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HYDRODYNAMICAL THEORY OF THREE-DIMENSIONAL DENSITY WAVE FOR SPIRAL STRUCTURE OF GALAXIES (II)

——GLOBAL MODE SOLUTION AND EFFECT OF THICKNESS

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Abstract

The present paper presents the solution of the equation governing density wave propagation in a galaxy disk with finite thickness, its global uniformly valid asymptotic solutions and the dispersion relation. By means of these solutions the influences of disk's thickness, the spiral arm's inclination and other physical factors at the corotation circle of the spiral galaxy are investigated. Results show that with other conditions kept invariant, the thicker the disk, the lower becomes the growth rate of the mode solution. From this it follows that in a thick lenticular or an elliptical galaxy, no spiral structure exists.

I. Introduction

This paper is a continuation of [1]. In [1], the perturbed density is expressed as,

$$\rho_1 = \hat{\rho}_1(r, z)H_m(\zeta)e^{i(\omega t - m\theta)}. \tag{1.1}$$

Then, from Poisson's equation and hydrodynamic equation we have derived the perturbed gravitational potential on the symmetric plane of galaxy in the first-order approximation of the thickness parameter ε and the "quasi-monochromatic wave" parameter ε_*,

$$\begin{cases} \psi_1|_{z=0} = \phi_1 H_m(\zeta)e^{i(\omega t - m\theta)}, \quad \tilde{\Phi}_1 = \tilde{\Phi}_{10} = \dfrac{z_{*1}^{(0)}}{k}(1 + \psi_{\varepsilon_0} + \psi_{\varepsilon_{*0}}) \\ \psi_{\varepsilon_0} = -D_0(r)k - D_1(r), \\ D_0(r) = 4.1588 z_{*1}^{(0)}\varepsilon(r), \quad D_1(r) = 1.8100 z_{*1}^{(0)}\dfrac{\varepsilon(r)C(r)}{\sigma_0(r)} + (8\ln 4)z_{*1}^{(0)}G\dfrac{d\varepsilon}{dr}, \\ \psi_{\varepsilon_{*0}} = 0(\varepsilon_*). \quad \left(\varepsilon_* = \dfrac{1}{k}\dfrac{d\ln k}{dr}\right) \end{cases} \tag{1.2}$$

We further obtained the equation governing wave propagation as follows:

$$G(1+\mu)\frac{dk}{dr} + F(1+\mu)k^2 + Gk\left[\frac{2k_0}{Q}D_0' - \frac{2k_0}{Q}\frac{F}{G}(1 - D_1) + R_{10}\right] - k_0^2(1-\nu^2)$$
$$+ R_{00} + \frac{2k_0}{Q}\left[G(D_1' - \psi_{\varepsilon_{*0}}') - GR_2(1 + \psi_{\varepsilon_0} + \psi_{\varepsilon_{*0}}) + \frac{R_{30}}{k}(1 + \psi_{\varepsilon_0} + \psi_{\varepsilon_{*0}})\right]$$

$$
\begin{aligned}
&- \frac{2k_0}{Q}\left[\frac{d}{dr}\beta(1+\phi_{\varepsilon_0}+\phi_{\varepsilon_{*0}}) + G\beta k(1+\phi_{\varepsilon_0}+\phi_{\varepsilon_{*0}}) + R_2\beta(1+\phi_{\varepsilon_0}+\phi_{\varepsilon_{*0}})\right] \\
&= -\frac{1}{\kappa\nu}\left(\frac{2m\Omega_0}{r}\right)\left\{\frac{2k_0}{Q}\left(-\varepsilon_* + \frac{R_{3*}}{k}\right) + GkR_{1*} + R_{0*}\right. \\
&\left. + \frac{2k_0}{Q}\left[\frac{d}{dr}\left(\frac{\phi_{\varepsilon_0}+\phi_{\varepsilon_{*0}}}{k}\right) + R_{3*}\frac{\phi_{\varepsilon_0}+\phi_{\varepsilon_{*0}}}{k} - \beta(1+\phi_{\varepsilon_0}+\phi_{\varepsilon_{*0}})\right]\right\}, \qquad (1.3)
\end{aligned}
$$

where

$$
\begin{cases}
Q = \dfrac{2\kappa M_r}{\varepsilon\rho_{00} z_{*1}^{(0)}}, \quad k_0 = \dfrac{\kappa}{M_r}, \quad \mu = \dfrac{2k_0 D_0}{Q}, \quad G = H'_m/H_m, \quad F = G^2 + G', \quad e = \dfrac{M_r}{M_\theta}, \\
s = \dfrac{d\ln\Omega_0}{d\ln r}, \quad \beta = \dfrac{1}{k}\dfrac{d\ln\tilde{\phi}_1}{dr} \approx \varepsilon_* - \dfrac{W_{*1}^{(0)}}{z_{*1}^{(0)}k}\dfrac{d(\varepsilon k)}{dr}, \quad \varepsilon\rho_{00} = \rho_{00}^{(0)} + \varepsilon\rho_{00}^{(1)} + \cdots
\end{cases} \qquad (1.4)
$$

