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TUNNEL EFFECT ON DENSITY WAVES OF GALAXIES AT COROTATION CIRCLE AND THE SWITCH CHARACTER OF THE "WASER" MECHANISM

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ABSTRACT

In this paper a nonlinear complex eigenvalue problem posed in galactic dynamics is studied. The uniformly valid asymptotic solutions and the quantum condition are obtained. In terms of these solutions, we have analyzed the wave propagation near the corotation circle. The tunnel effect on density waves through the potential barrier at corotation circle and the "switch character" of a certain type possessed by the "waser" mechanism are explained.

These physical characters have important singnificance in the dynamic theory of spiral structure of galaxies.

I. INTRODUCTION

Investigation on the persistence and origin of spiral structure of galaxies and the propagation of three-dimensional density wave over the galactic symmetric plane usually ends up in a nonlinear complex eigenvalue problem of a second-order ordinary differential equation of the following form:

$$\begin{cases} \dfrac{d^2u}{dr^2} + k_3^2(r, \omega)u = 0, \text{[1]} & (1) \\ k_3^2(r, \omega) = \lambda^2[f_0^2(r)(\omega - \Omega(r))^2 + f_1(r)]. & (2) \end{cases}$$

The boundary conditions can reasonably be taken as follows:

1) As $r \to 0$, u decays quickly; (3)

2) As $r \to \infty$, $ue^{i\omega t}$ behaves like an outgoing wave. (4)

In formula (2), λ is a large parameter, ω is a complex frequency, $\Omega(r) > 0$, $\Omega'(r) < 0$, $f_0^2(r) > 0$ are given as real functions; the function $f_1(r) = O\left(\dfrac{1}{\lambda}\right)$ may be complex. In common case, it is assumed that the wave number function $k^2(r)$ has a simple zero turning point at $r = r_{cc}$, the point r_{cc} being sufficiently far away from

1) In this paper, for simplicity, we rewrite k_3 as k.

the corotation point $r_{co}(\Omega(r_{co}) = \omega_R)$.

The particular case of $f_1(r) \equiv 0$ has been studied in detail by C. C. Lin *et al.* (1976)[1-3]. Clearly, in this case the wave number $k^2(r)$ has a double zero turning point at $r = r_{co}$; the case $f_1(r) \not\equiv 0$ has also been treated by C. B. Li, *et al.* (1976)[4], but the results obtained are available only to the special case $f_1(r_{co}) = 0$. If $f_1(r_{co}) \neq 0$, the integral in the formula (3.8) of [4] would have a logarithmic divergence. In this special case, their result shows that the existence of function $f_1(r)$ does not seriously influence the wave propagation behaviour. Recently, Nishimoto (1979)[5] has given the quantum condition of this problem for the case where the principal term of $k^2(r)$ is a third-degree polynomial.

In the present paper, the general case of the problems (1)—(4) is treated and uniformly valid solutions and dispersion relation are obtained. The behaviour of solutions is discussed. It should be pointed out that as $f_1(r_{co}) \neq 0$, a complex situation would arise. In this case, the function $k^2(r)$ has two simple zero turning points situated closely by each other. The existence of small non-zero term $f_1(r)$ will bring a considerable effect on the whole wave propagation. The parameter $f_1(r_{co})$ at corotation circle will cause a "tunnel effect" on the density wave and lead to a "switch character" of the "waser" machanism.

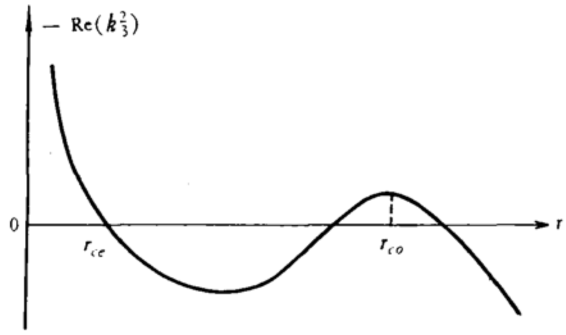

Fig. 1

II. Uniformly Valid Asymptotic Solutions Near $r = r_{co}$

We investigate Eq. (1) on a complex plane (r). The analytical functions $\Omega(r)$, $f_0^2(r)$ all take real values on the real axis. To determine the single-valued branch of function $k(r)$, we should, first of all, find out the positions of its branch points. To begin with, let us suppose $f_1(r) = 0$. Then for a growth mode, the frequency possesses a small imaginary part,

$$\omega = \omega_R + i\omega_I, \quad |\omega_I| \ll |\omega_R|, \quad \omega_I < 0.$$

At the branch point r_*, we have

$$\omega - \Omega(r_*) = 0,$$

$$\Omega(r_{co}) = \omega_R, \Omega(r_*) = \Omega(r_{co}) + \Omega'(r_{co})(r_* - r_{co}),$$

thus,

$$r_* = r_{co} + i\frac{\omega_I}{\Omega'(r_{co})}, \quad (r_*)_R = r_{co}, \quad (r_*)_I = \frac{\omega_I}{\Omega'(r_{co})} > 0.$$

