VARIATIONAL PRINCIPLE FOR STABILITY OF SPINNING SOLID BODIES WITH LIQUID-FILLED CAVITIES AND ITS APPLICATIONS

Xu Shuochang (徐硕昌)
(*Institute of Mechanics, Academia Sinica*)
Received January 3, 1978.

Abstract

This paper presents a system of linear differential-integral perturbation equations derived for the stability of uniform rotation around the principal axis of inertia of any shaped solid body with cavities containing viscous liquid. This system cannot be treated directly by using methods based on Lyapunov's theory of stability. In the present paper, a general variational principle is established as presented earlier in [1], and a series of criteria for stability are obtained. The results obtained previously by others[3~6] are found to be special cases of the present investigation.

The variational methods so obtained may be widely used in treating the stability of motion of solid bodies containing liquid, and may be applied, in principle, to deal with the linear non-selfadjoint eigenvalue problems.

I. Introduction

The problem of motion of a solid body with cavities completely or partly filled with liquid is a classical subject[2]. Though its study was started over a century ago, it is only during the last three decades that the research of this subject has been rapidly developed owing to the needs in engineering[3~6].

It is particularly effective to apply the first approximation method and Lyapunov's direct method to cases dealing with the stability of motion of a system with finite number of degrees of freedom. According to Zhukovskii's theorem, a system with an infinite number of degrees of freedom can be reduced to one with finite number of degrees of freedom. The major related results were shown in [4]. But in reality viscosity can never be neglected. Up to now only special cases with large or small viscosity have been examined[6]. For instance, Rumyantsey has put forward his stability theory with respect to a part of the variables which may be used to reduce a system with infinite degrees of freedom into that with finite degrees of freedom. And yet the sufficient condition obtained in this special case may be neither sufficient nor necessary for a general case[4]. Another instance may be referred to Greespan's theory but it is limited to solving the boundary layer equation on condition that the motion of solid shell is given[7].

The equations of motion referred to small perturbation derived in this paper are non-selfadjoint differential-integral equations which are difficult to solve directly. The aforementioned methods and classical variational method[4,8] cannot be used to treat these equations. It has been shown that the problem with $\infty + 3$ degrees of freedom

may be solved by the variational method of non-selfadjoint operator derived in [1]. The advantage of such a method consists in that the existence of eigenvalues of instability can be determined without knowing the concrete form of the eigenfunctions. It is, in fact, a generalization of the adjoint variational method in solid mechanics[9]. Although this method as well as the one referred to in [9] both resort to the same mathematical principle and provide the variational basis for the approximate calculation of stability, the former has an advantage over the latter by having the merits of Lyapunov's direct method.

The stability of motion of a solid body with cavities completely filled with viscous liquid rotating at a fixed point around one of its principal axes of inertia with constant angular velocity $\boldsymbol{\Omega}$ has been investigated on the assumption that the geometrical shape of the solid body and its cavity is arbitrary, their principal axes of inertia coincide with each other, and viscosity of liquid is arbitrary as well. The cases considered seem to be more general. In so doing the results obtained by some previous authors[4,6,10—12] may be regarded as special cases of our investigation. It should be mentioned that our method may be applied to problem of stability of motion of a solid body containing liquid in a wider scope. This point will be discussed in Sec. VI.

II. Mathematical Formulation of the Problem

1. *Fundamental assumptions.* Suppose that a solid body with cavities completely filled with liquid in equilibrium rotates uniformly around a principal axis of inertia through a fixed point with angular velocity $\boldsymbol{\Omega}$. We shall study the stability of motion with small perturbation.

First of all, we introduce three fundamental coordinate systems:

(1) The fixed coordinate system $\{o;\ \xi_1,\ \xi_2,\ \xi_3\}$. Its origin coincides with the fixed point o. The rotating axis ξ_3 directs vertically. The unit vectors of the coordinate are $\boldsymbol{i}_1^0$, $\boldsymbol{i}_2^0$ and $\boldsymbol{i}_3^0$. Suppose that the distances of the centers of gravity of the solid shell and liquid in the cavity from the fixed point o are h_1 and h_2 and the total mass of the solid shell and liquid is M_1 and M_2 respectively.

(2) The coordinate system associated with equilibrium motion $\{o;\ x_1, x_2, x_3\}$. x_3 axis coincides with ξ_3 axis. The system rotates with angular velocity $\boldsymbol{\Omega}_0(0,\ 0,\ \Omega_0)$ relative to the fixed system and its unit vectors are $\boldsymbol{i}_1$, $\boldsymbol{i}_2$ and $\boldsymbol{i}_3$ respectively.