$$
\begin{cases}
R_{00} = -\dfrac{d^2\ln\varepsilon\rho_{00}}{dr^2} - \dfrac{d\ln\varepsilon\rho_{00}}{dr}\dfrac{d}{dr}\ln\dfrac{M_r^2 r}{\kappa^2(1-\nu^2)} - \left(\dfrac{m}{er}\right)^2 + \dfrac{4m\Omega_0}{e^2 r}\dfrac{\nu'}{\kappa(1-\nu^2)}, \\
R_{0*} = \dfrac{d\ln\varepsilon\rho_{00}}{dr} + \dfrac{1}{e^2}\dfrac{d}{dr}\ln\left(\dfrac{M_\theta^2\Omega_0}{\kappa^2}\right), \\
R_{10} = \dfrac{d}{dr}\ln\left(\dfrac{M_r^2 r}{\varepsilon\rho_{00}\kappa^2}\right), \quad R_{1*} = \left(1 - \dfrac{1}{e^2}\right) \\
R_2 = \dfrac{d}{dr}\ln\left(\dfrac{\varepsilon\rho_{00} r}{\kappa^2(1-\nu^2)}\right), \quad R_{30} = \left(\dfrac{m}{r}\right)^2 + \left(\dfrac{4m\Omega_0}{r}\right)\dfrac{\nu'}{\kappa(1-\nu^2)}, \\
R_{3*} = \dfrac{d}{dr}\ln\left(\dfrac{\varepsilon\rho_{00}\Omega_0}{\kappa^2}\right), \quad \dfrac{4m\Omega_0\nu'}{\kappa r} = -\left[\left(\dfrac{2m\Omega_0}{\kappa r}\right)s + \dfrac{4m\Omega_0}{\kappa r}\nu\left(\dfrac{d\ln\kappa}{dr}\right)\right].
\end{cases} \qquad (1.5)
$$

In this paper. We adopt the "quasi-monochromatic wave" approximation (i.e. put $\varepsilon_* = 0$ in Eq. (1.3)), and, for the time being, owing to physical factors (according to C. C. Lin), do not consider the corotation response term in the right-hand side of Eq. (1.3). Now, set $\hat{x} = k/k_0$, then Eq. (1.3) can be rewritten as follows:

$$
\frac{d\hat{x}}{dr} = f\hat{x}^2 + g\hat{x} + h, \qquad (1.6)
$$

$$
\begin{cases}
f = -\dfrac{F}{G}k_0, \quad g = \dfrac{2Fk_0}{G}\dfrac{1-D_2}{(1+\mu)Q} - \dfrac{R_{10}+\mu R_2}{1+\mu} - \dfrac{d\ln k_0}{dr} \\
h = \dfrac{k_0}{(1+\mu)G}(1-\nu^2) + \dfrac{2}{(1+\mu)Q}\left[R_2(1-D_2) - D_2'\right] - \dfrac{R_{00}}{(1+\mu)Gk_0} \\
\qquad - \dfrac{2R_{30}}{(1+\mu)GQ}\left[\dfrac{1-D_2}{k_0\hat{x}} - D_0\right]; \quad D_2 = D_{2R} + iD_{2I} = \dfrac{G}{F}D_0' + D_1.
\end{cases} \qquad (1.7)
$$

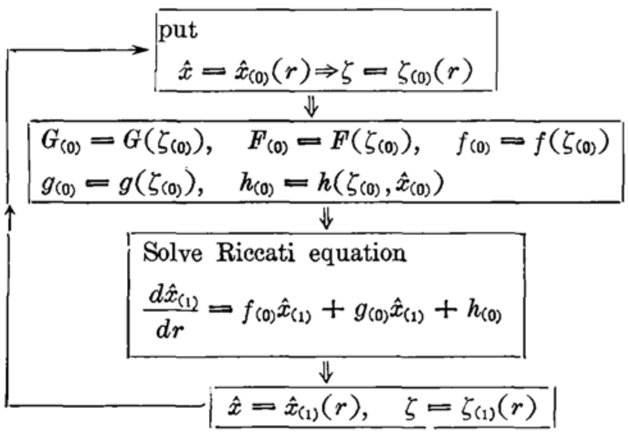
Obviously, if we conld find out the function $k(r, \varepsilon)$ and $\zeta(r, \varepsilon)$ from Eq. (1.6), then the perturbed density wave (1.1) will be completely determined. In [1], we have discussed the local approximate solutions of this equation. Now we shall deal with its global mode solutions and analyse the influence of disk's thickness on them.

II. Iterative Diagram for Treatment

For the equation governing wave propagation in this section, we shall present a diagram for the treatment of the complex nonlinear ordinary differential equation (1.6). It should be noted, to begin with, that the coefficients of this equation depend on parameter ε, while the unknown quantities (ζ, k, ω) are also the functions of ε. We are attempting to find out the first-order approximate solution in ε. As, in a disk galaxy, the relation $k_0 \gg 1$ generally holds, so

$$\varepsilon \ll \varepsilon k_0. \tag{2.1}$$

Now we can proceed to simplify further the first-order approximate solution in ε, so that there remain only the terms containing the parameter $\mu = O(\varepsilon k_0)$ without considering the other terms of $O(\varepsilon)$. In the calculation of the quantities of basic state, it is allowable to put $\varepsilon = 0$. Furthermore, it should also be noted that the unknown quantities (ζ, $\hat{x}$) are included in the functions G, F, f, g and h. In so doing Eq. (1.6) is not a common Riccati equation. For its solution, we adopt the following iterative process:



After iterating several times, the needed solution may be expected. But actually, as coefficients f, g, h are not sensitive to the change of $\hat{x}$, only one iteration would give a sufficiently good qualitative physical picture of the problem.

The problem on hand is closely related to solving a nonlinear Riccati equation. Now, let us put

$$\hat{x}_{(0)} = 1,\ \zeta_{(0)} = (kr)_{\max} + \int_{r_{\max}}^{r} kr\,dr, \tag{2.2}$$

(the subscript (k) in this paper denotes the kth iteration. Do not confuse it with the same notation in [1]), and make the following transformation,

$$W(r) = \exp\left[-\int \hat{x} f\,dr\right], \quad k = k_0\hat{x} = \frac{G}{F}\frac{d\ln W(r)}{dr}, \tag{2.3}$$

then Eq. (1.6) can be reduced to a second-order linear ordinary differential equation as follows:

$$W'' - \left(g + \frac{d\ln f}{dr}\right)W' + fhW = 0. \tag{2.4}$$

By further letting

$$W = u \exp\left[\frac{1}{2}\int\left(g + \frac{d\ln f}{dr}\right)dr\right], \tag{2.5}$$

