERNING EQUATION

<u>1. The Geometrical Meaning of α</u> Eqn. (1) can be written in the form

$$\alpha\hat{B} = \nabla \ln B \times \hat{B} + \nabla \times \hat{B}. \tag{9}$$

Dot-multiplying this by $\bar{\bar{B}}$ gives

$$\alpha = \hat{B} \cdot \nabla \times \hat{B}. \tag{10}$$

This shows that α is a function of $\bar{\bar{B}}$ only, in fact, the component in the $\bar{\bar{B}}$ direction of $\nabla \times \bar{\bar{B}}$, and is not directly related to $\bar{B}$.

Cross-multiplying (9) by $\bar{\bar{B}}$ gives

$$\nabla \ln B = (\nabla \times \hat{B}) \times \hat{B} + \nabla_{\parallel} \ln B. \tag{11}$$

the suffix $_{\parallel}$ denotes the component parallel to $\bar{\bar{B}}$. From (3), we have

$$\hat{B} \cdot \nabla \ln B = -\nabla \cdot \hat{B}. \tag{12}$$

Hence (11) can be written as

$$\nabla \ln B = \hat{n}/R_n - \nabla \cdot \hat{B}\hat{B}. \tag{13}$$

Here we have used the following relation:

$$(\nabla \times \hat{B}) \times \hat{B} = \hat{B} \cdot \nabla\hat{B} = \hat{n}/R_n. \tag{14}$$

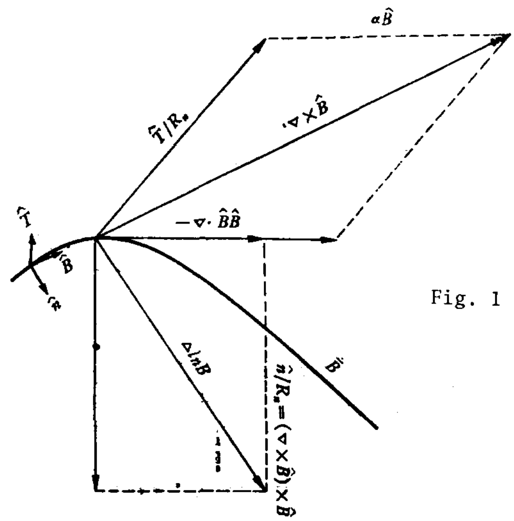

Fig. 1

in which R_n is the radius of curvature of the magnetic line, $\hat{n}$ is the unit normal vector, directed towards the centre of curvature. Define

$$\hat{B} \times \hat{n} = \hat{T}. \tag{15}$$

Substituting (13) in (9), we have

$$\nabla \times \hat{B} = \alpha\hat{B} + \hat{T}/R_n. \tag{16}$$

The various quantities in (13) and (16) are shown in Fig. 1.

From Fig. 1, we see that $\nabla \ln P$ is the ratio of the magnetic pressure gradient to B^2, while $\hat{n}/R_n$ is the ratio of the $\hat{n}$-component of the magnetic line tension (or the "magnetic centripetal force") to B^2. The $\hat{n}$-component of $\nabla \ln B$ is equal to the magnetic centripetal force divided by B^2 while its B-component is equal to magnetic tension gradient along $\hat{B}$ divided by B^2. The $\hat{B}$-component of $\nabla \times \hat{B}$ is $\alpha\hat{B}$ and this refers to the ratio of the work-less current to $(c/4\pi)B$. For a static magnetic fluid with ∇P, the $\nabla \ln B$ should be replaced by $\nabla \ln B + 4\pi\nabla P/B^2$, which expresses the balance between the Lorentz force and ∇P and in that case there is a working current to balance ΔP. Whether the field is force-free or otherwise, the force-free parameter α is the $\hat{B}$-component of $\nabla \times \hat{B}$ and is related to the spatial variation of $\hat{B}$ and has no direct relation with B.

