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THE BASIC STATE OF A DISK GALAXY WITH FINITE THICKNESS

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ABSTRACT. This paper deals with the steady basic state of a disc galaxy with finite thickness. A hydrodynamical model is used and the zeroth and first order solutions in the small parameter thickness/(overall radius) are obtained.

1. MATHEMATICAL FORMULATION OF THE PROBLEM NON-DIMENSIONAL EQUATIONS

We adopt the simplifying method of simulating a galactic disk by a self-gravitating, "gaseous disk". Turbulent stress is simulated by the gas pressure, and the dispersion of star velocities by the "equivalent velocity of sound", and the basic equations consist of the equation of continuity, the hydrodynamical equations and Poisson's equation for the gravitational potential. Let the total mass of the system be M and, with the gravitational force balancing the centrifugal force and the "pressure", the matter be concentrated in a disc of finite thickness: $0 \leqslant r' \leqslant R$; $\delta = \delta(r)$; $\varepsilon = \dfrac{\delta}{R} \ll 1$. We adopt the orthogonal curvilinear coordinates (r', θ, z') shown in Fig. 1, in which z'= const. is a family of curved surfaces parallel to the boundary profile of the disk. In terms of the basic parameters M, G, R, and $\delta(r)$, we define the non-dimensional radius r, height z, time t, volume and surface densities ρ and $\sigma(r)$, velocities in the 3-coordinate directions (u, v, w), equivalent sound velocities $(a_r^2, a_\theta^2, a_z^2)$, reciprocal Mach numbers $(M_r^2, M_\theta^2, M_z^2)$ and gravitational potential ψ as follows:

$$\left.\begin{aligned}
&V_0 = \left(\frac{4\pi GM}{R}\right)^{1/2},\\
&r' = Rr,\ z' = \delta(r)\cdot z,\ t' = \left(\frac{R}{V_0}\right)t.\\
&\rho' = \left(\frac{M}{R^3}\right)\rho;\ \sigma'(r) = \left(\frac{M}{R^2}\right)\sigma(r).\\
&(u', v', w') = V_0(u, v, w).\\
&a_r'^2 = M_r^2V_0^2a_r^2;\ a_\theta'^2 = M_\theta^2V_0^2a_\theta^2,\ a_z'^2 = M_z^2V_0^2a_z^2.\\
&M_r^2 = \frac{a_r^{*2}}{V_0^2};\ M_\theta^2 = \frac{a_\theta^{*2}}{V_0^2};\ M_z^2 = \frac{a_z^{*2}}{V_0^2},\\
&\psi' = \left(\frac{4\pi GM}{R}\right)\psi.
\end{aligned}\right\}\quad(1.1)$$

First we suppose the disk profile to vary slowly $\left(\left|\frac{d\ln\delta(r)}{dr}\right| \ll 1\right)$, so we may neglect the additional small quantities due to transformation of curvilinear coordinates in the Lamé coefficients and obtain the following set of basic non-dimensional equations:

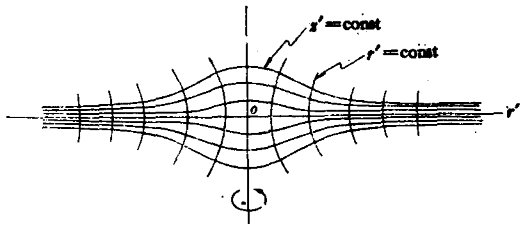

Fig. 1 Curvilinear coordinate system for a disk galaxy

$$\left.\begin{aligned}
&\varepsilon\left(\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v)}{\partial\theta}+\frac{\rho u}{r}\right)+\frac{\partial(\rho w)}{\partial z}=0,\\
&\varepsilon\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}-\frac{v^2}{r}\right)+w\frac{\partial u}{\partial z}+\varepsilon\left(M_r^2a_r^2\frac{\partial\ln\rho}{\partial r}+\frac{\partial\psi}{\partial r}\right)=0,\\
&\varepsilon\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\theta}+\frac{uv}{r}\right)+w\frac{\partial v}{\partial z}+\varepsilon\left(\frac{M_\theta^2a_\theta^2}{r}\frac{\partial\ln\rho}{\partial\theta}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)=0,\\
&\varepsilon\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\theta}\right)+w\frac{\partial w}{\partial z}+M_z^2a_z^2\frac{\partial\ln\rho}{\partial z}+\frac{\partial\psi}{\partial z}=0,\\
&\varepsilon^2\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}-\rho\right)+\frac{\partial^2\psi}{\partial z^2}=0.
\end{aligned}\right\}\quad(1.2)$$

