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ON FROZEN-IN AND RESISTIVE FORCE-FREE MAGNETIC FIELDS

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ABSTRACT

By taking the variation of the magnetic energy of a given system $\frac{1}{8\pi}\int_v (\nabla \times \mathbf{A})^2 d\tau$ with the constraint that $\int_v \mathbf{A}\cdot\nabla\times\mathbf{A}d\tau = $ const. and under the condition that the potential part $\nabla\phi$ of E is defined as $\nabla\phi = \frac{c}{4\pi\sigma}\nabla\int \alpha\,\mathrm{B}.d\mathrm{l}$ which ensures that $\frac{\partial \mathrm{A}}{\partial t}.\mathrm{B} = 0$, it will be shown that the force-free factor α is a constant for a stable magnetic field.

1. INTRODUCTION

A tenuous ionised gas carries a strong magnetic field; when the pressure gradient ∇P is less than the magnetic pressure gradient $\nabla B^2/8\pi$ by more than one order of magnitude its electromagnetic body force will be nearly zero, and we have a force-free field, whose equations are

$$\nabla \times \mathbf{B} = \alpha\mathbf{B}, \tag{1}$$

$$\frac{4\pi\mathbf{J}}{c} = \nabla \times \mathbf{B}. \tag{2}$$

α is the force-free factor, and is, in general, a function of position and time. It is a topic of interest in both astrophysics and controlled thermonuclear reactions.

It was proved by Lundquist [1] that for a static fluid whose magnetic field decays without distortion α must be a constant. Chandrasekhar and Woltjer [2] proved that a force-free field with a constant α is a field with a given magnetic energy that has the smallest Ohmic dissipation. Woltjer [3], by taking the variation of the magnetic energy of a system under a given constraint, found that for a frozen-in, closed system, a constant α represents a state of least magnetic energy and also that if the fluid is static, then α is a constant. Jette [4] proved that for a resistive force-free field, if it is static, then α is a constant.

No matter what its initial state is, a mass of tenuous conducting gas constrained by a strong magnetic field will evolve towards a state of stable equilibrium when its potential energy (that is, magnetic energy) will be at a minimum. In this paper, by taking the variation of the magnetic energy in a fixed volume of space, it will be shown that, whether the gas is static or in motion, and whether its conductivity is infinite or finite, a stable force-free field must have a constant α. Also, by taking the variation of the Ohmic dissipation in a fixed volume, it will be shown that a constant α represents minimum Ohmic dissipation in some particular cases.

2. VARIATIONAL TREATMENT

Following [3], we take the variation of the magnetic energy Q^B under the constraint of a force-free field. The magnetic energy is

$$Q^B = \frac{1}{8\pi}\int_v \mathbf{B}^2 d\tau. \quad (3)$$

where v represents a fixed volume of space.

We shall prove below that the constraint representing a force-free field should be

$$\int_v \mathbf{A}\cdot\mathbf{B}d\tau = \text{const} \quad (4)$$

where A is the magnetic vector potential, that is

$$\mathbf{B} = \nabla\times\mathbf{A}. \quad (5)$$

Introducing arbitrary variation ∇A and Lagrange multiplier $-\alpha/8\pi$, we have

$$\begin{aligned}\delta Q &= \delta Q^B - \frac{\alpha}{8\pi}\delta\int_v \mathbf{A}\cdot\nabla\times\mathbf{A}d\tau \\ &= \frac{1}{8\pi}\int_\Gamma \hat{n}\cdot(-2\nabla\times\mathbf{A} + \alpha\mathbf{A})\times\delta\mathbf{A}d\sigma + \frac{1}{4\pi}\int_v [\nabla\times(\nabla\times\mathbf{A}) \\ &\quad - \alpha\nabla\times\mathbf{A}]\delta\mathbf{A}d\tau = 0\end{aligned} \quad (6)$$

where $d\sigma$ is an elemental area on the boundary Γ and $\hat{n}$, the unit vector along its outward normal. On Γ, δA should be set to zero, hence (6) simplifies into

$$\mathbf{J} = \alpha\mathbf{B} \quad (7)$$

where α is a constant. From the foregoing, we see that (4) represents the constraint of a force-free field. Hence for a stable force-free field, α must be a constant, and this can also be understood as the final state of a force-free field.