2. Basic Equation Governing the Behaviour of α Let

$$\bar{D} = \nabla \times (\bar{V} \times \bar{B}). \tag{17}$$

Partial differentiating (1) with respect to time, we have

$$\nabla \times \frac{\partial \bar{B}}{\partial t} = \frac{\partial \alpha}{\partial t}\bar{B} + \alpha\frac{\partial \bar{B}}{\partial t}. \tag{18}$$

Taking the curl of (4) and noting (5), we have

$$\frac{\partial \bar{B}}{\partial t} = \bar{D} - \frac{c^2}{4\pi\sigma}\nabla \times (\alpha\bar{B}). \tag{19}$$

Substituting the last in (18) and using (13), we obtain

$$\frac{\partial\alpha}{\partial t}\hat{B} = \frac{\nabla\times\bar{D}-\alpha\bar{D}}{B} + \frac{c^2}{4\pi\sigma}\left(\nabla^2\alpha\hat{B} + \frac{2\hat{n}\cdot\nabla\alpha}{R_n}\hat{B} + 2\nabla\alpha\cdot\nabla\hat{B}\right). \tag{20}$$

Let us resolve this equation along and perpendicular to $\bar{\bar{B}}$. Along $\bar{\bar{B}}$, we have

$$\frac{\partial\alpha}{\partial t} = \bar{B}\cdot(\nabla\times\bar{D}-\alpha\bar{D})/B^2 + \frac{c^2}{4\pi\sigma}\left(\nabla^2\alpha + \frac{2\hat{n}\cdot\nabla\alpha}{R_n}\right), \tag{21}$$

while perpendicular to $\bar{\bar{B}}$, we have

$$\frac{\nabla_\perp\times\bar{D}-\alpha\bar{D}_\perp}{B} + \frac{c^2}{2\pi\sigma}\nabla\alpha\cdot\nabla\hat{B} = 0. \tag{22}$$

Here, $\nabla_\perp\times\bar{D}$ means the component of $\nabla\times\bar{D}$ perpendicular to $\bar{\bar{B}}$. Eqn. (21) shows that the local rate of change of α are controlled by two factors: the first term on the right represents the effect of the motion of the gas, also interpretable as the effect of an effective field of current; in the second term, $(c^2/4\pi\sigma)\ \nabla^2\alpha$ represents the dispersing effect of the resistance on α while $(c^2/4\pi\sigma)\ (\bar{\bar{n}}.\nabla\alpha/R_n)$ represents the contribution to $\partial\alpha/\partial t$ by the electrical resistance, the curvature and the gradient of α along $\bar{\bar{n}}$.

4. NECESSARY AND SUFFICIENT CONDITIONS FOR α TO BE A CONSTANT

We shall discuss the necessary and sufficient condition for α to be a constant separately for the resistive and frozen-in fields.

1. Force-Free Field of Resistive Type

Eqn.(21) shows that a necessary condition for α to be a constant is

$$\nabla\times\bar{D}-\alpha\vec{D} = 0. \tag{23}$$

According to the definition of $\bar{D}$ at (17), it is divergence-free, that is

$$\nabla\cdot\bar{D} = 0. \tag{24}$$

Comparing these last two equations with (1) and (2), we see that $\bar{D}$ is governed by the same basic equations as $\bar{B}$ is, and that $\bar{D}$ is also perpendicular to $\nabla\alpha$,

$$\bar{D}\cdot\nabla\alpha = 0. \tag{25}$$

The next question is: what is the necessary and sufficient condition for α to be constant? Before answering this question, we say a few words on the coordinate unit vectors to be used

Eqns. (21) and (22) can be more simply written as

$$\frac{\partial\alpha}{\partial t} = \frac{c^2}{4\pi\sigma}\left(\nabla^2\alpha + \frac{2\hat{n}\cdot\nabla\alpha}{R_1}\right), \tag{26}$$

$$\nabla\alpha\cdot\nabla\hat{B} = 0. \tag{27}$$

Here we have re-written the R_n in (21) as R_1, the suffix 1 referring to the first radius of curvature. Define

$$\left.\begin{aligned}\hat{B}_t &= \frac{\partial\hat{B}}{\partial t}\Big/\left|\frac{\partial\hat{B}}{\partial t}\right|,\\ \hat{a} &= \nabla\alpha/|\nabla\alpha|.\end{aligned}\right\} \tag{28}$$

Using (6) and (25), and dot-multiplying (19) by $\bar{\bar{\alpha}}$, we have

$$\hat{B}_t \cdot \hat{\alpha} = 0. \tag{29}$$

The property that $\hat{B}$ is a *unit* vector means that

$$\hat{B} \perp \hat{B}_t. \tag{30}$$

In the following discussion, we shall use the orthogonal coordinate system ($\hat{B}$, $\hat{\alpha}$, $\hat{B}_t$). $\hat{n}$ is in the same direction as $\hat{B}_t$.