If P° denotes the pressure of the liquid, in an equilibrium case, it is evident that

$$P^\circ = p_0 + \frac{\rho}{2}\Omega_0^2(x_1^2 + x_2^2) - \rho g x_3, \tag{2.1}$$

where ρ is the density of the liquid and g is the gravitational acceleration.

(3) The coordinate system associated with perturbed motion $\{o;\ x_1',\ x_2',\ x_3'\}$. Suppose that the perturbed motion of a solid body with cavities completely filled with liquid and deviated from equilibrium, is a rotation around the fixed point o with argular velocity $\boldsymbol{\omega}(\omega_1,\ \omega_2,\ \omega_3)$ relative to the system $\{o;\ x_1,\ x_2,\ x_3\}$. During the process of

perturbed motion, system $\{o;\ x_1',\ x_2', x_3'\}$ is fixed to the solid body with cavities, and its unit vectors are $\boldsymbol{i}_1',\ \boldsymbol{i}_2',\ \boldsymbol{i}_3'$ which coincide with its principal axes of inertia.

Assume that the unit vectors of $\{o;\ x_1',\ x_2',\ x_3'\}$ and $\{o;\ x_1,\ x_2,\ x_3\}$ systems satisfy the following equations:

$$\left.\begin{aligned} \boldsymbol{i}_1' &= \boldsymbol{i}_1 + \gamma_{12}\boldsymbol{i}_2 + \gamma_{13}\boldsymbol{i}_3, \\ \boldsymbol{i}_2' &= \gamma_{21}\boldsymbol{i}_1 + \boldsymbol{i}_2 + \gamma_{23}\boldsymbol{i}_3, \\ \boldsymbol{i}_3' &= \gamma_{31}\boldsymbol{i}_1 + \gamma_{32}\boldsymbol{i}_2 + \boldsymbol{i}_3, \end{aligned}\right\} \tag{2.2}$$

where $\gamma_{ij} = \cos(\boldsymbol{i}_i,\ \boldsymbol{i}_j)$. By orthogonal condition, we have $\gamma_{ij} = -\gamma_{ji}$.

2. *Basic equations and boundary conditions.* We shall formulate the basic equations and boundary conditions referred to the coordinate system $\{o;\ x_1,\ x_2,\ x_3\}$[4]:

(1) The equations of motion of the solid shell. If the principal moments of inertia of the solid shell and liquid in the cavity are $A_1,\ B_1,\ C_1$ and $A_2,\ B_2,\ C_2$ about the fixed point o, and the total moment of momentum of the solid shell $\boldsymbol{G}$ is

$$\left.\begin{aligned} G_{x_1} &= \omega_1 A_1 - \Omega_0(A_1 - C_1)\gamma_{31}, \\ G_{x_2} &= \omega_2 B_1 - \Omega_0(B_1 - C_1)\gamma_{32}, \\ G_{x_3} &= (\omega_3 + \Omega_0)C_1, \end{aligned}\right\} \tag{2.3}$$

then the equation of moment of momentum of the solid shell is

$$\begin{aligned} \frac{d\boldsymbol{G}}{dt} + \boldsymbol{\Omega}_0 \times \boldsymbol{G} &= -(M_1 g h_1 + M_2 g h_2)(\gamma_{32},\ -\gamma_{31},\ 0) \\ &\quad + \Omega_0^2[\gamma_{23}(B_2 - C_2),\ -\gamma_{13}(A_2 - C_2),\ 0] - \iint_s \boldsymbol{r} \times (-p_1, \boldsymbol{I} + \boldsymbol{\tau}) \cdot \boldsymbol{n} ds, \end{aligned} \tag{2.4}$$

where s is the surface of the cavity, $\boldsymbol{n}$ is outward unit normal vector, p_1 is the perturbed pressure, $\boldsymbol{\tau}$ is the viscous stress tensor, and $\boldsymbol{I}$ the unit tensor.

(2) The equation of motion of the liquid in the cavity is

$$\frac{\partial \boldsymbol{v}}{\partial t} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{v} = \frac{1}{\rho} \nabla \cdot (-p_1, \boldsymbol{I} + \boldsymbol{\tau}), \tag{2.5}$$

where

$$p_1 = p - p^\circ = p + \rho g x_3 - \frac{\rho}{2}\Omega_0^2(x_1^2 + x_2^2) - p_0. \tag{2.6}$$

The equation of the continuity of liquid is

$$\nabla \cdot \boldsymbol{v} = 0. \tag{2.7}$$

(3) Boundary conditions

$$\boldsymbol{v}|_s = \boldsymbol{\omega} \times \boldsymbol{r} \tag{2.8}$$

and

$$\frac{d\boldsymbol{i}_l'}{dt} = \boldsymbol{\omega} \times \boldsymbol{i}_l', \quad (l = 1, 2, 3). \tag{2.9}$$

The mathematical formulation of the whole problem is to solve $\boldsymbol{v}$, p_1, $\boldsymbol{\omega}$, $\boldsymbol{i}_1'$, $\boldsymbol{i}_2'$ and $\boldsymbol{i}_3'$ from Eqs. (2.4), (2.5), (2.7) and (2.9) under the boundary condition (2.8) and the initial conditions.