We can change Eq. (2.4) into following standard form:

$$\frac{d^2u}{dr^2} + k_3^2(r, \omega)u = 0, \tag{2.6}$$

in which

$$\begin{cases} k_3^2(r, \omega) = fh - \frac{1}{4}\left(g + \frac{d\ln f}{dr}\right)^2 + \frac{1}{2}\frac{d}{dr}\left(g + \frac{d\ln f}{dr}\right), \\ \left(g + \frac{d\ln f}{dr}\right) = \frac{2k_0}{(1+\mu)Q}\left[\frac{F}{G}(1 - D_2)\right] - \frac{R_{10} + \mu R_2}{1+\mu} + \frac{d\ln F}{dr} - \frac{d\ln G}{dr}. \end{cases} \tag{2.7}$$

Returning to the original variable, we have

$$k = \frac{G}{F}\left[\frac{d\ln u}{dr} + \frac{1}{2}\left(g + \frac{d\ln f}{dr}\right)\right]. \tag{2.8}$$

For the solution of Eq. (2.6), we adopt the following boundary conditions which are the same as those used by C. C. Lin.

1) As $r \to 0$, $ue^{i\omega t}$ decays quickly; (2.9)

2) As $r \to \infty$, $ue^{i\omega t}$ satisfies radiation condition as an outgoing wave. (2.10)

In such case, our problem is reduced to an eigenvalue problem of a second-order ordinary differential equation.

III. Tightly Wound Spiral Global Mode Solution for "Lin's Disk"

From the discussion of local solution, it is known that provided that there is no other mechanism of instability working, if $\varepsilon \to 0$, we should have $Q \equiv 1$[1]. This means that when a disk of galaxy collapses entirely to a plane disk, it will be in a marginal Jeans' stable state everywhere. Now let us imagine, starting from this plane disk state, keeping the values of M_r, σ_0(or Q, σ_0) invariant, but gradually increasing the Z-direction dispersion velocity M_z in disk's central part, then we can get a galaxy disk of such a type that in the central part the thickness would gradually increase to a nuclear ball (as shown in Fig. 1). Such a type of galaxy disk is rather similar to that galaxy model as envisaged by Lin with a central rigid nuclear ball added to an infinitesimally thin disk, which may be called "Lin's disk". But, it should be pointed out that there is a difference between our treatment and Lin's. For with ours, the whole galaxy disk is taken to be a self-gravitational material system without dividing it into disk component and nuclear ball component and without assuming the central part to be rigid.

For a tightly wound spiral wave, $|\zeta| \gg$ holds, so we could have the asymptotic expansions as follows:

$$\begin{cases} G(\zeta) \sim i\varepsilon - \dfrac{1}{2\zeta} + i\varepsilon \dfrac{1/4 - m^2}{2\zeta^2} + \cdots, \\ F(\zeta) \sim -1 - i\varepsilon \dfrac{1}{\zeta} + \dfrac{1/2 - m^2}{\zeta^2} + \cdots \end{cases} \tag{3.1}$$

Taking only the first term of them, and neglecting the higher order small terms $O(1/\zeta)$ in (2.7), we get,

$$k_3^2 \approx \frac{k_0^2}{1+\mu}\left[\frac{(1-D_{2R})^2}{(1+\mu)Q^2} - (1-\nu^2)\right]. \tag{3.2}$$

By introducing symbols,

$$Q_E = \frac{(1+\mu)^{1/2}}{1-D_{2R}}Q, \quad k_{0c} = \frac{k_0}{(1+\mu)^{1/2}}, \tag{3.3}$$

$$k_{3c} = k_{0c}^2\left[\frac{1}{Q_E^2} - 1 + \nu^2\right], \tag{3.4}$$

Eq. (2.5) is reduced to

$$\frac{d^2u}{dr^2} + k_{3c}^2 u = 0. \tag{3.5}$$

The form of this equation is the same as that of C. C. Lin[3]. But here the function $Q_E(r)$, due to $Q=1$, is completely determined by the profile of galaxy. At the corotation circle ($r = r_{co}$), because $\varepsilon = 0$, $Q(r_{co}) = 1$ holds. When r approaches the central part of galaxy, the value of function $Q_E(r)$ will keep on increasing with the increase of ε. As a special numerical example, we take the basic-state model (II) by Mark, *et al.*[4], and let $Z_{*1}^{(0)} = 0.5$, $\varpi = (0.95, 0.05)$, $a_1 = 1.2$, $a_2 = 0.8$, $a_s = 0.175$, $M_1 = 2.0$, $M_2 = 0.8$, $M_s = 0.54$, $\varepsilon(r) = \varepsilon_1 + \varepsilon_0 \exp\left[-\frac{1}{2}\left(\frac{r}{R_H}\right)^2\right]$, $x_j = \left[1 + \left(\frac{r}{a_j}\right)^2\right]^{1/2}$ ($j = 1, 2, s$).

And then,

$$\sigma_0(r) = \frac{4.5M_1}{\pi a_1^2}\frac{1}{x_1^{11}} - \frac{4.5M_2}{\pi a_2^2}\frac{1}{x_2^{11}} + \frac{M_s}{2\pi a_s^2}\frac{1}{x_s^3},$$

$$\Omega_0^2(r) = \frac{16}{105}\left[\frac{M_1}{4\pi a_1^3}H(x_1) - \frac{M_2}{4\pi a_2^3}H(x_2)\right] + \frac{M_s}{4\pi a_s^3}\frac{1}{x_s^3},$$