Dot-multiplying by $\hat{\alpha}$ gives

$$\hat{\alpha} \cdot \nabla \hat{B} \cdot \hat{\alpha} = \hat{n}_\alpha \cdot \hat{B}/R_{1\alpha} = 0. \tag{31}$$

$\hat{n}_\alpha$ and $R_{1\alpha}$ represent respectively the components along $\hat{\alpha}$ of the principal normal unit vector and the first radius of curvature. If $R_{1\alpha} \neq 0$, then

$$\left.\begin{array}{l} \hat{n}_\alpha \perp \hat{B}, \\ \hat{n}_\alpha \perp \hat{\alpha}. \end{array}\right\} \tag{32}$$

We can take $\hat{n}_\alpha$ and $\hat{B}_t$ to be parallel or anti-parallel, without loss of generality.

First, we discuss the *sufficient* condition for α to be a constant and this will be done in three steps (A), (B) and (C).

(A) Calculation of $\nabla\hat{B}$, $\nabla\hat{\alpha}$ and $\hat{B}_t$ Their derivatives along $\hat{\alpha}$ can be found from Frenet's formula, thus,

$$\left.\begin{array}{l} \hat{\alpha} \cdot \nabla \hat{B} = -\hat{n}_\alpha / R_{2\alpha} = 0, \\ \hat{\alpha} \cdot \nabla \hat{\alpha} = \hat{n}_\alpha / R_{1\alpha}, \\ \hat{\alpha} \cdot \nabla \hat{n}_\alpha = \hat{\alpha} \cdot \nabla \hat{B}_t = -\hat{\alpha}/R_{1\alpha}. \end{array}\right\} \tag{33}$$

The first of these equations shows that the second radius of curvature $R_{2\alpha}$ along the $\hat{\alpha}$-axis must be infinite in order to satisfy (25), and this shows that this line is located in the plane perpendicular to $\hat{B}$. The derivatives along $\hat{B}$ of the above three tensors are calculated as follows:

$$\hat{B} \cdot \nabla \hat{B} = \hat{n}/R, \tag{34}$$

$$\begin{aligned} \hat{B} \cdot \nabla \hat{\alpha} &= \hat{B} \cdot \nabla \hat{\alpha} \cdot \hat{B}\hat{B} + \hat{B} \cdot \nabla \hat{\alpha} \cdot \hat{B}_t \hat{B}_t \\ &= -\hat{B} \cdot \nabla \hat{B} \cdot \hat{\alpha} \hat{B} + \{\nabla \hat{\alpha} \cdot \hat{B} - \hat{B} \times (\nabla \times \hat{\alpha})\} \cdot \hat{B}_t \hat{B}_t \\ &= -\frac{\hat{n} \cdot \hat{\alpha}}{R_1} \hat{B} + \alpha \hat{B}_t, \end{aligned} \tag{35}$$

$$\begin{aligned} \hat{B} \cdot \nabla \hat{B}_t &= \hat{B} \cdot \nabla \hat{B}_t \cdot \hat{\alpha}\hat{\alpha} + \hat{B} \cdot \nabla \hat{B}_t \cdot \hat{B}\hat{B} \\ &= -(\hat{n} \cdot \hat{B}_t / R_1) \hat{B} - \alpha \hat{\alpha}. \end{aligned} \tag{36}$$

In the calculation (35), we have used the relation

$$\hat{\alpha} \cdot \nabla \times \hat{\alpha} = 0$$

Similarly, their derivatives along $\hat{B}_t$ are

$$\hat{B}_t \cdot \nabla \hat{B} = -\alpha \hat{\alpha} - (\hat{n}_t \cdot \hat{B}/R_{1t}) \hat{B}_t. \tag{37}$$

$$\hat{B}_t \cdot \nabla \hat{\alpha} = \alpha \hat{B} - (\hat{n}_t \cdot \hat{\alpha}/R_{1t}) \hat{B}_t, \tag{38}$$

$$\hat{B}_t \cdot \nabla \hat{B}_t = \hat{n}_t / R_{1t}. \tag{39}$$

Here $\hat{n}_t$ and R_{1t} are respectively the principal normal unit vector of the $\hat{B}_t$-axis and the first radius of curvature. The expressions for the 3 tensors calculated accordingly are