3. *The statement of the eigenvalue problem.* The above mathematical problem can be treated by the stability theory of the viscous fluid mechanics[13,14]. Since these equations contain only the derivatives with respect to time t but not the time t itself, it is expected that there may be a solution proportional to the exponential function $e^{i\lambda t}$. Furthermore, as the equation of boundary condition for perturbed motion is homogeneous, so the eigenvalue problem of λ can be derived. If the imaginary parts of all the eigenvalues are positive, the motion is stable; otherwise, there is at least an imaginary part of an eigenvalue being negative, and then the motion is unstable[13,14].

Suppose that

$$\left.\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\omega}_0 e^{i\lambda t} = (\omega_{10}, \omega_{20}, \omega_{30}) e^{i\lambda t}, \\ \boldsymbol{v} &= \boldsymbol{V}(\boldsymbol{r}) e^{i\lambda t}, \quad p_1 = p(\boldsymbol{r}) e^{i\lambda t}. \end{aligned}\right\} \tag{2.10}$$

Substituting the first equation of (2.10) into (2.9), we have

$$\left.\begin{aligned} \boldsymbol{i}_1' &= \left(1, \ -\frac{i\omega_{30}}{\lambda} e^{i\lambda t}, \frac{i\omega_{20}}{\lambda} e^{i\lambda t}\right), \\ \boldsymbol{i}_2' &= \left(\frac{i\omega_{30}}{\lambda} e^{i\lambda t}, \ 1, \ -\frac{i\omega_{10}}{\lambda} e^{i\lambda t}\right), \\ \boldsymbol{i}_3' &= \left(-\frac{i\omega_{20}}{\lambda} e^{i\lambda t}, \frac{i\omega_{10}}{\lambda} e^{i\lambda t}, \ 1\right). \end{aligned}\right\} \tag{2.11}$$

Then substituting (2.10) and (2.11) into (2.4) (2.5), (2.7), (2.9) and (2.8), we obtain the following eigenvalue problem:

$$\left\{\begin{aligned} & i\lambda^2(A_1\omega_{10}, B_1\omega_{20}, C_1\omega_{30}) + \lambda(A_1 + B_1 - C_1)\boldsymbol{\Omega}_0 \times \boldsymbol{\omega}_0 + i(M_1gh_1 + M_2gh_2)(\omega_{10}, \omega_{20}, 0) \\ & \qquad + i\Omega_0^2[(B_1 - C_1 + B_2 - C_2)\omega_{10}, (A_1 - C_1 + A_2 - C_2)\omega_{20}, 0] \\ & \qquad + \lambda \iint_s \boldsymbol{r} \times (-p\boldsymbol{I} + \boldsymbol{\tau}) \cdot \boldsymbol{n} ds = 0, && (2.12) \\ & i\lambda \boldsymbol{V} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{V} = \frac{1}{\rho} \nabla \cdot (-p\boldsymbol{I} + \boldsymbol{\tau}), && (2.13) \\ & \nabla \cdot \boldsymbol{V} = 0, && (2.14) \\ & \text{on the surface s, } \boldsymbol{V}|_s = \boldsymbol{\omega}_0 \times \boldsymbol{r}, && (2.15) \end{aligned}\right.$$

where $\boldsymbol{\tau}$ is the viscous stress tensor, $\tau_{ij} = \mu\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$, and $\boldsymbol{I}$ is the unit tensor.

It is easy to prove that if λ, $\boldsymbol{\omega}_0$ $\boldsymbol{V}$ and p are the solution of the eigenvalue problem defined by (2.12)—(2.15)(denoted as the problem A), then $i\lambda^*$, $\boldsymbol{\omega}_0^*$, $\boldsymbol{V}^*$ and p^* are the solution too. In the following, we shall treat the problem A by the variational method of non-selfadjoint operator[1].