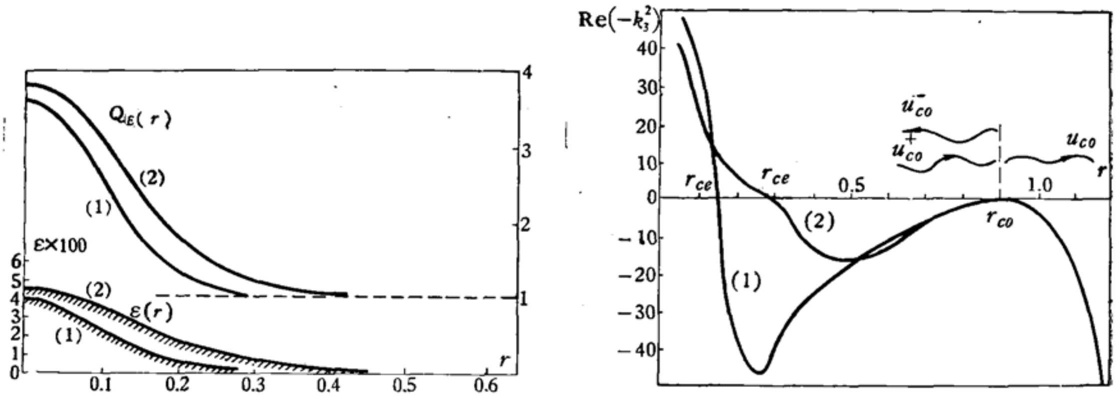
$$C(r) = \frac{4.5M_1}{\pi a_1^3}I(x_1) - \frac{4.5M_2}{\pi a_2^3}I(x_2) - \frac{M_s}{4\pi a_s^3}\left(\frac{1}{x_s^3} - \frac{3}{x_s^5}\right),$$

$$I(x) = \frac{5.5}{x^{13}} - \frac{2.5}{x^{11}} - \frac{5}{9x^9} - \frac{5}{21x^7} - \frac{5}{42x^5} - \frac{1}{18x^3},$$

$$H(x) = \frac{59.0625}{x^{11}} + \frac{26.25}{x^9} + \frac{16.875}{x^7} + \frac{11.25}{x^5} + \frac{6.5625}{x^3}.$$

Then we may draw the corresponding curves of function $Q_E(r)$ and those of function $k_{3c}^2(r)$ from various profiles of galaxy (see Fig. 1 a and b). It should be pointed out that in the density wave theory of plane disk, the distribution of the function $Q_E(r)$ is an uncertain one. To eliminate this uncertainty is of important theoretical significance. As has been noted by Panatoni (1979)[5], the global mode solution is very sensitive to the distribution of this function. Now, starting from the three-dimensional model of galaxy disk, we proposed an approach to the solution of this problem.

It can be seen from Fig. 1 b that the function $k_{3c}^2(r)$ has a double zero turning



a b

Fig. 1. The distribution of functions.

a) $Q_E(r)$; b) $K_3^2(r)$ for various "Lin's disks" (with the basic state model (I) by Mark, *et al.* and $[M]=10^{10}M_\odot$, $[R]=10\text{kpc}$).

point at $r=r_{co}$, and may also have a simple zero turning point at $r=r_{ce}$ inside the corotation circle. Analogous to the Schrödinger wave, the density wave will go to and fro by reflection in the region $r_{ce}\leqslant r\leqslant r_{co}$, but will be amplified continually at r_{co} to form a "waser". Therefore, for "Lin's disk" model, all the mathematical results obtained by C. C. Lin for density wave of plane disk may be brought here in a proper way. The quantum condition by him is changed to

$$\int_{r_{ce}}^{r_{co}} k_{3c}dr=\left(n+\frac{1}{2}\right)\pi+\frac{i}{4}\ln 2. \tag{3.6}$$

IV. Global Mode Solution of Galaxy Disk With Finite Thickness

In the treatment of "Lin's disk", we neglected the thickness effect, the effect of loosely wound character of spiral structure, and other higher order effects. Now suppose $Q=1+\eta\ (\eta>0)$ holds, then the thickness of disk at the corotation circle is larger than zero, i.e., $\varepsilon(r_{co})>0$. We further retain the first three terms of asymptotic expansions (3.1) for the consideration of the higher order effects. Thus, we have

$$\begin{cases}\dfrac{F}{G}\sim i\varepsilon-\dfrac{1}{2\zeta}-i\varepsilon\dfrac{m^2+3/4}{2\zeta^2}+\cdots,\\[2mm] \dfrac{1}{G}\sim -i\varepsilon-\dfrac{1}{2\zeta}-i\varepsilon\dfrac{m^2-3/4}{2\zeta^2}+\cdots,\\[2mm] \dfrac{d\ln F}{d\zeta}\sim -i\varepsilon\dfrac{1}{\zeta^2}+\cdots,\quad \dfrac{d\ln G}{d\zeta}\sim -i\varepsilon\dfrac{1}{2\zeta^2}+\cdots\end{cases} \tag{4.1}$$

and

$$\begin{aligned}k_3^2=k_{0c}^2\Big\{&\frac{1}{Q_E^2}\left(1+\frac{i\varepsilon}{\zeta}-\frac{m^2+1}{\zeta^2}\right)-(1-\nu^2)\left(1-\frac{1}{2\zeta^2}\right)+\frac{R_{00}}{k_0^2}\\&+\frac{2R_{30}}{k_0^2}\left(\frac{1-D_2}{Q\hat{x}_{(0)}}-\frac{\mu}{2}\right)+\frac{1}{Qk_0}\left(i\varepsilon-\frac{1}{2\zeta}\right)\left[\frac{1-D_2}{1+\mu}(R_{10}+\mu R_2)\right.\\&\left.+\frac{d}{dr}\ln\left(\frac{k_{0c}}{Q_E}\right)-2R_2(1-D_2)+2D_2'\right]-\frac{(R_{10}+\mu R_2)^2}{4k_0^2(1+\mu)}\end{aligned}$$

$$-\frac{1+\mu}{2k_0^2}\frac{d}{dr}\left(\frac{R_{10}+\mu R_2}{1+\mu}\right)\Big\},\tag{4.2}$$

$$k=-\left(i\sigma+\frac{1}{2\zeta}\right)\left[\frac{k_{0c}}{Q_E}\left(i\sigma-\frac{1}{2\zeta}\right)-\frac{R_{10}+\mu R_2}{1+\mu}+\frac{d\ln u}{dr}\right].\tag{4.3}$$