$$\left.\begin{aligned}
\nabla\hat{B} &= \frac{1}{R_1}\hat{B}\hat{n} - \alpha\hat{B}_t\hat{a} - \frac{\hat{n}_t\cdot\hat{B}}{R_{1t}}\hat{B}_t\hat{B}_t,\\
\nabla\hat{a} &= -\frac{\hat{n}\cdot\hat{a}}{R_1}\hat{B}\hat{B} + \frac{\hat{a}\hat{B}_t}{R_{1a}} + \alpha\hat{B}\hat{B}_t + \alpha\hat{B}_t\hat{B} - \frac{\hat{n}_t\cdot\hat{a}}{R_{1t}}\hat{B}_t\hat{B}_t,\\
\nabla\hat{B}_t = \nabla\hat{n}_a &= -\frac{\hat{n}\cdot\hat{B}_t}{R_1}\hat{B}\hat{B} - \frac{1}{R_{1a}}\hat{a}\hat{a} + \frac{1}{R_{1t}}\hat{B}_t\hat{n}_t - \alpha\hat{B}\hat{a}.
\end{aligned}\right\}\tag{40}$$

(B) Relation between $\frac{\partial\alpha}{\partial t}$, $\left|\frac{\partial\hat{B}}{\partial t}\right|$ and $|\nabla\alpha|$. The time derivative of (10) is

$$\begin{aligned}
\frac{\partial\alpha}{\partial t} &= \frac{\partial\hat{B}}{\partial t}\cdot\nabla\times\hat{B} + \hat{B}\cdot\nabla\times\frac{\partial\hat{B}}{\partial t}\\
&= \left|\frac{\partial\hat{B}}{\partial t}\right|(\hat{B}_t\cdot\nabla\times\hat{B} + \hat{B}\cdot\nabla\times\hat{B}_t) + \hat{B}\cdot\nabla\left|\frac{\partial\hat{B}}{\partial t}\right|\times\hat{B}_t\\
&= \left|\frac{\partial\hat{B}}{\partial t}\right|[2\hat{B}_t\cdot\nabla\times\hat{B} - \nabla\cdot(\hat{B}\times\hat{B}_t)] + \hat{B}\cdot\nabla\left|\frac{\partial\hat{B}}{\partial t}\right|\times\hat{B}_t\\
&= \left|\frac{\partial\hat{B}}{\partial t}\right|(\nabla\cdot\hat{a} + 2\hat{B}_t\cdot\hat{T}/R_1) + \hat{a}\cdot\nabla\left|\frac{\partial\hat{B}}{\partial t}\right|.
\end{aligned}\tag{41}$$

Eqn. (26) can be reduced to a similar form, thus,

$$\frac{\partial\alpha}{\partial t} = \frac{c^2}{4\pi\sigma}\left[\left(\nabla\cdot\hat{a} + \frac{2\hat{n}\cdot\hat{a}}{R_1}\right)|\nabla\alpha| + \hat{a}\cdot\nabla|\nabla\alpha|\right].\tag{42}$$

Because $\hat{n}\perp\hat{T}$, $\hat{B}_t\perp\hat{a}$, and all these 4 unit vectors are perpendicular to $\bar{\bar{B}}$, we have

$$\hat{B}_t\cdot\hat{T} = \hat{n}\cdot\hat{a}.\tag{43}$$

Hence (42) can also be written as

$$\frac{\partial\alpha}{\partial t} = \frac{c^2}{4\pi\sigma}[(\nabla\cdot\hat{a} + 2\hat{B}_t\cdot\hat{T}/R_1)|\nabla\alpha| + \hat{a}\cdot\nabla|\nabla\alpha|].\tag{44}$$

This is reduced to the same form as (41) if we replace $(c^2/4\pi\sigma)$ $|\nabla\alpha|$ by $|\partial B/\partial t|$. The relation between the two is

$$\frac{|\nabla\alpha|c^2}{4\pi\sigma} = \left|\frac{\partial\hat{B}}{\partial t}\right| + G.\tag{45}$$

in which G satisfies the equation

$$G(\nabla\cdot\hat{a} + 2\hat{B}_t\cdot\hat{T}/R_1) + \hat{a}\cdot\nabla G = 0.\tag{46}$$

Using (45), we re-write (19) in the form

$$\begin{aligned}
\bar{D} &= \frac{\partial\bar{B}}{\partial t} + \frac{c^2}{4\pi\sigma}\nabla\times(\alpha\bar{B})\\
&= \frac{\partial B}{\partial t}\hat{B} + B\left(\left|\frac{\partial\hat{B}}{\partial t}\right| - \frac{c^2}{4\pi\sigma}|\nabla\alpha|\right)\hat{B}_t + \frac{c^2\alpha^2}{4\pi\sigma}B\hat{B}\\
&= B\left[\left(\frac{\partial\ln B}{\partial t} + \frac{c^2\alpha^2}{4\pi\sigma}\right)\hat{B} - G\hat{B}_t\right].
\end{aligned}\tag{47}$$

This shows that $-GB$ is the $\bar{\bar{B}}_t$-component of $\bar{D}$; is the gas is at rest, or $\bar{D}$ is parallel to $\bar{\bar{B}}$, then G will be zero.