3. BOUNDARY CONDITIONS SATISFYING (4)

We now seek boundary conditions that satisfy (4). From the latter, we have

$$\frac{\partial}{\partial t}\int_v \mathbf{A}\cdot\nabla\times\mathbf{A}d\tau = \int_\Gamma \hat{n}\cdot\frac{\partial\mathbf{A}}{\partial t}\times\mathbf{A}d\sigma + 2\int_v \frac{\partial\mathbf{A}}{\partial t}\cdot\nabla\times\mathbf{A}d\tau. \quad (8)$$

Introduce Ohm's Law:

$$\frac{\mathbf{J}}{\sigma} = \mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}. \quad (9)$$

The electric field E should be

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi. \quad (10)$$

$\nabla\phi$ is potential gradient and, in the usual electrodynamical theory, it is used to adjust $\nabla\cdot A$ to zero. But here, we shall use a gauge that will ensure

$$\frac{\partial \mathbf{A}}{\partial t}\cdot\nabla\times\mathbf{A}=0. \tag{11}$$

Scalar multiply Eq (9) by B and using the last, we have

$$\frac{\mathbf{J}\cdot\mathbf{B}}{\sigma}=-\mathbf{B}\cdot\nabla\phi. \tag{12}$$

Integrating this and using (1), we have

$$\phi=-\int_0^l\frac{c\alpha\mathbf{B}\cdot d\mathbf{l}}{4\pi\sigma}. \tag{13}$$

Here dl is an element of the field line, and the integration begins at the point where $\phi = 0$. This is the special gauge used in the present paper. Substituting (11) into (8), we have

$$\frac{\partial}{\partial t}\int_v\mathbf{A}\cdot\nabla\times\mathbf{A}d\tau=\int_\Gamma\hat{n}\cdot\frac{\partial\mathbf{A}}{\partial t}\times\mathbf{A}d\sigma. \tag{8'}$$

This agrees with the result given in [3] on frozen-in, force-free fields. We now introduce unit vector $\hat{A}$ and generalize the above into

$$\frac{\partial}{\partial t}\int_v\mathbf{A}\cdot\nabla\times\mathbf{A}d\tau=\int_\Gamma A^2\hat{n}\cdot\frac{\partial\hat{A}}{\partial t}\times\hat{A}d\sigma. \tag{8''}$$

Eq. (8") shows that on the boundary Γ, one of the two following conditions must be satisfied in order that (4) may hold:

$$\begin{cases}(a)\ A_\Gamma=0, & (14)\\ (b)\ \left.\dfrac{\partial\hat{A}}{\partial t}\right|_\Gamma=0. & (15)\end{cases}$$

(i) Physical Meaning of $A_\Gamma = 0$

From (5) we know that

$$\mathbf{B}=\nabla A\times\hat{A}+A\nabla\times\hat{A}. \tag{16}$$

Introducing condition (a), we have

$$\mathbf{B}_\Gamma=\nabla A|_\Gamma\times\hat{A}_\Gamma. \tag{17}$$

Hence the boundary represented by $A_\Gamma = 0$ is a magnetic surface which is, moreover, fixed in space; this is one kind of boundary in a stable force-free field.

(ii) Physical Meaning of $\left.\dfrac{\partial\hat{A}}{\partial t}\right|_\Gamma=0$

Substituting (13) into (10), then substituting the result into (9) and using condition (b), we have, on Γ,

$$\mathbf{J}_\Gamma/\sigma=-\frac{1}{c}\left.\frac{\partial A}{\partial t}\right|_\Gamma\hat{A}_\Gamma+\frac{c}{4\pi\sigma}\nabla_\Gamma\int_0^l\alpha\mathbf{B}\cdot d\mathbf{l}+\mathbf{V}_\Gamma\times\mathbf{B}_\Gamma/c. \tag{18}$$

which can be written as

$$-\frac{1}{c}\frac{\partial A}{\partial t}\bigg|_{\Gamma}\hat{A}_{\Gamma}+\frac{c}{4\pi\sigma}\nabla_{\Gamma}\int_0^l \alpha\mathbf{B}dl-\mathbf{J}_{\Gamma}/\sigma=\mathbf{B}_{\Gamma}\times\mathbf{V}_{\Gamma}/c,$$