In this case, the double zero point r_* of function $k^2(r)$ is above the real-axis. Generally, if $f_1(r) \neq 0$, we may rewrite

$$k^2(r) = \lambda^2[A_0^2(r)(r - r_*)^2 + f_1(r)], \tag{5}$$

and expand functions $A_0(r) f_1(r)$ as a Taylor series in the neighbourhood of $r = r_*$,

$$\begin{cases} A_0^2(r) = A_0^2 + O(r - r_*), \\ f_1(r) = -d_0 + d_1(r - r_*) + d_2(r - r_*)^2 + \cdots, \end{cases} \tag{6}$$

which implies that

$$A_0 = O(1), \quad d_N = \frac{1}{N!} f_1^{(N)}(r_*) = O\left(\frac{1}{N!\,\lambda}\right).$$

Then near r_{co} the function $k^2(r)$ can be expressed as

$$\begin{aligned} k^2(r) &= \lambda^2[A_0^2(r - r_*)^2 - d_0 + d_1(r - r_*) + d_2(r - r_*) + \cdots] \\ &= \lambda'^2[(r - r'_*)^2 - d'_0 + \cdots], \end{aligned} \tag{7}$$

where

$$\begin{cases} \lambda'^2 = \lambda^2(A_0^2 + d_2) = O(\lambda^2), \\ r'_* = r_* - \dfrac{d_1}{2(A_0^2 + d_2)}, \quad d'_0 = \dfrac{d_1^2}{4(A_0^2 - d_2)} + \dfrac{d_0}{A_0^2 + d_2}. \end{cases} \tag{8}$$

From Formula (7), it can be seen that because $f_1(r) \neq 0$, the double zero point r_* of wave number $k^2(r)$ will be split into two simple zoro points situated closely by each other:

$$r_{*1} = r'_* + \sqrt{d'_0}; \quad r_{*2} = r'_* - \sqrt{d'_0} \quad (0 \leqslant arg(d'_0) < 2\pi). \tag{9}$$

Provided $|f_1(r)|$ is sufficiently small, the points r'_*, r_{*1}, r_{*2} would all approach the point r_*, but still remain above the real axis. In order to choose the single-valued branch of the function $k(r)$, we make a cut on the complex plane (r): $r_{*2} \to r_{*1}$ with its continuation line. When $|f_1(r)|$ is very small, this cut will not cross the realn axis, and we can choose the real axis as an integral path to make an analytic continuation of $k(r)$. In so doing, we can determine the single-valued branch of the function $k(r)$ in the following way: Assume the principal term of $k(r)$ near r_{co} to be

$$k(r) \approx -\lambda'(r - r_{*1})^{1/2}(r - r_{*2})^{1/2}, \tag{10}$$

and let $\mathrm{Re}(k) > 0$ (say, $\arg(k) \approx 0$) as $r \ll r_{co}$, then we can make its analytic continuation along the real axis. Consequently, as $r \gg r_{co}$, $\mathrm{Re}(k) < 0$ (say, $\arg(k) \approx \pi$) holds. Of course, if $|f_1|$ is gradually increasing, the positions of points (r'_*, r_{*1}, r_{*2}) will continuously change, so that they would possibly go under the real axis, and the cut of complex plane (r) would also possibly intersect the real axis. In this case, the integral path that makes analytic continuation of the function $k(r)$, will continuously evolve itself as a curve (C) passing below points (r_{*1}, r_{*2}) (see Fig. 2).

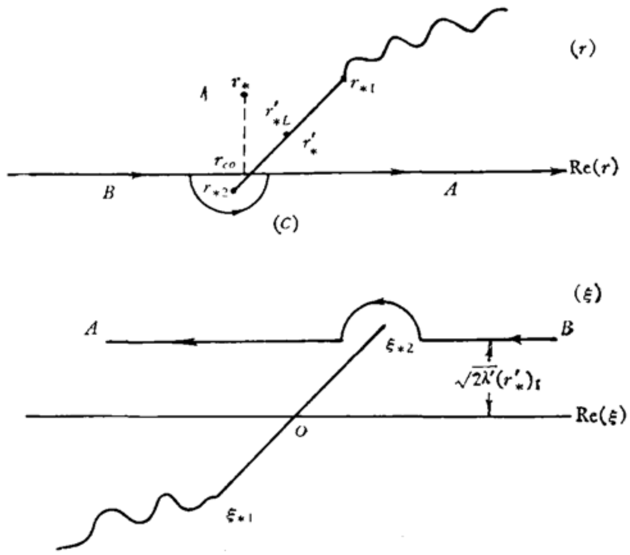

Fig. 2

After determining the function $k(r)$, we may now introduce a transformation $\xi=\xi(r)$ from the plane (r) to the plane (ξ).

$$\begin{cases}\tau = -\int_{r_{*2}}^{r} kdr = \int_{\xi_{*2}}^{\xi}\left(\frac{\xi_1^2}{4}-a\right)^{1/2} d\xi_1 = a[t\sqrt{t^2-1}-\mathrm{ch}^{-1}t] \\ \quad = a[t\sqrt{t^2-1}-\ln(t+\sqrt{t^2-1})], \\ t=\xi/\xi_{*2}.\end{cases} \tag{11}$$

We demand that

$$\xi_{*1}=\xi(r_{*1}), \quad \xi_{*2}=\xi(r_{*2}). \tag{12}$$

Naturally, on the plane (ξ) there is also a corresponding cut: $\xi_{*2}\rightarrow\xi_{*1}$ with its continuation line (see Fig. 2).