(C) Relations Between $\frac{\partial\hat{a}}{\partial t}$, $\left|\frac{\partial\hat{B}}{\partial t}\right|$. $\frac{\partial\alpha}{\partial t}$ and $\frac{\partial\hat{B}}{\partial t}$ From experience gained in [1], we take the gauge for $\nabla\phi$ so that the term $\partial\bar{A}/\partial t$ in (8) satisfy the relation

$$\frac{\partial \bar{A}}{\partial t} \cdot \bar{B} = 0. \tag{48}$$

Let

$$\frac{\partial \hat{A}}{dt} = 0. \tag{49}$$

Substituting in (48), we have

$$\hat{A} \cdot \hat{B} = 0. \tag{50}$$

$\partial/\partial t$ (50) and using (49), we obtain

$$\frac{\partial \hat{B}}{\partial t} \cdot \hat{A} = 0. \tag{51}$$

In other words, if $\left|\frac{\partial \hat{B}}{\partial t}\right| \neq 0$ then

$$\hat{B}_t \cdot \hat{A} = 0 \tag{52}$$

From this and (50) we see that $\bar{\bar{A}}$ is perpendicular to both $\bar{\bar{B}}_t$ and $\bar{\bar{B}}$, hence

$$\hat{A} = \hat{\alpha}. \tag{53}$$

Then, on using (49), we have

$$\left.\frac{\partial \hat{\alpha}}{\partial t}\right|_{\frac{\partial \hat{A}}{\partial t}=0} = 0. \tag{54}$$

$\partial/\partial t(\alpha.\nabla B)$ and using (27) and (54), we have, for $\partial\bar{\bar{A}}/\partial t = 0$,

$$\hat{\alpha} \cdot \nabla \frac{\partial \hat{B}}{\partial t} = \hat{\alpha} \cdot \nabla \left|\frac{\partial \hat{B}}{\partial t}\right| \hat{B}_t + \left|\frac{\partial \hat{B}}{\partial t}\right| \hat{\alpha} \cdot \nabla \hat{B}_t = 0. \tag{55}$$

Using the third expression in (40), we reduce the above to

$$\hat{\alpha} \cdot \nabla \frac{\partial \hat{B}}{\partial t} = \hat{\alpha} \cdot \nabla \left|\frac{\partial \hat{B}}{\partial t}\right| \hat{B}_t - \frac{1}{R_{1\alpha}} \left|\frac{\partial \hat{B}}{\partial t}\right| \hat{\alpha} = 0. \tag{56}$$

The $\bar{\bar{\alpha}}$-component of this is

$$\left.\left|\frac{\partial \hat{B}}{\partial t}\right|\right|_{\frac{\partial \hat{A}}{\partial t}=0} = 0. \tag{57}$$

Substituting in (41), we have

$$\left.\frac{\partial \alpha}{\partial t}\right|_{\frac{\partial \hat{A}}{\partial t}=0} = 0. \tag{58}$$

The above analysis shows that if $\partial\bar{\bar{A}}/\partial t = 0$, then so are $|\partial\bar{\bar{B}}/\partial t|$, $\partial\bar{\bar{\alpha}}/\partial t$ and $\partial\alpha/\partial t$. Substituting (57) in (45), we have

$$\frac{c^2}{4\pi\sigma} |\nabla\alpha|_{\frac{\partial \hat{A}}{\partial t}=0} = G, \tag{59}$$

Hence

$$|\nabla\alpha|_{\substack{\frac{\partial \hat{A}}{\partial t}=0 \\ G=0}} = 0. \tag{60}$$

Putting $G = 0$ in (47), we have

$$\bar{D}_{G=0} = \beta\bar{B}. \tag{61}$$

where β is independent of position. The above analysis shows that a sufficient condition for α to be constant is

$$\left.\begin{aligned}\bar{D} &= \beta\bar{B},\\ \frac{\partial\hat{A}}{\partial t} &= 0.\end{aligned}\right\} \tag{62}$$

of which the gauge for $\nabla\phi$ is (48).