Introduce the definitions

$$\mathbf{E}_{\Gamma}=-\frac{1}{c}\frac{\partial A}{\partial t}\bigg|_{\Gamma}\hat{A}_{\Gamma}+\frac{c}{4\pi\sigma}\nabla_{\Gamma}\int_0^l \alpha\mathbf{B}dl,$$

$$\mathbf{E}_{\Gamma}^{(e)}=\mathbf{E}_{\Gamma}-\mathbf{J}_{\Gamma}/\sigma=-\frac{1}{c}\frac{\partial A}{\partial t}\bigg|_{\Gamma}\hat{A}_{\Gamma}+\frac{c\alpha}{4\pi\sigma}\nabla_{\Gamma\perp}\int_0^l \bar{B}\cdot d\bar{l} \tag{19}$$

where $E_{\Gamma}^{(e)}$ is the effective electric field and $\nabla_{\Gamma\perp}$ is the gradient perpendicular to $\hat{B}$. Taking the components of this last along the $\hat{A}$, $\hat{B}$, $\hat{T}$ directions, $\hat{T}$ being defined by

$$\hat{A}\times\hat{B}=\hat{T}. \tag{20}$$

we have

$$\begin{cases} E_{\Gamma\hat{B}}^{(e)}=0, & (21)\\ E_{\Gamma\hat{A}}^{(e)}=-\frac{1}{c}\frac{\partial A}{\partial t}\bigg|_{\Gamma}+\frac{c}{4\pi\sigma}\hat{A}\cdot\nabla\int_0^{l_\Gamma}(\alpha B)\hat{B}\cdot d\mathbf{l}-\mathbf{J}_{\Gamma\hat{A}}/\sigma=B_{\Gamma}V_{\Gamma\hat{T}}/c & (22)\\ E_{\Gamma\hat{T}}^{(e)}=+\frac{c\alpha}{4\pi\sigma}\hat{T}\cdot\nabla\int_0^{l_\Gamma}B\hat{B}\cdot d\mathbf{l}-\mathbf{J}_{\Gamma\hat{T}}/\sigma=-B_{\Gamma}V_{\Gamma\hat{T}}/c & (23)\end{cases}$$

These tell us that when $\frac{\partial\hat{A}}{\partial t}\Big|_{\Gamma}=0$, $\hat{A}_{\Gamma}$ is perpendicular to $\hat{B}$ and the $\hat{T}$-component of the effective electric field exists only in the resistive case, and vanishes as $\sigma\to\infty$, when the form of E_{Γ} does not change. This is the other set of boundary conditions that a stable force-free field must have. The physical meaning of $\frac{\partial\hat{A}}{\partial t}=0$ will be discussed in another paper.

4. PHYSICAL PICTURES

No further physical quantities can be evaluated by the variational method used here. To clarify the physical picture of a stable force-free field, I now introduce two assumptions (their proof has been obtained by me in another work).

(A) If α is a constant, then $\left|\frac{\partial\hat{B}}{\partial t}\right|$ is zero, that is ,

$$\frac{\partial\hat{B}}{\partial t}\bigg| \quad =0; \tag{24}$$

(B) The necessary and sufficient conditions for a constant α are

$$\frac{\partial}{\partial t}\hat{A}=0, \tag{25}$$

$$\nabla\times D=\alpha D, \tag{26}$$

where D is defined by

$$\mathbf{D}=\nabla\times(\mathbf{V}\times\mathbf{B})=\beta\mathbf{B}. \tag{27}$$

Under these assumptions, we take the curl of Eq.(9) and obtain

$$\frac{c\alpha^2\mathbf{B}}{4\pi\sigma}=-\frac{1}{c}\frac{\partial B}{\partial t}\hat{B}+\mathbf{D}/c. \tag{28}$$

From (27) and (26) we have

$$\nabla \cdot \mathbf{D} = 0. \tag{29}$$

hence

$$\hat{\mathcal{B}} \cdot \nabla\beta = 0. \tag{30}$$

Substituting (27) into (26) gives

$$\nabla\beta \times \hat{\mathcal{B}} = 0. \tag{31}$$

(30) and (31) show that β is a position-independent function. Hence on integrating (28), we have

$$\mathbf{B} = \mathbf{B}_0 \exp\left(-\frac{c^2\alpha^2}{4\pi\sigma}t + \int_0^t \beta\, dt\right). \tag{32}$$

If the fluid is static, then $\beta = 0$, and the result in [1] will be recovered. B_0 is a function of space only and (32) states that because of the resistence, the magnetic field decays with the factor exp $(-c^2\alpha^2 t/4\pi\sigma)$ and the motion of the fluid produces a Poyting energy flux, causing B to increase by the factor exp $\int_0^t \beta dt$.