Formula (10) holds as $|r-r_{co}|\ll 1$. From (10) and (11), it may be deduced that near r_{co},

$$-\lambda'(r-r_{*1})^{1/2}(r-r_{*2})^{1/2}\approx\left(\frac{\xi^2}{4}-a\right)^{1/2}\left(\frac{d\xi}{dr}\right)$$
$$\approx -\lambda'[(r-r'_*)^2-d'_0]^{1/2}.$$

Thus we get,

$$\begin{cases} a=\frac{\lambda' d'_0}{2}, \quad \xi_{*1}=-2\sqrt{a}, \quad \xi_{*2}=2\sqrt{a}, \\ \xi(r)\approx-\sqrt{2\lambda'}(r-r'_*).\end{cases} \tag{13}$$

In Table 1, we have shown the phase relations of relative quantities.

Table 1

	$r \ll r_{co}$	$r \gg r_{co}$
$\arg(r - r_{*2}) \approx$	$-\pi$	0
$\arg(\xi) \approx$	0	π
$\arg(k) \approx$	0	π
$\arg(\tau) \approx$	0	2π

Now we may turn to the discussion of Eq. (1). Make a Langer transformation,

$$\xi = \phi(r), \quad v = \psi(r)u, \tag{14}$$

nd Eq. (1) can be reduced to

$$\frac{d^2v}{d\xi^2} + \frac{1}{\phi'^2}\left(\phi'' - \frac{2\phi'\psi'}{\psi}\right)\frac{dv}{d\xi} + \frac{1}{\phi'^2}\left[k^2 - \psi\left(\frac{\psi'}{\psi^2}\right)^2\right]v = 0, \tag{15}$$

Letting

$$\psi = [\phi'(r)]^{1/2} \text{ and } \phi(r) = \xi(r), \tag{16}$$

then from (15) we get,

$$\frac{d^2v}{d\xi^2} + \left[\frac{k^2}{\xi'^2} - \hat{Q}(\xi)\right]v = 0, \tag{17}$$

where

$$\hat{Q}(\xi) = \frac{\psi'^2}{\phi'^2}\frac{1}{\psi^3} = \frac{1}{4}\frac{\xi''^2}{(\xi')^{9/2}}. \tag{18}$$

From the definition of $\xi(r)$ i.e. (11), it follows that

$$\xi'(r) = k(r)\Big/\left(\frac{\xi^2}{4} - a\right)^{1/2}. \tag{19}$$

It can be seen that, in the neighbourhood of r_{co} the function $\xi'(r)$ is regular with neither singular point nor zero point, and $\xi' = O(\lambda'^{1/2})$. Then, we obtain

$$\frac{k^2}{\xi'^2} = \left(\frac{\xi^2}{4} - a\right), \quad \hat{Q}(\xi) = O(\lambda'^{-5/4}) \ll 1. \tag{20}$$

As the lowest order approximation, $\hat{Q}(\xi)$ can be neglected, and we get

$$\frac{d^2v}{d\xi^2} + \left(\frac{\xi^2}{4} - a\right)v = 0, \tag{21}$$

which is the well-known parabolic cylinder equation. Its two linear independent complex solutions are $\{E(a, \xi); E_*(a, \xi)\}$. Correspondingly, the uniformly valid asymptotic solutions of Eq. (1) are

$$u = k^{-1/2}\left(\frac{\xi^2}{4} - a\right)^{1/4} \cdot \{E(a, \xi); E_*(a, \xi)\}. \tag{22}$$

From the Weber function theory, we get the following formulas: (see Ref. [6], p. 693)

$$\begin{cases} E(a,\xi) = \sqrt{2}\, e^{\frac{\pi a}{4}+\frac{i\pi}{8}+\frac{i\phi_2}{2}} U(ia, ze^{-\frac{i\pi}{4}}), \\ E_*(a,\xi) = \sqrt{2}\, e^{\frac{\pi a}{4}-\frac{i\pi}{8}-\frac{i\phi_2}{2}} U(-ia, ze^{i\frac{\pi}{4}}), \\ U(a,z) = \dfrac{\Gamma\left(\frac{1}{2}-a\right)}{2\pi i} e^{-\frac{z^2}{4}} \int_\alpha \exp\left(zs - \dfrac{s^2}{2}\right) s^{a-\frac{1}{2}} ds \end{cases} \tag{23}$$

From the connection formulas of the function $U(a, z)$ (see Ref. [6], p. 687) we can easily derive the following connection formulas of the functions $E(a, z), E_*(a, z)$ as follows:

$$\begin{cases} E_*(a, -z) = ie^{\pi a} E_*(a, z) - i\dfrac{\sqrt{2\pi}e^{\frac{\pi a}{2}-i\phi_2}}{\Gamma\left(\frac{1}{2}-ia\right)} E(a, z), \\ E(a, z) = i\dfrac{\sqrt{2\pi}e^{\frac{\pi a}{2}+i\phi_2}}{\Gamma\left(\frac{1}{2}+ia\right)} E_*(a, z) - ie^{\pi a} E(a, z). \end{cases} \tag{24}$$