<u>(D) To Prove that (62) is Also Necessary</u> From (45) we know that when α is constant, we have

$$G\Big|_{\alpha=\text{const}} = -\left|\frac{\partial\hat{B}}{\partial t}\right|_{\alpha=\text{const}} \tag{63}$$

Expression (47) can be simply written as

$$\bar{D} = \beta\bar{B} - BG\hat{B}_t. \tag{64}$$

Substituting this in (20), we have

$$\nabla\beta\times\bar{B} - \nabla(BG)\times\hat{B}_t - BG\nabla\times\hat{B}_t = -\alpha BG\hat{B}_t. \tag{65}$$

Using the third expression of (40), $\bar{\bar{B}}$.(65) gives

$$\hat{\alpha}\cdot\nabla\ln(BG) = \hat{\alpha}\cdot\hat{n}_t/R_{1t}. \tag{66}$$

Substituting in (46), we have

$$G\left(\nabla\cdot\hat{\alpha} + \frac{2\hat{B}_T\cdot\hat{T}}{R_1} - \hat{\alpha}\cdot\nabla\ln B + \frac{\hat{\alpha}\cdot\hat{n}_t}{R_{1t}}\right) = 0. \tag{67}$$

Hence

$$G = 0. \tag{68}$$

Substituting in (63), we obtain

$$\left|\frac{\partial\hat{B}}{\partial t}\right|_{\alpha=\text{const}} = 0. \tag{69}$$

Hence, let

$$\bar{B} = \bar{B}_0 f(t). \tag{70}$$

where $\bar{B}_0$ is the value of $\bar{B}$ at $t = 0$, hence $f(0) = 1$. Substituting the last in (19) and using $\alpha = \text{const.}$, we have

$$\bar{D} = \beta\bar{B} = \beta f\bar{B}_0. \tag{71}$$

This integrates into

$$\bar{\nabla}\times\bar{B} = \frac{\beta}{\alpha}\bar{B} + \nabla\Phi. \tag{72}$$

Here, the gauge relation for Φ is

$$\Phi = -\frac{\beta}{\alpha}\int_0^l \bar{B}\cdot dl. \tag{73}$$

where $d\bar{l}$ is the line element along the magnetic field. Inserting this in (4) and using (8), we have

$$\frac{\partial\bar{A}}{\partial t} = \left(\frac{c^2\alpha}{4\pi\sigma} - \frac{\beta}{\alpha}\right)f(t)\nabla_\perp\int_0^l \bar{B}_0\cdot d\bar{l} \tag{74}$$

From this, we see that $\bar{A}$ does not change shape (the shape of the electric field does not change), that is,

$$\left.\frac{\partial \hat{A}}{\partial t}\right|_{\alpha=\mathrm{const}} = 0.$$

2. Force-Free Field of Frozen-In Type

By "Frozen-in Type", we mean the case where σ is infinite and it might appear that the above proof would no longer hold, but any actual gas has some finite σ, so the above conclusion still applies to a force-free field of the frozen-in type. Also, according to [1], the state of minimum magnetic energy is characterized by a constant α. According to the physical mechanism revealed in (21) and (22), we see that the transition from a variable α to a constant α requires the agency of a finite σ.

5. CONCLUSIONS

The conclusions reached up to this point are the following:

1. In [1], I proved that, whether the force-free field is of the resistive or the frozen-in type, and whether the gas is at rest or in motion, a constant α characterizes the state of minimum magnetic energy of the system, or the stable force-free field, which is the final fate of any force-free field.

2. This paper is a continuation of [1]. Besides proving the above proposition once again, a new result is this: if $\nabla\phi$ obeys the gauge relation (48), then eqns. (62) are the necessary and sufficient conditions for α to be constant, and then the local electric and magnetic fields will not change shape.

3. Eqn. (21) shows that the mechanism that makes a non-constant α evolve into a constant one, (or the system from some initial state evolve into a stable force-free field) is the dispersive action of the electrical resistance on the α-field. Without this mechanism, we can not explain how a force-free field of the frozen-in type can evolve into the state of minimum magnetic energy.

4. For a discussion on the physical parameters of the force-free field with constant α, the reader is referred to [1].

REFERENCES

[1] Pan Liang-ru, *Acta Astronomica Sinica* 19 (1978) p. 172. English translation in *Chinese Astronomy* vol. 3, no. 3.