By denoting the typical value of $k_0(r)$ with $\lambda(\lambda\gg 1)$, the expression (4.2) may be rewritten as,

$$k_3^2=\lambda^2\left[f_0^2+\frac{f_1}{\lambda}\right],\tag{4.4}$$

in which the principal term

$$\lambda^2 f_0^2=k_{0c}^2\left[\frac{1}{Q_E^2}-\left(\frac{1}{Q_E^2}\right)_{r_{co}}-1+\nu_R^2\right]\tag{4.5}$$

is a real function and its behaviour is the same as the function $k_{3c}^2(r)$ of "Lin's disk" with a double zero point at the corotation circle r_{co}. Generally speaking, the function $k_3(r)$ itself has, instead of a double zero turning point near r_{co}, but two simple zero points situated close-by. Specifically, for a neutral tightly wound spiral wave, because

$$\begin{cases} Q_E(r_{co})>1, \quad k_3^2(r)\approx k_{3c}^2(r)=k_{0c}^2\left(\dfrac{1}{Q_E^2}-1+\nu^2\right),\\ k_3^2(r_{co})\approx k_{0c}^2\left(\dfrac{1}{Q_E^2}-1\right)\Big|_{r=r_{co}}<0,\end{cases}\tag{4.6}$$

these two simple zero points are all on the real axis (see Fig. 2b). In this case, our problem becomes analogous to Schrödinger's wave with penetration through a potential barrier.

Like Lin's disk, the principal term $\lambda^2 f_0^2$ may be assumed to have another simple zero point at r_{ce} inside the corotation circle, then the function $k_3^2(r)$ will have a complex simple zero turning point r'_{ce} near the point r_{ce}. The behaviours of the global mode solutions of galaxy disk with finite thickness are just determined by the characters and the distribution of these turning points. In the following the asymptotic solutions

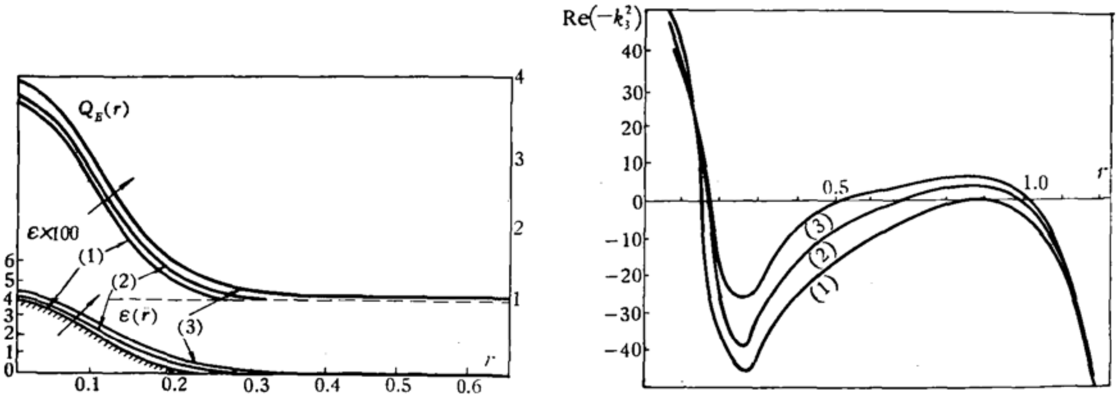


Fig. 2. The distribution of function.

a) $Q_E(r)$; b) $K_3^2(r)$ for disk-shaped galaxies with various thicknesses (with the basic state model (I) by Mark, *et al.* and $[M]=10^{10}M_\odot$, $[R]=10$ kpc).

of Eq. (2.6) with formula (4.4) should be found near $r = r_{co}$ and $r = r_{c*}$ respectively, and then have them smoothly connected in overlapped region for a global mode solution of this equation. This has been done in [2] for general case with the following results:

(1) *Uniformly valid asymptotic solutions near* $r = r_{co}$

We make the following power series expansions near r_{co},

$$\begin{cases}\lambda^2 f_0^2(r) = \lambda^2 A_0^2 (r - r_{co})^2 + \cdots, \\ \lambda f_1(r) = \lambda^2[-d_0 + d_1(r - r_{co}) + d_2(r - r_{co})^2 + \cdots].\end{cases} \tag{4.7}$$

Then it follows that,

$$k_3^2 = \lambda'^2[(r - r'_{co})^2 - d'_0 + \cdots], \tag{4.7$'$}$$

in which

$$\begin{cases}\lambda'^2 = \lambda^2(A_0^2 + d_2), \quad r'_{co} = r_{co} - \dfrac{d_1}{2(A_0^2 + d_2)}, \\ d'_0 = \dfrac{d_1^2}{4(A_0^2 - d_2)^2} + \dfrac{d_0}{A_0^2 + d_2} \approx \dfrac{d_0}{A_0^2} = -\dfrac{k_3^2(r_{co})}{\lambda'^2}.\end{cases} \tag{4.8}$$

From (4.7)$'$ we have derived two simple zero turning points r_{*1}, r_{*2} of $k_3^2(r)$ as

$$r_{*1} = r'_{co} + \sqrt{d'_0}, \quad r_{*2} = r'_{co} - \sqrt{d'_0}. \quad (0 \leqslant \arg(d'_0) < 2\pi) \tag{4.9}$$

Introduce a transformation $\xi = \xi(r)$, so that

$$\begin{cases}\tau = -\displaystyle\int_{r_{*2}}^{r} k_3 dr = a\,[t\sqrt{t^2 - 1} - \ln(t + \sqrt{t^2 - 1})], \\ t = \xi/\xi_{*2}.\end{cases} \tag{4.10}$$

in which the single-valued branch of the function $k_3(r)$ may be obtained in the same way as shown in [2].