Integrating (27), we have

$$\mathbf{V} \times \mathbf{B} = \beta\mathbf{B}/\alpha + \nabla\Theta. \tag{33}$$

Similar to the gauge (13) imposed on ϕ, we now set

$$\Theta = -\frac{\beta}{\alpha}\int_0^l \mathbf{B} \cdot d\mathbf{l}. \tag{34}$$

Substituting (33) and (34) into (18) gives

$$\left.\frac{c\alpha\mathbf{B}}{4\pi\sigma}\right|_r = \left[-\frac{\partial A}{c\partial t}\hat{A} + \nabla\int_0^l \left(\frac{c\alpha}{4\pi\sigma} - \frac{\beta}{\alpha c}\right)\mathbf{B} \cdot d\mathbf{l} + \beta\mathbf{B}/\alpha c\right]_r. \tag{35}$$

From (32), (35) and (28), we then have

$$\left.\begin{aligned} A &= A_0 \exp\left(-\frac{c^2\alpha^2}{4\pi\sigma}t + \int_0^t \beta\, dt\right). \\ \mathbf{A}_0 &= \mathbf{B}_0/\alpha - \nabla\int_0^l \mathbf{B}_0 \cdot d\mathbf{l}/\alpha. \end{aligned}\right\} \tag{36}$$

Substituting these into (10) gives

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \exp\left(-\frac{c^2\alpha^2}{4\pi\sigma}t + \int_0^t \beta\, dt\right), \\ \mathbf{E}_0 &= \left(\frac{c\alpha^2}{4\pi\sigma} - \beta/c\right)\mathbf{A}_0 + \frac{c\alpha}{4\pi\sigma}\nabla\int_0^l \mathbf{B}_0 \cdot d\mathbf{l}. \end{aligned}\right\} \tag{37}$$

From these and (9), we have

$$\mathbf{V} = c\mathbf{E}_0 \times \hat{\mathcal{B}}/B_0. \tag{38}$$

Note that V is independent of time, so is a stationary drift field; this shows that the magnetic field and the plasma do not exchange mechanical energy, which is a property that should be possessed by a force-free field. From (37), we see that, as $\sigma \to \infty$, E_0 does not change its form, in agreement with the results at (22) and (23). Also, (37) shows that the $\hat{B}$ component of the electric field E balances the potential fall of the current, and its component perpendicular to $\hat{B}$ is along $\hat{A}$, and is controlled by the Joule current and the fluid motion.

5. MINIMUM OHMIC DISSIPATION

As in the variational treatment of the magnetic energy, we can, under the constraint,

$$\int_v \mathbf{B} \cdot \nabla \times \mathbf{B} d\tau = \text{const} \tag{39}$$

take the variation of the Ohmic dissipation

$$Q^a = \frac{c^2}{16\pi^2\sigma} \int_v (\nabla \times \mathbf{B})^2 d\tau \tag{40}$$

in a fixed volume of space v. With variation δB and Lagrange's multiplier $-c^2\alpha/16\pi^2\sigma$, we have corresponding to (6),

$$\begin{aligned} \delta Q &= \delta Q^{(a)} + \delta\left[-\frac{c^2\alpha}{16\pi^2\sigma}\int_v \mathbf{B} \cdot \nabla \times \mathbf{B} d\tau\right] \\ &= \frac{c^2}{16\pi^2\sigma}\int_v [\nabla \times (\nabla \times \mathbf{B}) - \alpha\nabla \times \mathbf{B}] \cdot \delta\mathbf{B} d\tau = 0. \end{aligned} \tag{41}$$

This gives

$$\nabla \times \mathbf{J} = \alpha \mathbf{J}, \tag{42}$$

with α a constant.