III. General form of Variational Principle

1. *Adjoint eigenvalue problem (denoted as problem A^+)*

$$\begin{cases} i\mu^2(A_1\sigma_{10}, B_1\sigma_{20}, C_1\sigma_{30}) - \mu(A_1 + B_1 - C_1)\boldsymbol{\Omega}_0 \times \boldsymbol{\sigma}_0 + i(M_1gh_1 + M_2gh_2)(\sigma_{10}, \sigma_{20}, o) \\ \quad + i\Omega_0^2[(B_1 - C_1 + B_2 - C_2)\sigma_{10}, (A_1 - C_1 + A_2 - C_2)\sigma_{20}, o] \\ \quad + \mu \iint_s \boldsymbol{r} \times (-Q\boldsymbol{I} + \boldsymbol{T}) \cdot \boldsymbol{n}ds = 0, & (3.1) \\ i\mu\boldsymbol{U} - 2\boldsymbol{\Omega}_0 \times \boldsymbol{U} = \dfrac{1}{\rho}\nabla \cdot (-Q\boldsymbol{I} + \boldsymbol{T}), & (3.2) \\ \nabla \cdot \boldsymbol{U} = 0, & (3.3) \\ \text{on the surface } s,\ \boldsymbol{U}|_s = \boldsymbol{\sigma}_0 \times \boldsymbol{r}, & (3.4) \end{cases}$$

where the components of $\boldsymbol{T}$ are $T_{ij} = \mu\left(\dfrac{\partial U_i}{\partial x_j} + \dfrac{\partial U_j}{\partial x_i}\right)$ and $\boldsymbol{\sigma}_0 = (\sigma_{10}, \sigma_{20}, \sigma_{30})$.

The problems A and A^+ have their clear physical meaning. If A is referred to the case of the right-handed rotation, then A^+ is referred to the left-handed rotation. It is easily proved that if the eigenvalues $\mu, \boldsymbol{\sigma}, \boldsymbol{U}, Q$ are the solutions of the problem A^+ then $i\mu^*, \boldsymbol{\sigma}^*$ and Q^* are also its solutions. Later on, we shall prove that A and A^+ have the same series of eigenvalues.

2. *Derivation of the variational equation.* Suppose that λ_l, ω_l, $\boldsymbol{V}_l$ and p_l are solutions of the problem A and that μ_n, $\boldsymbol{\sigma}_n$ $\boldsymbol{U}_n$ and Q_n are solutions of the problem A^+. Multiplying the both sides of (2.13) by $\boldsymbol{U}_n$, integrating over the volume τ occupied by the liquid, and using (2.14) (3.3) and (3.4), we get

$$\begin{aligned} & i\lambda_l \iiint_\tau \rho \boldsymbol{V}_l \cdot \boldsymbol{U}_n d\tau + 2\iiint_\tau \rho(\boldsymbol{\Omega}_0 \times \boldsymbol{V}_l) \cdot \boldsymbol{U}_n d\tau \\ & \quad + \frac{1}{2}\iiint_\tau \mu\left(\frac{\partial V_{l,i}}{\partial x_j} + \frac{\partial V_{l,j}}{\partial x_i}\right)\left(\frac{\partial U_{n,i}}{\partial x_j} + \frac{\partial U_{n,j}}{\partial x_i}\right) d\tau \\ & = \iint_s (\boldsymbol{\sigma}_n \times \boldsymbol{r}) \cdot (-p_l\boldsymbol{I} + \boldsymbol{\tau}) \cdot \boldsymbol{n}d\tau. \end{aligned} \tag{3.5}$$

Again, multiplying the both sides of Eq. (2.12) by $\boldsymbol{\sigma}_n$, we have

$$\begin{aligned} & i\lambda_l^2(A_1\omega_{l,1}\sigma_{n,1} + B_1\omega_{l,2}\sigma_{n,2} + C_1\omega_{l,3}\sigma_{n,3}) - \lambda\Omega_0(A_1 + B_1 - C_1)(\omega_{l,2}\sigma_{n,1} \\ & \quad - \omega_{l,1}\sigma_{n,2}) + i(M_1gh_1 + M_2gh_2)(\omega_{l,1}\sigma_{n,1} + \omega_{l,2}\sigma_{n,2}) + i\Omega_0^2[(B_1 \\ & \quad - C_1 + B_2 - C_2)\omega_{l,1}\sigma_{n,1} + (A_1 - C_1 + A_2 - C_2)\omega_{l,2}\sigma_{n,2}] \\ & \quad + \lambda_l \iint_s (\boldsymbol{\sigma}_n \times \boldsymbol{r}) \cdot [-R\boldsymbol{I} + \boldsymbol{\tau}] \cdot \boldsymbol{n}ds = 0. \end{aligned} \tag{3.6}$$