Furthermore,

$$\xi_{*1} = \xi(r_{*1}) = -2\sqrt{a}, \quad \xi_{*2} = \xi(r_{*2}) = 2\sqrt{a}, \quad a = \frac{\lambda' d'_0}{2}. \tag{4.11}$$

We may make Langer's transformation,

$$\xi = \xi(r), \quad v = [\xi'(r)]^{\frac{1}{2}} u. \quad \left(\xi' = k_3(r)\Big/\left(\frac{\xi^2}{4} - a\right)^{\frac{1}{2}}\right) \tag{4.12}$$

As the lowest order approximation, Eq. (2.6) can be reduced to the following parabolic cylindrical equation:

$$\frac{d^2 v}{d\xi^2} + \left(\frac{\xi^2}{4} - a\right) v = 0. \tag{4.13}$$

Then the uniformly valid asymptotic solutions of Eq. (2.6) near r_{co} can be obtained:

$$u = k_3^{-\frac{1}{2}} \left(\frac{\xi^2}{4} - a\right)^{\frac{1}{4}} \cdot \{E(a, \xi); E_*(a, \xi)\},$$

where $E(a,\xi)$, $E_*(a,\xi)$ are parabolic cylindrical functions. When $|\xi| \gg 1$, its asymptotic expansions are:

$$E(a,\xi) \sim \sqrt{\frac{2}{\xi}}e^{i\Theta}, \quad E_*(a,\xi) \sim \sqrt{\frac{2}{\xi}}\,e^{-i\Theta}\Big(\,|\arg(\xi)| \leqslant \frac{\pi}{4}\Big), \tag{4.15}$$

$$\Theta = \frac{\xi^2}{4} - a\ln\xi + \frac{\pi}{4} + \frac{\phi_2}{2}, \quad \phi_2 = \arg\Gamma\left(\frac{1}{2} + ia\right). \tag{4.16}$$

Owing to

$$\lambda^2 f_0^2 = k_{0c}^2\left[\frac{m^2\Omega_0'^2(r_{co})}{\kappa^2}(r - r_{co})^2 + \cdots\right] = \lambda^2 A_0^2(r - r_{co})^2 + \cdots, \tag{4.17}$$

we get

$$\lambda'^2 = \lambda^2 A_0^2 = \left(\frac{k_{0c} m\Omega_0'}{\kappa}\right)^2_{r_{co}} \text{ or } \lambda' = \left|\frac{k_{0c} m\Omega_0'}{\kappa}\right|_{r_{co}}. \tag{4.18}$$

From this it follows that

$$a \approx -\frac{1}{2}\frac{k_3^2(r_{co})}{\lambda'} = -\left(\frac{\kappa}{2m\Omega_0}\frac{k_{0c}r}{|s|}\frac{k_3^2}{k_{0c}^2}\right)_{r_{co}}. \tag{4.19}$$

Specifically, for tightly wound spiral waves, from (4.6) we have

$$a = \left\{\frac{\kappa}{2m\Omega_0}\frac{k_{0c}r}{|s|}\left(1 - \frac{1}{Q_E^2} + \nu_l^2\right)\right\}_{r_{co}}. \tag{4.20}$$

Since $Q_E(r_{co}) > 1$, the value of a is positive real. The thicker the disk, the larger becomes the value of a.

After finding out solutions of $u(r)$, we may turn to the solution of the function $k(r)$ or $\hat{x}(r)$. To this end we need only use formula (4.3). For tightly wound spiral waves, the relation is simple. Indeed, as

$$u \propto \exp\left(\pm i\int_{r_{*2}}^{r} k_3 dr\right), \tag{4.21}$$

we have,

$$k \approx \frac{k_{0c}}{Q_E} \pm \epsilon k_3. \tag{4.22}$$

Furthermore, we know the group velocity C_g of density waves (1.1) can be written as,

$$C_g = -\epsilon\left(\frac{\partial\omega}{\partial k}\right) = \mp\left(\frac{\partial k_3}{\partial\omega}\right)^{-1} = \mp\frac{\kappa}{k_{0c}^2}\left(\frac{k_3}{\nu}\right). \tag{4.23}$$

From the way the single-valued function $k_3(r)$ was obtained, we know that the real part of the multiplier (k_3/ν) is always negative (i.e. $\mathrm{Re}(k_3/\nu) < 0$). Therefore, the group velocity of the branch wave corresponding to positive sign (+) in front of $k_3(r)$ in (4.21) is always positive. In this case, if $\epsilon = +1$ (which corresponds to a leading wave), its wave number k will take a larger value, so that this branch wave will be a short wave; If $\epsilon = -1$, (which corresponds to a trailing wave), its wave number k will take a smaller value, and then the branch wave will be a long wave. In Table 1, we have shown the types and directions of these branch waves. Table 1 shows that for trailing waves, the long wave will go to the corotation circle, while the short wave will depart from it; for leading waves, the situation is the reverse.

Table 1

	$\epsilon=-1$ (Trailing Wave)		$\epsilon=+1$ (Leading Wave)	
ν_R	<0	>0	<0	>0
$\mathrm{Re}(k_3)$	>0	<0	>0	<0
$\frac{d\ln u}{dr}=+ik_3$	$C_g>0$ (Long)	$C_g>0$ (short)	$C_g>0$ (Short)	$C_g>0$ (Long)
$\frac{d\ln u}{dr}=-ik_3$	$C_g<0$ (Short)	$C_g<0$ (Long)	$C_g<0$ (Long)	$C_g<0$ (Short)

From the above, it is obvious that if we want the boundary condition (2.10) to be satisfied and the density wave to become an outgoing short wave outside the corotation circle (i.e. take the radiation condition of short wave to simulate the short wave absorbed by outer Lindblad response as was done by Lin), then we must take $\epsilon=-1$, and

$$u_{co}=k_3^{-\frac{1}{2}}\left(\frac{\xi^2}{4}-a\right)^{\frac{1}{4}}\cdot E_*(a,\xi e^{-i\pi}). \tag{4.24}$$