Besides, as $|\arg(z)| \leqslant \frac{\pi}{4}$ we may also get asymptotic expansions as,

$$E(a, z) \sim \sqrt{\frac{2}{z}}\, e^{i\Theta}, \quad E_*(a, z) \sim \sqrt{\frac{2}{z}}\, e^{-i\Theta}, \tag{25}$$

where

$$\begin{cases} \Theta = \dfrac{z^2}{4} - a\ln z + \dfrac{\pi}{4} + \dfrac{\phi_2}{2}, \\ \phi_2 = \arg \Gamma\left(\dfrac{1}{2} + ia\right). \end{cases} \tag{26}$$

In order to satisfy the boundary condition (4) so that we can get a solution corresponding to an outgoing wave in the region $r \gg r_{co}$, we must first make a study of (11). When $t = |\xi/\xi_{*2}| \gg 1$,

$$\tau(\xi) \sim \frac{\xi^2}{4} - a\ln\xi + \frac{a}{2}(\ln a - 1). \tag{27}$$

Then we get

$$\Theta \sim \tau + \frac{\pi}{4} + \frac{\phi_2}{2} - \frac{a}{2}(\ln a - 1). \tag{28}$$

It has been shown that as $r \gg r_{co}$, $\arg(\xi) \approx \pi$ holds, in such case the solution

$$E_*(a, \xi e^{-i\pi}) \sim e^{i\int_{r_{*2}}^{r} k dr}$$

must correspond to an outgoing wave.

In the region $r \ll r_{co}$, as $\arg(\xi) \approx 0$, using the connection formula (24), we have

$$E_*(a, \xi e^{-i\pi}) = ie^{\pi a}E_*(a, \xi) - i\frac{\sqrt{2\pi}e^{\frac{\pi a}{2} - i\phi_2}}{\Gamma\left(\frac{1}{2} - ia\right)}E(a, \xi).$$

Thus, the uniformly valid asymptotic solution near $r = r_{co}$ may be expressed as

$$\left\{\begin{aligned} &u_{co} = u_{co}^+ + u_{co}^- \\ &u_{co}^+ = ie^{\pi a}E_*(a, \xi) \\ &u_{co}^- = -i\frac{\sqrt{2\pi}e^{\frac{\pi a}{2} - i\phi_2}}{\Gamma\left(\frac{1}{2} - ia\right)}E(a, \xi) \\ &u_{co} = E_*(a, \xi e^{-i\pi}) \end{aligned}\right\} \times k^{-1/2}\left(\frac{\xi^2}{4} - a\right)^{1/4}, \tag{29}$$

in which, it can be easily shown, that u_{co}^+ corresponds to an outgoing wave, and u_{co}^- an ingoing wave.

It should be noted that the lower limit of the integral in the transformation (11) as introduced above is the branch point r_{*2}. We may also choose another point, e.g. r'_{*L} as the lower limit of the integral; the point r'_{*L} is situated on the left side closely by the point r'_* along the direction of the cut line $r_{*2} \to r_{*1}$. In this time, we may define

$$\tau_* = -\int_{r'_{*L}}^{r} kdr = \tau - \tau_0,$$

then

$$\left\{\begin{aligned} &\tau_0 = -\int_{r_{*2}}^{r'_{*L}} kdr = a[t_0\sqrt{t_0^2 - 1} - \ln(t_0 + \sqrt{t_0^2 - 1})], \\ &t_0 = 0, \quad t_0 - 1 = e^{-i\pi}. \end{aligned}\right.$$

Now, we have

$$\tau_0 = \frac{i\pi a}{2}, \quad \Theta \sim \tau_* + \frac{\pi}{4} + \frac{\phi_2}{2} - \frac{a}{2}(\ln a - 1) + \frac{ia\pi}{2}. \tag{30}$$

To show the behaviour of wave interaction near the corotation circle r_{co}, we now discuss a special case. Let $r'_* = r_{co}$, and the parameter a is real. Using the formula

$$\left|\Gamma\left(\frac{1}{2} \pm ia\right)\right| = \sqrt{\frac{\pi}{\operatorname{ch}(\pi a)}}, \quad \arg\Gamma\left(\frac{1}{2} - ia\right) = -\phi_2, \tag{31}$$

we get

$$\left.\begin{aligned} &u_{co}^+ = ie^{\pi a}E_*(a, \xi) \\ &u_{co}^- = -i\sqrt{1 + e^{2\pi a}}E(a, \xi) \\ &u_{co} = E_*(a, \xi e^{-i\pi}) \end{aligned}\right\} \times k^{-\frac{1}{2}}\left(\frac{\xi^2}{4} - a\right)^{\frac{1}{4}}. \tag{32}$$

If $a > 0$, we take r_{*2} as the lower limit of integral. Then we have

$$\tau = -\int_{r_{*2}}^{r} kdr,$$

because the value of r_{*2} is a real number, as $r < r_{*2}$ the value τ is also a real one, and so is Θ. If $a < 0$, we take r'_{*L} as the integral lower limit. Thus we have

$$\tau^* = -\int_{r'_{*L}}^{r} kdr,$$

and the quantity τ_* is a real number. Using formula (30) and letting $a = |a|e^{i\pi}$, we derive

$$\Theta \sim \tau_* + \frac{\pi}{4} + \frac{\phi_2}{2} - \frac{a}{2}(\ln|a| - 1),$$

then the value Θ is still a real one.