Eliminating the surface integral terms involving pressure from Eqs. (3.5) and (3.6), we get

$$(i\lambda_l)^2\left[(A_1\omega_{l,1}\sigma_{n,1}+B_1\omega_{l,2}\sigma_{n,2}+C_1\omega_{l,3}\sigma_{n,3})+\iiint_\tau \rho \boldsymbol{V}_l\cdot\boldsymbol{U}_l d\tau\right]$$
$$+(i\lambda_l)\left[-(A_1+B_1-C_1)\Omega_0(\omega_{l,2}\sigma_{n,1}-\omega_{l,1}\sigma_{n,2})+2\iiint_\tau \rho(\boldsymbol{\Omega}_0\times\boldsymbol{V}_l)\right.$$
$$\left.\cdot\boldsymbol{U}_n d\tau+\iiint_\tau \frac{\mu}{2}\left(\frac{\partial V_{l,i}}{\partial x_j}+\frac{\partial V_{l,j}}{\partial x_i}\right)\left(\frac{\partial U_{n,i}}{\partial x_j}+\frac{\partial U_{n,j}}{\partial x_i}\right)d\tau\right]-(M_1gh_1$$
$$+M_2gh_2)(\omega_{l,1}\sigma_{n,1}+\omega_{l,2}\sigma_{n,2})-\Omega_0^2[(B_1-C_1+B_2-C_2)\omega_{l,1}\sigma_{n,1}$$
$$+(A_1-C_1+A_2-C_2)\omega_{l,2}\sigma_{n,2}]=0. \tag{3.7}$$

Similarly, multiplying the both sides of (3-1) by $\boldsymbol{\omega}_l$, integrating (3.2) with respect to τ after multiplying its two sides by $\boldsymbol{V}_l$, and eliminating the surface integral term in these resulting equations, we have

$$(i\mu_n)^2\left[(A_1\omega_{l,1}\sigma_{n,1}+B_1\omega_{l,2}\sigma_{n,2}+C_1\omega_{l,3}\sigma_{n,3})+\iiint_\tau \rho \boldsymbol{V}_l\cdot\boldsymbol{U}_n d\tau\right]$$
$$+(i\mu_n)\left[-(A_1+B_1-C_1)\Omega_0(\omega_{l,2}\sigma_{n,1}-\omega_{l,1}\sigma_{n,2})+2\iiint_\tau \rho(\boldsymbol{\Omega}_0\times\boldsymbol{V}_l)\right.$$
$$\left.\cdot\boldsymbol{U}_n d\tau+\iiint_\tau \frac{\mu}{2}\left(\frac{\partial V_{l,i}}{\partial x_j}+\frac{\partial V_{l,j}}{\partial x_i}\right)\cdot\left(\frac{\partial U_{n,i}}{\partial x_j}+\frac{\partial U_{n,j}}{\partial x_i}\right)d\tau\right]$$
$$-(M_1gh_1+M_2gh_2)(\omega_{l,1}\sigma_{n,1}+\omega_{l,2}\sigma_{n,2})-\Omega_0^2[(B_1-C_1$$
$$+B_2-C_2)\omega_{l,1}\sigma_{n,1}+(A_1-C_1+A_2-C_2)\omega_{l,2}\sigma_{n,2}]=0. \tag{3.8}$$

From Eqs. (3.7) and (3.8), it can be seen that $\lambda_l=\mu_l$ and thus we have proved that the problems A and A^+ have the same series of eigenvalues.

In Eq. (3.7), by setting $l=n$ and introducing the notations:

$$A(\boldsymbol{\omega}_l,\boldsymbol{\sigma}_l)=A_1\omega_{l,1}\sigma_{l,1}+B_1\omega_{l,2}\sigma_{l,2}+C_1\omega_{l,3}\sigma_{l,3}, \tag{3.9}$$

$$I(\boldsymbol{V}_l,\boldsymbol{U}_l)=\iiint_\tau \rho\boldsymbol{V}_l\cdot\boldsymbol{U}_l d\tau, \tag{3.10}$$

$$B(\boldsymbol{\omega}_l,\boldsymbol{\sigma}_l)=(A_1+B_1-C_1)(\boldsymbol{\Omega}_0\times\boldsymbol{\omega}_l)\cdot\boldsymbol{\sigma}_l, \tag{3.11}$$

$$\Psi(\boldsymbol{V}_l,\boldsymbol{U}_l)=2\iiint_\tau \rho(\boldsymbol{\Omega}_0\times\boldsymbol{V}_l)\cdot\boldsymbol{U}_l d\tau, \tag{3.12}$$

$$\Phi(\boldsymbol{V}_l,\boldsymbol{U}_l)=\iiint_\tau \frac{\mu}{2}\left(\frac{\partial V_{l,i}}{\partial x_j}+\frac{\partial V_{l,j}}{\partial x_i}\right)\cdot\left(\frac{\partial U_{l,i}}{\partial x_j}+\frac{\partial U_{l,j}}{\partial x_i}\right)d\tau, \tag{3.13}$$

$$L(\boldsymbol{\omega}_l,\boldsymbol{\sigma}_l)=\Omega_0^2[(C_1-B_1+C_2-B_2)\omega_{l,1}\sigma_{l,1}+(C_1-A_1$$
$$+C_2-A_2)\omega_{l,2}\sigma_{l,2}]-(M_1gh_1+M_2gh_2)(\omega_{l,1}\sigma_{l,1}+\omega_{l,2}\sigma_{l,2}). \tag{3.14}$$