Inside the corotation circle, owing to $\arg(\xi)\approx 0$, we get

$$\left.\begin{aligned} u_{co}&=u_{co}^{+}+u_{co}^{-},\\ u_{co}^{+}&=ie^{\pi a}E_*(a,\xi),\\ u_{co}^{-}&=i-\frac{\sqrt{2\pi}\exp\left(\frac{\pi a}{2}-i\phi_2\right)}{\Gamma\left(\frac{1}{2}-ia\right)}E(a,\xi)\end{aligned}\right\}\times k_3^{-\frac{1}{2}}\left(\frac{\xi^2}{4}-a\right)^{\frac{1}{4}}, \tag{4.25}$$

where u_{co}^{+} corresponds to the outgoing trailing long wave, and u_{co}^{-} to the ingoing trailing short wave. For the tightly wound spiral wave, because the value of a is real, and

$$\Gamma\left(\frac{1}{2}\pm ia\right)=\sqrt{\frac{\pi}{\mathrm{ch}(\pi a)}}\,e^{\pm i\phi_2}, \tag{4.26}$$

we get

$$\left.\begin{aligned} u_{co}^{+}&=ie^{\pi a}E_*(a,\xi)\\ u_{co}^{-}&=i-\sqrt{1+e^{2\pi a}}E(a,\xi)\end{aligned}\right\}\times k_3^{-\frac{1}{2}}\left(\frac{\xi^2}{4}-a\right)^{\frac{1}{4}}. \tag{4.25$'$}$$

In this cose, there exists a "potential barrier" near the corotation circle (in the region $|\xi|\leqslant 2\sqrt{a}$). The thicker the disk, the higher becomes the "potential barrier". There exists no wave within the barrier. When an outgoing long wave u_{co}^{+} meets the potential barrier, one part of it penetrates through the potential barrier due to the "tunnel effect", and becomes an outgoing short wave, while the other part will be reflected and becomes an ingoing short wave u_{co}^- (see Fig. 3). The reflection ratio R and the transmission ratio T are respectively as follows:

$$R = \sqrt{1 + e^{-2\pi a}}, \quad T = e^{-\pi a}. \tag{4.27}$$

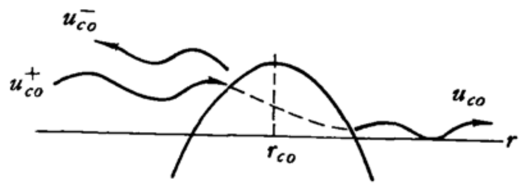


Fig. 3. The sketch of wave's propagation near the corotation circle potential barrier.

(2) *Uniformly valid asymptotic solutions near* r_{ce}

Let

$$\zeta_c = \int_{r'_{ce}}^{r} k_3 dr,$$

Then the asymptotic solution near r_{ce} satisfying the boundary condition (2.8) is,

$$u_{ce} = \left(\frac{\zeta_c}{k_3}\right)^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)} \ (\zeta_c e^{-i\pi}). \tag{4.28}$$

In the region $r > r_{ce}$, we have

$$u_{ce} = u_{ce}^+ + u_{ce}^-,$$
$$u_{ce}^+ = \left(\frac{\zeta_c}{k_3}\right)^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)}(\zeta_c), \qquad u_{ce}^- = \left(\frac{\zeta_c}{k_3}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{3}} H^{(2)}{}_{\frac{1}{3}}(\zeta_c). \tag{4.29}$$

(3) *Dispersion relation*

In the overlapped region ($r_{ce} \ll r \ll r_{co}$), the solutions u_{co}, u_{ce} are all available. By smoothly connecting them in this region, we obtain the desired global mode solutions and the following quantum condition:

$$\phi = \int_{r'_{ce}}^{r_{*1}} k_3 dr = \left(n + \frac{1}{2}\right)\pi + \frac{a}{2}(\ln a - 1) - \frac{ia\pi}{4} - \frac{i}{2}\ln\left[\frac{\Gamma\left(\frac{1}{2} - ia\right)}{\sqrt{2\pi}}\right], \tag{4.30}$$

from which we can solve the eigenfrequencies $\{\omega\}$.

For the tightly wound spiral wave in particular, from (4.30), we obtain

$$\phi = \left(n + \frac{1}{2}\right)\pi + \frac{a}{2}(\ln a - 1) - \frac{\phi_2}{2} + \frac{i}{4}\ln(1 + e^{-2\pi a}). \tag{4.31}$$

For the eigenfrequencies with small imaginary part (i.e. $\omega = \omega_R + i\omega_I$, $|\omega_I| \ll |\omega_R|$), the above expression can be simplified to:

$$\begin{cases} \int_{r'_{ce}}^{r_{*2}} (k_3)_{\omega_R} dr = \left(n + \frac{1}{2}\right)\pi + \frac{\phi_0 - \phi_2}{2}, \\ \omega_I \tau_* = \frac{1}{2} \ln \Lambda, \end{cases} \tag{4.32}$$

in which we have adopted the following relations:

$$\phi_0 = \frac{a}{2}(\ln a - 1), \quad \Lambda = \sqrt{1 + e^{-2\pi a}}, \quad \tau_* = \int_{r'_{ce}}^{r_{*2}} \left(\frac{\partial k_3}{\partial \omega}\right)_{\omega_R} dr. \tag{4.33}$$

For "Lin's disk", as $\mu(r_{co}) = a = 0$, $\Lambda = \sqrt{2}$, then formula (4.31) is reduced to formula (3.6).