Thus, after an incident wave u_{co}^+ has arrived at the corotation circle, an ingoing wave u_{co}^- will be reflected and an outgoing wave u_{co} will be transmitted from the corotation circle with reflection ratio R and transmission ratio T respectively,

$$\begin{cases} R = \left|\dfrac{u_{co}^-}{u_{co}^+}\right| = \sqrt{1 + e^{-2\pi a}}, \\ \\ T = \left|\dfrac{u_{co}}{u_{co}^+}\right| = e^{-\pi a}. \end{cases} \tag{33}$$

From the above, it can be seen that if $a > 0$, the reflection ratio will quickly tend to 1 with increasing $|a|$; if $a < 0$, the situation is reversed, and the reflection ratio R and transmission ratio T will both quickly tend to infinity (see Fig. 3).

We may now consider the character of these solutions. When a is real, we have formulas (see [6], p. 693)

$$\begin{cases} E(a, x) = k_*^{-1/2} W(a, x) + i k_*^{1/2} W(a, -x), \\ E_*(a, x) = \overline{E(a, x)} = k_*^{-1/2} W(a, x) - i k_*^{1/2} W(a, -x), \\ k_* = \sqrt{1 + e^{2\pi a}} - e^{\pi a}. \end{cases} \tag{34}$$

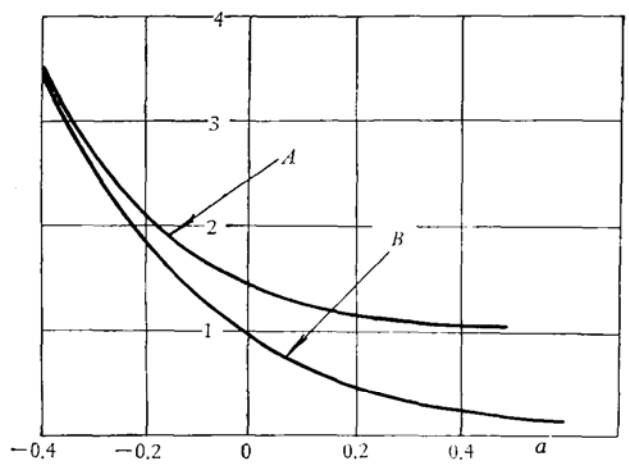

Fig. 3

For $a=0, \pm 0.5, \pm 1$, we draw the curves of the solution $W(a, x)$ (see Fig. 4(A)—(E)).