Eq. (3.7) may be written as

$$[A(\boldsymbol{\omega}_l,\boldsymbol{\sigma}_l)+I(\boldsymbol{V}_l,\boldsymbol{U}_l)](i\lambda_l)^2+[B(\boldsymbol{\omega}_l,\boldsymbol{\sigma}_l)+\Psi(\boldsymbol{V}_l,\boldsymbol{U}_l)$$
$$+\Phi(\boldsymbol{V}_l,\boldsymbol{U}_l)](i\lambda_l)+L(\boldsymbol{\omega}_l,\boldsymbol{\sigma}_l)=0. \tag{3.15}$$

or

$$(i\lambda_l) = \frac{1}{A(\boldsymbol{\omega}_l, \boldsymbol{\sigma}_l) + I(\boldsymbol{V}_l, \boldsymbol{U}_l)} \left\{ -\frac{[B(\boldsymbol{\omega}_l, \boldsymbol{\sigma}_l) + \Psi(\boldsymbol{V}_l, \boldsymbol{U}_l) + \Phi(\boldsymbol{V}_l, \boldsymbol{U}_l)]}{2} \right.$$
$$\left. \pm \sqrt{\left[\frac{B(\boldsymbol{\omega}_l, \boldsymbol{\sigma}_l) + \Psi(\boldsymbol{V}_l, \boldsymbol{U}_l) + \Phi(\boldsymbol{V}_l, \boldsymbol{U}_l)}{2}\right]^2 - [A(\boldsymbol{\omega}_l, \boldsymbol{\sigma}_l) + I(\boldsymbol{V}_l, \boldsymbol{U}_l)] \cdot L(\boldsymbol{\omega}_l, \boldsymbol{\sigma}_l)} \right\}. \tag{3.16}$$

Eqs. (3.15) and (3.16) are the required variational equations. Therefore the problems A and A^+ are reduced to the functional variation problems. For any $\varepsilon > 0$, all $\frac{|\boldsymbol{\omega}|}{\Omega_0} < \varepsilon$ and $\frac{|\boldsymbol{\sigma}|}{\Omega_0} < \varepsilon$ form a region R in six-dimensional space. If $\boldsymbol{V}$ and $\boldsymbol{U}$ are the continuous and differentiable functions defined on τ, and satisfy the additional conditions $\nabla \cdot \boldsymbol{V} = 0$, $\nabla \cdot \boldsymbol{U} = 0$ and $\boldsymbol{V}|_s = \boldsymbol{\omega} \times \boldsymbol{r}$, $\boldsymbol{U}|_s = \boldsymbol{\sigma} \times \boldsymbol{r}$, then the whole of $\boldsymbol{U}$ and $\boldsymbol{V}$ forms a functional space M, and the functional $i\lambda(\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{\omega}, \boldsymbol{\sigma})$ is a functional defined on $M + R$. This is a variational problem of nonlinear functional under the fixed boundary conditions.

3. *Proof of the variational principle.* Let $\delta\boldsymbol{V}$, $\delta\boldsymbol{U}$, $\delta\boldsymbol{\omega}$ and $\delta\boldsymbol{\sigma}$ be the variations of $\boldsymbol{V}$, $\boldsymbol{U}$, $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$, and satisfy $\boldsymbol{U} + \delta\boldsymbol{U} \in M$, $\boldsymbol{V} + \delta\boldsymbol{V} \in M$, $\boldsymbol{\omega} + \delta\boldsymbol{\omega} \in R$ and $\boldsymbol{\sigma} + \delta\boldsymbol{\sigma} \in R$, then it may be proved that when the functional $i\lambda$ reaches its extreme values, the following relations can be obtained:

$$\begin{aligned}
&\{2i\lambda[A(\boldsymbol{\omega}, \boldsymbol{\sigma}) + I(\boldsymbol{V}, \boldsymbol{U})] + B(\boldsymbol{\omega}, \boldsymbol{\sigma}) + \Psi(\boldsymbol{V}, \boldsymbol{U}) + \Phi(\boldsymbol{V}, \boldsymbol{U})\} i\delta\lambda \\
&= -\delta\boldsymbol{\sigma}\{(i\lambda)^2(A_1\omega_{10}, B_1\omega_{20}, C_1\omega_{30}) + (i\lambda)(A_1 + B_1 - C_1)\boldsymbol{\Omega}_0 \times \boldsymbol{\omega}_0 \\
&\quad - (M_1gh_1 + M_2gh_2)(\omega_{10}, \omega_{20}, 0) + i\lambda \iint_s \boldsymbol{r} \times (-p\boldsymbol{I} + \boldsymbol{\tau}) \cdot \boldsymbol{n} ds \\
&\quad - \Omega_0^2[(B_1 - C_1 + B_2 - C_2)\omega_{10}, (A_1 - C_1 + A_2 - C_2)\omega_{20}, 0]\} \\
&\quad - \iiint_\tau \delta\boldsymbol{U}(i\lambda\rho)\left\{(i\lambda)\boldsymbol{V} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{V} - \frac{1}{\rho}\nabla \cdot (-p\boldsymbol{I} + \boldsymbol{\tau})\right\} d\tau \\
&- \delta\boldsymbol{\omega}\{(i\lambda)^2(A_1\sigma_{10}, B_1\sigma_{20}, C_1\sigma_{30}) - i\lambda(A_1 + B_1 - C_1)\boldsymbol{\Omega}_0 \times \boldsymbol{\sigma}_0 \\
&\quad - (M_1gh_1 + M_2gh_2)(\sigma_{10}, \sigma_{20}, 0) + i\lambda \iint_s \boldsymbol{r} \times (-Q\boldsymbol{I} + \boldsymbol{T}) \cdot \boldsymbol{n} ds \\
&\quad - \Omega_0^2[(B_1 - C_1 + B_2 - C_2)\sigma_{10}, (A_1 - C_1 + A_2 - C_2)\sigma_{20}, 0]\} \\
&\quad - \iiint_\tau (i\lambda\rho)\left\{(i\lambda)\boldsymbol{U} - 2\boldsymbol{\Omega}_0 \times \boldsymbol{U} - \frac{1}{\rho}\nabla \cdot (-Q\boldsymbol{I} + \boldsymbol{T})\right\} \delta\boldsymbol{V} d\tau = 0.
\end{aligned} \tag{3.17}$$

From the above equation, by using the fundamental lemma of the variational method ([8], Vol. 1, Chap. 4., §3), it is concluded that when the functional $i\lambda(\boldsymbol{V}, \boldsymbol{U}, \boldsymbol{\omega}, \boldsymbol{\sigma})$ defined on the region $M + R$ takes its extreme value, the solutions of the eigenvalue problems A and A^+ are derived; on the other hand, the solutions of problems A and A^+ cause the functional $i\lambda$ to take its extreme values. Hence, the variational Eq. (3.15) or (3.16) provides the variational basis of the approximate method in the treatment of stability and it is also the fundamental starting point in the discussion of the signs of the imaginary parts of the eigenvalues.

4. *Integral relation of the eigenvalue*. Now, we shall determine the signs of the imaginary parts of the eigenvalues from the variational Eq. (3.16). Substituting $\boldsymbol{\omega}^*$ for $\boldsymbol{\sigma}$ and $\boldsymbol{V}^*$ for $\boldsymbol{U}$ in Eq. (3.15) or (3.16), the integral relation of the eigenvalue may be obtained as follows:

$$(A+I)(i\lambda)^2+(B+\Psi+\Phi)(i\lambda)+L=0, \tag{3.18}$$

or

$$i\lambda=\frac{1}{A+I}\left\{-\frac{B+\Psi+\Phi}{2}\pm\sqrt{\left[\frac{B+\Psi+\Phi}{2}\right]^2-(A+I)\cdot L}\right\}, \tag{3.19}$$

where

$$A(\boldsymbol{\omega},\boldsymbol{\omega}^*)=A_1\omega_{10}\omega_{10}^*+B_1\omega_{20}\omega_{20}^*+C_1\omega_{30}\omega_{30}^*, \tag{3.20}$$

$$I=I(\boldsymbol{V},\boldsymbol{V}^*)=\iiint_\tau \rho\boldsymbol{V}\cdot\boldsymbol{V}^*d\tau, \tag{3.21}$$

$$B=B(\boldsymbol{\omega},\boldsymbol{\omega}^*)=(A_1+B_1-C_1)(\boldsymbol{\Omega}_0\times\boldsymbol{\omega})\cdot\boldsymbol{\omega}^*, \tag{3.22}$$