For the case of galaxy disk with finite thickness, we can introduce a β_a-multiplier as defined below:

$$\beta_a = \frac{\omega_I \tau_*}{(\omega_I \tau_*)_{a=0}} = 2 \frac{\ln \Lambda}{\ln 2}. \tag{4.34}$$

In so doing, β_a-multiplier reflects the influence of the thickness of disk on growth rate of the mode, its curve being shown in Fig. 4. Provided other conditions are kept invariant, and let the galaxy disk gets thicker increasingly, the no-wave region near the corotation circle will get larger and larger (see Fig. 2 a and b), and the growth rate of the mode solution will become smaller and smaller, so that the spiral design becomes increasingly fainter until it vanishes. From the above reasoning it can be naturally deduced that in a thicker lenticular galaxy and an elliptical galaxy the spiral design cannot be present. This is in agreement with observations.

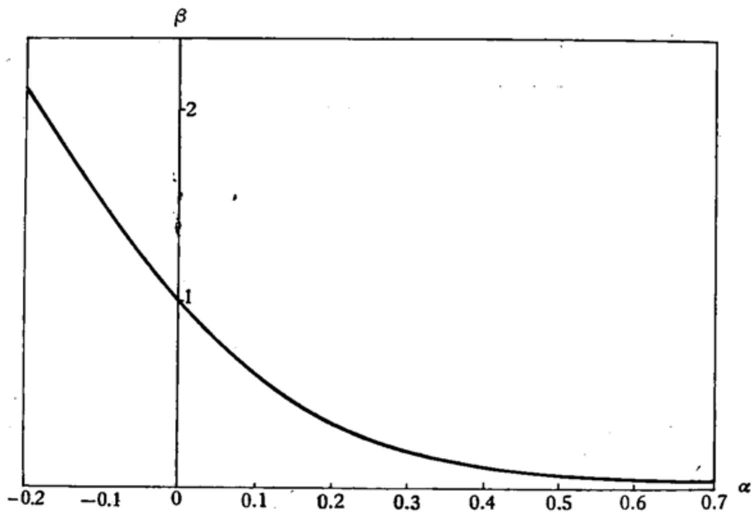


Fig. 4. The β_a-multiplier versus the parameter a.

V. Influence of Other Physical Factors at Corotation Circle on Spiral Structure

We have already discussed the effect of thickness of the dick for tightly wound spiral wave. Now we shall discuss the loosely wound spiral wave. From (4.2), and

(4.9), we get

$$a = a_R + i\varepsilon a_I, \tag{5.1}$$

$$a_R = X + Y + Z, \tag{5.2}$$

$$\begin{cases} X = \left(\dfrac{1+s/2}{1+\mu}\right)^{\frac{1}{2}} \dfrac{1}{i_a|s|}\left(1 - \dfrac{1}{Q_E^2} + \nu_I^2\right) > 0, \\ Y = -\left(\dfrac{1+s/2}{1+\mu}\right)^{\frac{1}{2}} \dfrac{i_a}{|s|}\left(2 - \dfrac{1}{e^2} - \dfrac{1}{Q_E^2}\right) < 0, \\ Z = \dfrac{i_a \mathrm{sgn}(s)}{(1+\mu)^{1/2}(1+s/2)^{1/2}}\left(\dfrac{1}{e^2} + \dfrac{2}{Q}\ (1 - D_{2R}) - \mu\right) < 0, \end{cases} \tag{5.3}$$

$$a_I = -\left(\frac{1+s/2}{1+\mu}\right)^{\frac{1}{2}} \frac{1}{i_a|s|}\left\{\frac{1}{Qk_0}\frac{d}{dr}\ln\left(\frac{Q^2(1-\nu^2)}{M_r^2}\right) + \frac{1-D_{2R}}{Q^2k_0}\left[\frac{1-Q-\mu}{r}\right.\right.$$
$$\left.\left. + Q\mu\frac{d}{dr}\ln\left(\frac{\varepsilon\rho_{00}k_0}{Q}\right)\right] - \frac{1}{Qk_0}\left[D_{2R}\frac{d}{dr}\ln\left(\frac{Q^2(1-\nu^2)}{\varepsilon\rho_{00}}\right) + \mu' - D'_{2R}\right]\right\}, \tag{5.4}$$

in which all physical quantities on the right-hand side are evaluated at $r = r_{co}$, and the following symbol has been adopted,

$$i_a = \frac{m}{k_0 r}. \tag{5.5}$$

It should be noted that when we derived the value of a from $k_3^2(r_{co})$, we neglected the higher order small terms $O(1/\zeta^2)$ but kept the terms $O(m^2/\zeta^2)$.

It may be assumed that $d/dr \ln(Q^2/M_r^2) \approx 0$ near the corotation circle. In this case, the imaginary part a_I is an order magnitude smaller than the real part a_R, and its influence may be ignored. From formula (5.3), it may be seen that (i) presence of parameters η (the degree of Jeans' stability) and μ (the thickness of disk) will enchance the value of a_R, thus decreasing the growth rate of the mode solution; (ii) presence of parameters i_a (the inclination of spiral arm) and $/s/$ (the shear multiplier) will decrease the value of a_R thus increasing the growth rate of the mode solution. This indicates again that the loosely wound effect can excite a bigger instability. In our previous work[3], we have discussed such a gravitational instability caused by loosely wound spiral wave, and called it "loosely wound instability".

VI. Conclusions

The results obtained in the present paper can be briefly summarized as follows:

(1) The uncertain function $Q_E(r)$ in the equation governing density wave propagation on a plane disk can be determined by the profile of galaxy disk with finite thickness.

(2) The "waser" mechanism and existence of growth-type mode solutions proposed by Lin, *et al.* for the plane disk galaxy exist also for galaxies of "Lin's disk" model. It must, however, be noted that for a general galactic disk with finite thickness, such "waser" mechanism will be modulated by the thickness of disk, the loosely wound character and other physical factors through the parameter a at the corotation circle.

(3) The presence of parameters (η, μ) will raise the potential barrier of the corotation circle, thus decreasing the growth rate of the mode solutions, while the presence of parameters $(i_a, /s/)$ will lower that potential barrier, thus increasing the growth rate of the mode solutions. When the other conditions remain invariant, the greater the thickness of a galactic disk, the fainter becomes its spiral design.

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