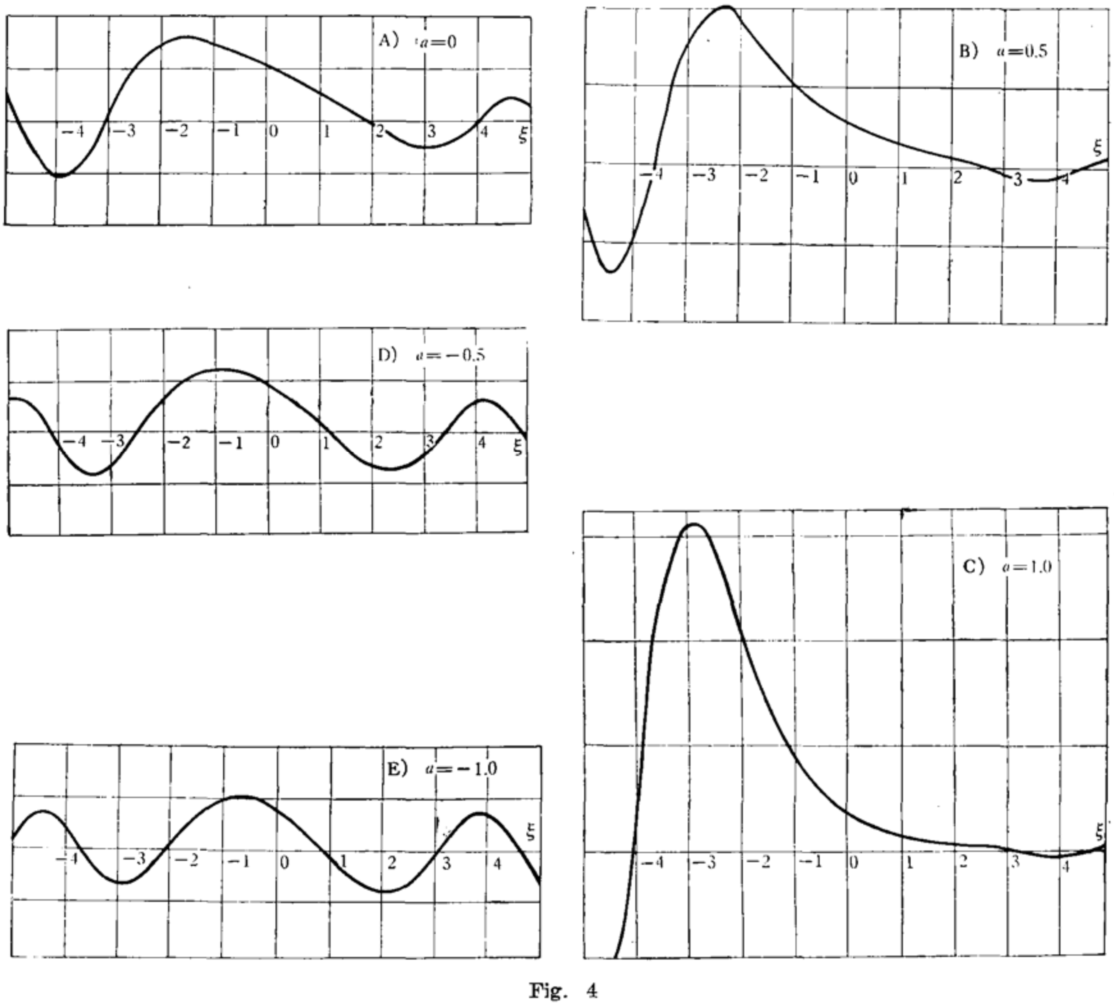

Fig. 4

These curves reflect the profiles of wave $\mathrm{Re}\{E_*(a, x)\}$. We see that the sign of parameter a exerts a great influence on the behaviour of wave propagation, which can be described as follows:

(1) when $a>0$, the wave number $k(r)$ has two real zero points near the corotation circle and between them there is a potential barrier, in which the solution $W(a, x)$ decays, denoting non-existence of wave. When an incident wave from the inner corotation circle arrives at the potential barrier, the one part of it, owing to the "tunnel effect", may penetrate through the barrier to the othrer side. The larger the value of a, the higher the potential barrier, the weaker the transmitted wave, and the

smaller the growth rate of reflected wave.

(2) When $a < 0$, the wave number k has two conjugate purely imaginary zero points, and the solution $W(a, x)$ has a wave pattern in the whole region. This indicates that an incident wave can easily pass over the corotation point to become a transmitted wave which is designated "the potential valley". Clearly, the larger the value a, the wider the "potential valley" and the larger the reflection ratio and transmission ratio of the wave (see Fig. 5).

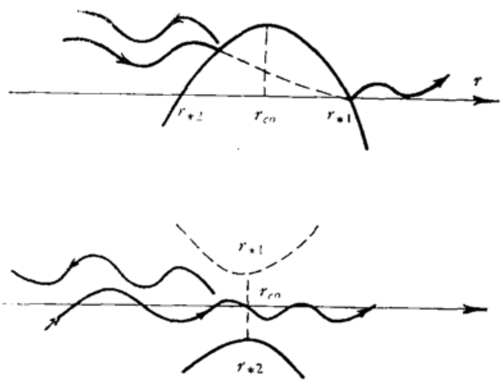

Fig. 5

From the above it can be deduced that for a general complex parameter $a = |a|e^{i\theta}$, if the value $|a|$ is fixed and phase angle θ is increasing, then as $\theta = 0$, the corotation circle will behave as a "shut state" to the incident wave, at this moment, both the rate of increase of reflected wave and the intensity of transmitted wave are very small; as $\theta \rightarrow \pi$, the corotation circle becomes an "open state" to incident wave, at the moment, the rate of increase of reflected wave and the intensity of transmitted wave are getting larger; finally, as $\theta \rightarrow 2\pi$, the corotation circle returns again to the "shut state" to the incident wave. The wave propagation characters are seriously modulated by the modulus and the phase of the parameter a, so that we may call such character a "switch character" of the "waser" machanism.

III. Uniformly Valid Asymptotic Solutions Near $r = r_{ce}$

The presence of the small term $f_1(x)$ cannot change the property that the wave number $k(r)$ has a simple zero turning point r'_{ce} near point r_{ce}. The uniformly valid asymptotic solution near a simple zero turning point has been sufficiently studied[2]. Here we need only cite the result of Ref. [2]. Assume

$$\zeta = \int_{r_{ce}}^{r} k dr, \tag{35}$$

and take $\arg(\zeta) = 0$ for $r > r_{ce}$, then make its analytic continuation along the real axis. Thus, for $r < r_{ce}$, $\arg(\zeta) \approx \frac{3}{2}\pi$ holds. To satisfy the boundary condition (3), the proper solution must be

$$u_{ce} = \left(\frac{\zeta}{k}\right)^{1/2} H_{1/3}^{(1)}(\zeta e^{-i\pi}). \tag{36}$$

For the region $r > r_{ce}$, by the connection formula of the Hankel functions, the solution can be dissociated as follows:

$$\begin{cases} u_{ce} = u_{ce}^{+} + u_{ce}^{-}, \quad u_{ce}^{+} = \left(\frac{\zeta}{k}\right)^{1/2} H_{1/3}^{(1)}(\zeta), \\ u_{ce}^{-} = \left(\frac{\zeta}{k}\right)^{1/2} e^{-\frac{1}{3}i\pi} H_{1/3}^{(2)}(\zeta). \end{cases} \tag{37}$$

IV. Dispersion Relation

In the overlapping region $(r_{cc} \ll r \ll r_{co})$ on the complex plane (r), both solution u_{co} and u_{ce} are available. Therefore, we can write

$$\begin{aligned} u = u_{co} &= Ae^{-i\tau} + Be^{i\tau} \\ &= u_{ce} = ce^{i\zeta} + De^{-i\zeta}, \end{aligned} \tag{38}$$

and it follows that

$$\frac{A}{B} = \frac{C}{D} e^{i2\phi}, \quad \phi = \int_{r_{ce}}^{r_{*2}} kdr. \tag{39}$$