Corresponding to (8), we now obtain from (39),

$$\begin{aligned} &\int_v \left(\frac{\partial \mathbf{B}}{\partial t} \cdot \nabla \times \mathbf{B} + \mathbf{B} \cdot \nabla \times \frac{\partial \mathbf{B}}{\partial t}\right) d\tau \\ &= \int_\Gamma \mathbf{B}^2 \hat{n} \cdot \frac{\partial \hat{B}}{\partial t} \times \hat{B} d\sigma + 2\int_v \frac{\partial \mathbf{B}}{\partial t} \cdot \nabla \times \mathbf{B} d\tau = 0. \end{aligned} \tag{43}$$

This last is satisfied by two sorts of conditions, one sort referring to the boundary:

$$(a) \quad \left.\frac{\partial \hat{B}}{\partial t}\right|_\Gamma = 0, \tag{44}$$

$$(b) \quad B_\Gamma = 0. \tag{45}$$

The condition (44) is the same as the condition (24), and is one that must be satisfied by all cases with constant α. Condition (45) states that there is no magnetic field on the boundary, and since according to the Virial Theorem [5], a force-free field cannot exist in the interior of a system, we shall not discuss this case further. The other sort of conditions come from equating the last integral of (43) to zero,

$$\int_v \frac{\partial \mathbf{B}}{\partial t} \cdot \nabla \times \mathbf{B} d\tau = \int_v \alpha B \frac{\partial B}{\partial t} d\tau = 0. \qquad (46)$$

which gives

$$(c)\ \alpha = 0, \qquad (47)$$

$$(d)\ \frac{\partial B}{\partial t} = 0. \qquad (48)$$

Condition (47) states that there is no current in the interior of the system; by combining this with (44) we see that (32) and (37) are the solution when $\alpha = 0$. Similarly, condition (48) is a particular case of (32) with $\beta = c^2\alpha^2/4\pi\sigma$.

The above analysis shows that α = const also represents the state of minimum Ohmic dissipation, and that it is a particular case of minimum energy when $\alpha = 0$ or $\partial B/\partial t = 0$.

6. DISCUSSION AND CONCLUSION

1. A constant α represents the state of minimum magnetic energy of a frozen-in and resistive field, a stable force-free field, or the end configuration of a force-free field. Its physical meanings are: (i) The magnetic field does not change its form, (ii) the field decays with the factor exp $(-c^2\alpha^2 t/4\pi\sigma)$, and also increases by the factor exp $\int_0^t \beta dt$, (iii) the $\hat{B}$ component of the electric field is the potential fall in the current, or the effective electric field is perpendicular to B, and (iv) the $V_\perp$ of the gas is a drift induced by the effective electric field, it is stationary, that is, it is not accelerated or deccelerated by the magnetic field, and there is no exchange of mechanical energy between the gas and the magnetic field.

2. A constant α also represents the state of minimum Ohmic dissipation of the force-free field and is a particular case of the state of minimum magnetic energy with $\alpha = 0$ or $\partial B/\partial t = 0$.

3. Ferraro and Plumpton [6] have surmised that a constant α is a natural end configuration, but our analysis shows that the only force-free field that can be maintained in a strong field is one with a constant α, otherwise such a field will be unstable, thus the end configuration cannot be asserted from the present analysis.

4. A word should be said about the gauge relation (12). On integration, it gives

$$\frac{c}{4\pi}\int \frac{\alpha B^2}{\sigma} d\tau = -\int_v \mathbf{B} \cdot \nabla\phi d\tau = -\int_\Gamma \hat{n} \cdot \mathbf{B}\phi d\sigma. \qquad (49)$$

If Γ is a magnetic surface and ϕ is a single-valued, then the right side of (49) is zero, hence so is the left side, giving either $\alpha = 0$ or $B = 0$, which is physically meaningless; but from (13) we see that ϕ can be multi-valued, and so this difficulty is by-passed.

5. A mass of gas always has some electrical resistance, and the present writer has found that the resistance not only causes the diffusion of the field but also that of α. When considering the long-term evolution, the resistance must be included; but if we consider the short-term behaviour of an unstabilised or quasi-stable force-free field, then we can treat it as a frozen-in field.

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