Next we note that the disk thickness δ is maintained by the velocity dispersion against the force of gravity, so that as $M_z \to 0$, we must have $\delta \to 0$. Hence we may take $M_z^2 = \varepsilon = \frac{Ra_z^{*2}}{4\pi GM}$. Lastly, we shall consider the axisymmetric basic state to be in a steady state with no outward material flow from the centre. Then from (1.2), we get

$$\frac{v^2}{r} = M_r^2 a_r^2 \frac{\partial\ln\rho}{\partial r} + \frac{\partial\psi}{\partial r}, \qquad (1.3)_1$$

$$0 = \varepsilon a_z^2 \frac{\partial\ln\rho}{\partial z} + \frac{\partial\psi}{\partial z}, \qquad (1.3)_2$$

$$\varepsilon^2\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \rho\right) + \frac{\partial^2\psi}{\partial z^2} = 0. \qquad (1.3)_3$$

The projected surface density is determined by

$$\sigma(r) = \int_{-\infty}^{+\infty} \varepsilon\rho\, dz. \qquad (1.4)$$

Boundary Conditions

1. At infinity, when $r \to \infty$; $|z| \to \infty$ $\rho, \psi \to 0$; (1.5)
2. On the plane of symmetry:

$$(1)\ \psi, \rho|_{z=0^+} = \psi, \rho|_{z=0^-};\ \frac{\partial\rho}{\partial z} = \frac{\partial\psi}{\partial z} = 0; \qquad (1.6)$$

(2) $\left.\frac{v^2}{r}\right|_{z=0} = f|_{z=0}$ is a known function (1.7)

or $\sigma(r)$ is a known function (1.8)

or $\rho(r, z)|_{z=0} = \rho(r, 0)$ is a known function (1.9)

2. SOLUTION OF POISSON'S EQUATION AND ITS DEVELOPMENT

Under the boundary conditions (1.5) and (1.6), the solution of Poisson's equation is: when $z > 0$,

$$\psi(r, z) \doteq -\frac{1}{2}\int_0^\infty\int_0^\infty\left[\int_{-\infty}^{+\infty} e^{-k|z-z_1|}\varepsilon\rho(r_1 z_1)dz_1\right] r_1 J_0(kr_1)J_0(kr)dr_1 dk_1; \quad (2.1)$$

and when $z < 0$,

$$\psi(r, z) = \psi(r, -z);\ \rho(r, z) = \rho(r, -z). \quad (2.2)$$

From the physical characteristics of the basic state of the disk galaxy, we specifically suppose that, as $r \to \infty$, $|z| \to \infty$, the density tends to zero sufficiently fast and sufficiently smoothly so as to satisfy the following mathematical conditions:

1. For any arbitrary, small positive number δ_1, provided r^* is sufficiently large, the following inequality holds for arbitrary (k,z):

$$\left|\int_{r^*}^\infty (\varepsilon\rho) r J_0(kr)dr\right| < \delta_1; \quad (2.3)$$

2. For any arbitrary small positive number δ_2, provided k^* is sufficiently large, the following inequality holds for arbitrary (k,r,z):

$$\frac{1}{N!}\left|\int_{k^*}^\infty k^N P(k, z)J_0(kr)dk\right| < \delta_2, \quad (2.4)$$

where

$$P(k, z) = \int_0^\infty (\varepsilon\rho) r J_0(kr)dr. \quad (2.5)$$

Under these assumptions, the solution (2.1) can be developed into

$$\psi(r, z) = -\frac{1}{2}\sum_{N=0}^\infty \frac{\varepsilon^N}{N!}\int_0^\infty\int_0^\infty\left[\int_{-\infty}^{+\infty} k^N|z - z_1|^N(\varepsilon\rho)dz_1\right] r_1 J_0(kr_1)J_0(kr)dr_1 dk. \quad (2.6)$$

3. THREE-DIMENSIONAL SOLUTION OF THE BASIC STATE

We now pass to the solution of the equation (1.3). Since the parameter $\varepsilon \ll 1$, we have the following developments in powers of ε:

$$\begin{cases} \varepsilon\rho = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\cdot\rho_2 + \cdots, \\ \sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \cdots, \\ \sigma_K = \int_{-\infty}^{+\infty}\rho_K dz, \\ \psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots, \\ f = \frac{v^2}{r} = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots. \end{cases} \quad (3.1)$$

Then, from (1.3) and (2.6), we can find approximations of various orders.