Using the asymptotic expansions of functions $H_{1/3}(\zeta)$, $E(a, \xi)$, $E_*(a, \xi)$ again, we obtain

$$\begin{cases} A = ie^{\pi a} \sqrt{\frac{2}{\xi}}\, k^{-1/2} \left(\frac{\xi^2}{4} - a\right)^{1/4} e^{-i\left[\frac{\pi}{4} + \frac{\phi_2}{2} - \frac{1}{2}a(\ln a - 1)\right]}, \\ B = -i \frac{\sqrt{2\pi} e^{\frac{\pi a}{2} - i\phi_2}}{\Gamma\left(\frac{1}{2} - ia\right)} \sqrt{\frac{2}{\xi}}\, k^{-1/2} \left(\frac{\xi^2}{4} - a\right)^{1/4} e^{i\left[\frac{\pi}{4} + \frac{\phi_2}{2} - \frac{1}{2}a(\ln a - 1)\right]}, \\ C = \left(\frac{\zeta}{k}\right)^{1/2} e^{-i\left(\frac{\pi}{6} + \frac{\pi}{4}\right)}, \quad D = \left(\frac{\zeta}{k}\right)^{1/2} e^{-\frac{i\pi}{3} + i\left(\frac{\pi}{6} + \frac{\pi}{4}\right)}. \end{cases} \tag{40}$$

From the above we can easily get

$$e^{i2\phi} = -\frac{\Gamma\left(\frac{1}{2} - ia\right) e^{\frac{\pi a}{2}}}{\sqrt{2\pi}} e^{ia(\ln a - 1)},$$

or

$$\phi = \left(n + \frac{1}{2}\right)\pi + \frac{a}{2}(\ln a - 1) - \frac{ia\pi}{4} - \frac{i}{2} \ln \left[\frac{\Gamma\left(\frac{1}{2} - ia\right)}{\sqrt{2\pi}}\right]. \tag{41}$$

This is the dispersion relation that we want.

As a simple numerical example, let

$$k = \lambda r^{1/2}[(r - \hat{\omega})^2 - d]^{1/2}, \quad \hat{\omega} = C - \omega. \tag{42}$$

In this case

$$r_{*1} = \hat{\omega} + d^{1/2}, \quad r_{*2} = \hat{\omega} - d^{1/2}, \quad r_{ce} = 0, \quad \lambda' = \lambda\hat{\omega}^{1/2}. \tag{43}$$

So, we have

$$\phi = \int_0^{r_{*2}} kdr = \lambda r_{*1}^{1/2} r_{*2}^2 \int_0^1 t^{1/2}(1 - t)^{1/2}(1 - tz)^{1/2} dt, \tag{44}$$

where

$$z = \frac{r_{*2}}{r_{*1}}. \tag{45}$$

In terms of hypergeometric function, (44) can be rewritten as,

$$\phi = \lambda r_{*1}^{1/2} r_{*2}^{2} \frac{\Gamma(\hat{b})\Gamma(\hat{c}-\hat{b})}{\Gamma(\hat{c})} F(\hat{a}, \hat{b}, \hat{c}, z),$$

where

$$\hat{a} = -\frac{1}{2}, \quad \hat{b} = \frac{3}{2}, \quad \hat{c} = 3, \quad |z| \approx 1.$$

Using the formula in Ref. [6] (p. 559, 15.3.11),

$$F(\hat{a}, \hat{b}, \hat{c}, z) = \frac{\Gamma(m)\Gamma(\hat{c})}{\Gamma(\hat{a}+m)\Gamma(\hat{b}+m)} \sum_{n=0}^{m-1} \frac{(\hat{a})_n(\hat{b})_n}{n!(1-m)_n}(1-z)^n - \frac{\Gamma(\hat{c})}{\Gamma(\hat{a})\Gamma(\hat{b})}(z-1)^m \cdot$$

$$\sum_{n=0}^{\infty} \frac{(\hat{a}+m)_n(\hat{b}+m)_n}{n!(n+m)!}(1-z)^n \cdot [\ln(1-z) - \psi(n+1) - \psi(n+m-1) + \psi(\hat{a}+n+m) + \psi(\hat{b}+n+m)],$$

$$\hat{c} = \hat{a} + \hat{b} + m, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

and the following relations,

$$\begin{cases} 1 - z \approx 2\left(\dfrac{d^{1/2}}{\hat{\omega}}\right)\left(1 - \dfrac{d^{1/2}}{\hat{\omega}}\right), \\ r_{*1}^{1/2} r_{*2}^{2} \approx \hat{\omega}^{5/2}\left[1 - \dfrac{25}{8}\left(\dfrac{d}{\hat{\omega}^2}\right)\right], \quad d = \dfrac{2a}{\lambda\hat{\omega}^{1/2}}, \end{cases} \tag{46}$$

we obtain

$$\phi \approx \lambda \frac{4}{15} \hat{\omega}^{5/2}\left(1 + \frac{3}{2}\frac{d^{1/2}}{\hat{\omega}} + \cdots\right),$$

and the dispersion relation becomes

$$\hat{\omega}^{5/2} + \sqrt{\frac{9a}{2\lambda}}\,\hat{\omega}^{5/4} = \frac{15}{4\lambda}\left[\left(a + \frac{1}{2}\right)\pi + R + iW\right].$$