1. Zero-Order Approximation. We have

$$\psi_0 = -\frac{1}{2}\int_0^\infty s_0(k)J_0(kr)dk; \quad s_0(k) = \int_0^\infty \sigma_0(r)rJ_0(kr)dr; \tag{3.2}$$

$$f_0 = M_r^2 a_r^2 \frac{\partial \ln \rho_0}{\partial r} + \frac{\partial \psi_0}{\partial r}. \tag{3.3}$$

Let, further,

$$\varphi_0 = \frac{\rho_0(r, z)}{\rho_0(r, 0)}; \quad \alpha = \frac{\rho_0(r, 0)}{a_z^2}, \tag{3.4}$$

We then derive the equation for the density distribution:

$$\begin{cases} \dfrac{d^2 \ln \varphi_0}{dz^2} + \alpha\varphi_0 = 0; \\ \varphi_0(0) = 1; \ \varphi_0(\infty) = 0. \end{cases} \tag{3.5}$$

Its solution is

$$\varphi_0(z) = \operatorname{sech}^2\left(\frac{z}{z_*}\right); \quad z_* = \sqrt{\frac{2}{\alpha}}. \tag{3.6}$$

The projected surface density is

$$\sigma_0(r) = 2\int_0^\infty \rho_0 dz = 2\rho_0(r, 0)\int_0^\infty \varphi_0(z)dz = 2z_*\rho_{00}. \tag{3.7}$$

Here we can suppose $\alpha = \text{const.}$, and z_* is then the non-dimensional characteristic thickness The disk profile is

$$z' = Rz_*\varepsilon = z_* R\frac{\alpha a_z'^2}{\rho_{00}U^2}, \tag{3.8}$$

showing that it is determined by $\rho_{00}(r) = \rho_0(r, 0)$ and the dispersion velocity $a_z'(r)$, The curve $\varphi_0(x)$ is shown in Fig. 2 and it agrees with the results of previous researchers. Also, putting $z = 0$ in (3.2) and (3.3), we have

$$f_0|_{z=0} = M_r^2 a_r^2 \frac{d \ln \sigma_0}{dr} - \frac{1}{2}\frac{dg_0}{dr}, \tag{3.9}$$

$$g_0(r) = \int_0^\infty s_0(k) J_0(kr) dk. \tag{3.10}$$

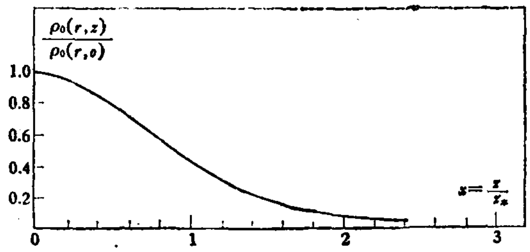

Fig. 2 Distribution of relative density along height in the zeroth approximation

<u>2. First-Order Approximation</u> Similar to the foregoing, we have

$$\phi_1 = -\frac{1}{2}\int_0^\infty s_1(k) J_0(kr) dk + \frac{1}{2}\int_{-\infty}^{+\infty} |z - z_1| \rho_0 dz_1; \tag{3.11}$$

$$s_1(k) = \int_0^\infty \sigma_1(r) r J_0(kr) dr; \tag{3.12}$$

$$\frac{\partial^2 \phi_2}{\partial z^2} = \rho_1 - \frac{1}{2}\int_0^\infty k^2 s_0(k) J_0(kr) dk; \tag{3.13}$$

$$f_1 = M_r^2 a_r^2 \frac{\partial}{\partial r}\left(\frac{\rho_1}{\rho_0}\right) + \frac{\partial \phi_1}{\partial r}. \tag{3.14}$$

Now let

$$\Phi_1(x) = \frac{a_z^2}{C(r)}\left(\frac{\rho_1}{\rho_0}\right), \quad C(r) = \frac{1}{2}\int_0^\infty k^2 s_0(k) J_0(kr) dk, \tag{3.15}$$

and we have, for the density distribution,

$$\begin{cases} \frac{d^2 \Phi_1}{dx^2} + \alpha \varphi_0 \Phi_1 = 1; \\ \Phi_1'(0) = 0. \end{cases} \tag{3.16}$$

If we take boundary condition (1.8), then

$$\sigma_1(r) = \sigma_2(r) = \cdots = 0. \tag{3.17}$$

From this we get as the second condition necessary for a definate solution of (3.16),

$$\int_0^\infty \varphi_0 \phi_1 dz = 0. \tag{3.18}$$

If we denote

$$x = \frac{z}{z_*};\ \frac{\alpha}{2}\phi_1(x) = Y(x), \tag{3.19}$$

then from (3.16) and (3.18) we obtain the general solution

$$Y(x) = C_1 \operatorname{th} x + C_2(1 - x\operatorname{th} x) + \Big[\operatorname{th} x \int_0^x (1 - x_1 \operatorname{th} x_1) dx_1 - (1 - x\operatorname{th} x)\int_0^x \operatorname{th} x_1 dx_1, \tag{3.20}$$