Thus it follows that

$$\hat{\omega}^{5/4} = -\sqrt{\frac{9a}{8\lambda}} \pm \sqrt{\frac{9a}{8\lambda} + \frac{15}{4\lambda}\left[\left(n + \frac{1}{2}\right)\pi + R + iW\right]}. \tag{47}$$

In the above, the following notation is used

$$R + iW = \frac{a}{2}(\ln a - 1) - \frac{ia\pi}{4} - \frac{i}{2}\ln\left[\frac{\Gamma\left(\frac{1}{2} - ia\right)}{\sqrt{2\pi}}\right]. \tag{48}$$

Taking $\lambda = 10, 20; C = 20, 2.0$ and the various values of $a = |a|e^{i\theta}$, we calculated formula (47) to obtain eigenvalues $\{\omega\}$. It is to be noted that there are 5×2 different complex roots of $\hat{\omega}$ in (47). However, we demand that the branch point $r'_* = \hat{\omega} = (C - \omega)$ sit on the left of point r_{ce} and $(r'_*)_R \gg (r'_*)_I$. In this case there are actually only two proper values out of ten. In the following we shall give the results for the case: $\lambda = 10, C = 20$.

In Fig. (6), we drew the curves of the growth rate of mode solution (ω_I/ω_{I0}) versus the real parameter a (in which $\omega_{I0} = \omega_I|_{a=0}$). It is shown in the figure, that as $a > 0$, the growth rate has to decrease with the increase of value of $|a|$, while as $a < 0$, the situation is just the reverse. For a fixed value of $|a|$, Fig. 7 shows the variation of ω_I/ω_{I0} with the value of θ, i.e. the growth rate will vary with the period (2π), as the phase angle θ increases. This is the "switch character" described above. Figs. 7 and 8 also show that besides the mode with a larger growth rate,

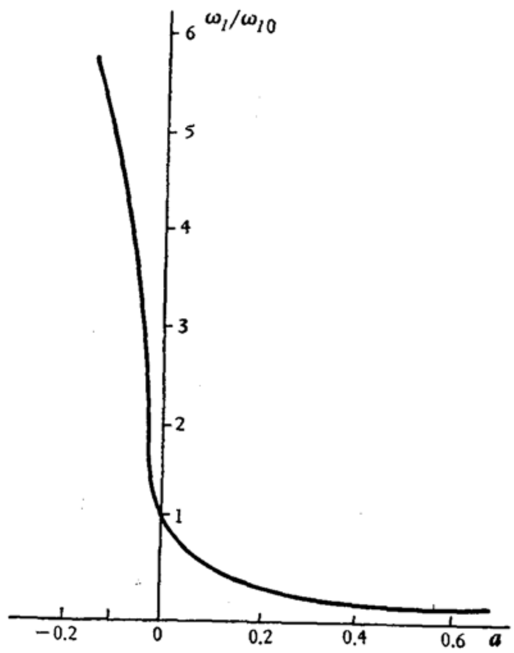

Fig. 6

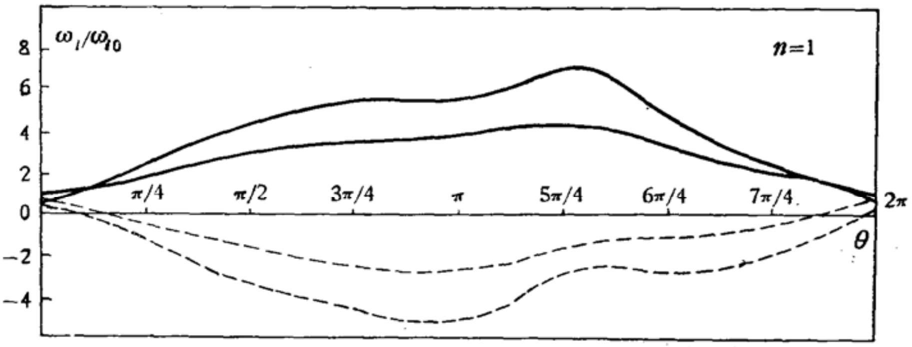

Fig. 7

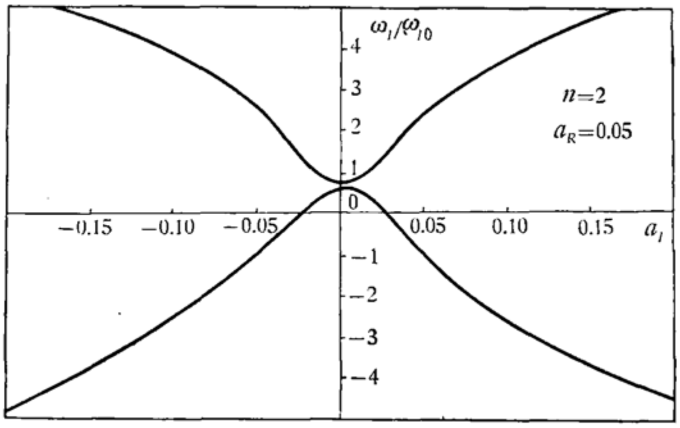

Fig. 8

there also exists another mode with a small growth rate. The relation of growth rate of the latter with the value of θ (shown as the imaginary line in Fig. 7) is just the reverse of that of the former. Only for $a=0$, the two modes are the same. This "split" phenomenon of the eigenfrequency spectrum caused by $|a| \neq 0$ is an interesting topic for further research.

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