with the constants of integration

$$C_1 = 0;\ C_2 = Y(0) = -0.705. \tag{3.21}$$

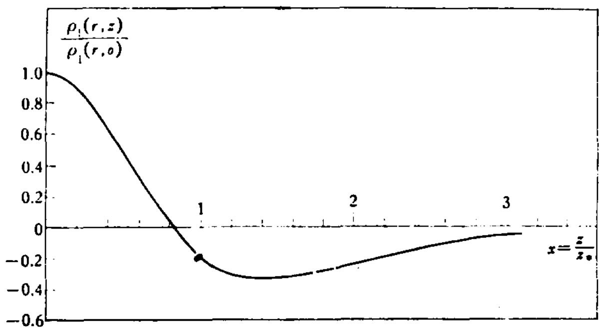

Fig. 3 Distribution of the relative density along height in the first-order approximation

Finally, we have

$$\begin{cases} \rho_1(r, x) = 2C(r)\varphi_0(x)Y(x); \\ \rho_1(r, 0) = -1.41C(r); \\ \dfrac{\rho_1(r, x)}{\rho_1(r, 0)} = \dfrac{\varphi_0(x)Y(x)}{Y(0)}, \end{cases} \tag{3.22}$$

while the first-order solution for the potential in the plane of symmetry is

$$\psi_1|_{z=0} = \int_0^\infty z\rho_0 dz = \rho_{00}\int_0^\infty z\varphi_0(z)dz = \frac{\ln 2}{\sqrt{2\alpha}}\sigma_0(r), \tag{3.23}$$

hence

$$f_1|_{z=0} = -\frac{4M_r^2a_r^2}{\sqrt{\alpha}}\frac{d}{dr}\left(\frac{C(r)}{\sigma_0(r)}\right) + \frac{\ln 2}{\sqrt{2\alpha}}\frac{d\sigma_0}{dr}. \tag{3.24}$$

To give a specific example, we suppose $\sigma_0(r)$ to have the form given by Toomre, that is,

$$\begin{cases} \sigma_0^{(0)} = \dfrac{C}{(a^2+r^2)^{1/2}}; \\ \dfrac{dg_0^{(0)}}{dr} = \dfrac{C}{r}\left[\dfrac{a}{(a^2+r^2)^{1/2}} - 1\right] \end{cases} \tag{3.25}$$

and

$$\sigma_0^{(N+1)} = \left|\left(\frac{\partial}{\partial a}\right)^N \frac{Ca}{(a^2+r^2)^{3/2}}\right| \quad (N = 0, 1, 2, \cdots). \tag{3.26}$$

Correspondingly,

$$\begin{cases} s_0^{(N+1)}(k) = |Ck^N e^{-ak}|; \\ g_0^{(N+1)}(r) = \displaystyle\int_0^\infty s_0^{(N+1)} J_0(kr)dk = \sigma_0^{(N)}(r); \\ C^{(N+1)}(r) = \dfrac{1}{2}\sigma_0^{(N+2)}(r). \end{cases} \tag{3.27}$$

Then we have

$$\begin{cases} f_0^{(N+1)}|_{z=0} = M_r^2a_r^2\dfrac{d}{dr}\ln\sigma_0^{(N+1)}(r) - \dfrac{1}{2}\dfrac{d\sigma_0^{(N)}}{dr}; \\ f_1^{(N+1)}|_{z=0} = -\dfrac{2M_r^2a_r^2}{\sqrt{\alpha}}\dfrac{d}{dr}\left[\dfrac{\sigma_0^{(N+2)}}{\sigma_0^{(N+1)}}\right] + \dfrac{\ln 2}{\sqrt{2\alpha}}\dfrac{d\sigma_0^{(N+1)}}{dr}. \end{cases} \tag{3.28}$$

For a "cold disk", $M_r^2a_r^2 \approx 0$, and we have

$$f^{(N+1)}|_{z=0} = -\frac{d\sigma_0^{(N)}}{dr} + \varepsilon\frac{\ln 2}{\sqrt{2\alpha}}\frac{d\sigma_0^{(N+1)}}{dr} + O(\varepsilon^2), \tag{3.29}$$

and for the density distribution along height,

$$\varepsilon\rho^{(N+1)}(r, z) = \left[\sqrt{\frac{\alpha}{8}}\,\sigma_0^{(N+1)}(r) + \varepsilon\sigma_0^{(N+2)}(r)Y(x)\right]\operatorname{sech}^2 x + O(\varepsilon^2). \qquad (3.30)$$

Obviously, as $\varepsilon \to 0$, this degenerates into Toomre's result for an infinitely thin disk.

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Translator's Note: An Appendix outlining a general analytical soltuion of Poisson's Equation in the non-axi-symmetric case is omitted in the translation.