$$\Psi=\Psi(\boldsymbol{V},\boldsymbol{V}^*)=2\iiint_\tau \rho(\boldsymbol{\Omega}_0\times\boldsymbol{V})\cdot\boldsymbol{V}^*d\tau, \tag{3.23}$$

$$\Phi=\Phi(\boldsymbol{V},\boldsymbol{V}^*)=\iiint_\tau \frac{\mu}{2}\left(\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}\right)^2 d\tau, \tag{3.24}$$

$$\begin{aligned}L=L(\boldsymbol{\omega},\boldsymbol{\omega}^*)=\Omega_0^2[&(C_1-B_1+C_2-B_2)\omega_{10}\omega_{10}^*+(C_1-A_1\\&+C_2-A_2)\omega_{20}\omega_{20}^*]-(M_1gh_1+M_2gh_2)(\omega_{10}\omega_{10}^*+\omega_{20}\omega_{20}^*).\end{aligned} \tag{3.25}$$

It may be proved that the eigenvalues determined from the variational Eqs. (3.15) and (3.18) have a one to one correspondence.

First, let $\boldsymbol{U}=\boldsymbol{U}-\boldsymbol{V}^*+\boldsymbol{V}^*=\boldsymbol{V}^*+\delta\boldsymbol{V}^*$, $\boldsymbol{\sigma}=\boldsymbol{\sigma}-\boldsymbol{\omega}+\boldsymbol{\omega}^*=\boldsymbol{\omega}^*+\delta\boldsymbol{\omega}^*$ in variational equation (3.15). Then we derive the transformation equation (3.15) by the same procedure as in deriving variational equation as follows:

$$\begin{aligned}&\{(A+I)(i\lambda)^2+(B+\Psi+\Phi)(i\lambda)+L\}\\&\quad+\delta\boldsymbol{\omega}^*\Big\{(i\lambda)^2(A_1\sigma_{10},B_1\sigma_{20},C_1\sigma_{30})-(i\lambda)(A_1+B_1-C_1)\boldsymbol{\Omega}_0\times\boldsymbol{\sigma}_0\\&\quad-(M_1gh_1+M_2gh_2)(\sigma_{10},\sigma_{20},0)+i\lambda\iint_s \boldsymbol{r}\times(-Q\boldsymbol{I}+\boldsymbol{T})\cdot\boldsymbol{n}ds\\&\quad-\Omega_0^2[(B_1-C_1+B_2-C_2)\sigma_{10},(A_1-C_1+A_2-C_2)\sigma_{20},0]\Big\}\\&\quad+\iiint_\tau (i\lambda)\rho\left\{(i\lambda)\boldsymbol{U}-2\boldsymbol{\Omega}_0\times\boldsymbol{U}-\frac{1}{\rho}\nabla\cdot(-Q\boldsymbol{I}+\boldsymbol{T})\right\}\delta\boldsymbol{V}^*d\tau=0.\end{aligned} \tag{3.26}$$

Since $\boldsymbol{\sigma}$, $\boldsymbol{U}$ and $\boldsymbol{Q}$ are the solutions of problem A^+, this just proves that the eigenvalue determined from Eq. (3.15) is the same as those determined from (3.18).

On the other hand, by the entire analogous procedure, Eq. (3.15) may be derived from (3.18). Hence, we may analyse the critical condition of stability only by the integral relation (3.19) of the eigenvalue.

VI. Stability Criteria of Motion of a Solid Body Containing Liquid

The various quantities in the integral relation of eigenvalue (3.19), as expressed by Eqs. (3.20)—(3.25), have a clear physical meaning: $A(\boldsymbol{\omega}, \boldsymbol{\omega}^*) > 0$ represents kinetic energy of the perturbed motion of the solid shell; $I(\boldsymbol{V}, \boldsymbol{V}^*) > 0$ represents kinetic energy of the perturbed motion of the liquid; $B(\boldsymbol{\omega}, \boldsymbol{\omega}^*)$ and $\Psi(\boldsymbol{V}, \boldsymbol{V}^*)$ are pure imaginary numbers corresponding formally to the work done by Coriolis force; $\Phi(\boldsymbol{V}, \boldsymbol{V}^*) > 0$ represents the dissipative work done by the viscous force of the liquid; $L(\boldsymbol{\omega}, \boldsymbol{\omega}^*)$ corresponds to the change in the potential energy of the system caused by perturbed motion.

Now, we are concerned with the general case of arbitrarily shaped solid body with cavities completely filled with incompressible viscous liquid, with only a limitation that the principal axes of inertia of the solid body and liquid coincide with each other. Let their principal moments of inertia be A_1, B_1, C_1 and A_2, B_2, C_2, their masses be M_1 and M_2, and distances of their centre of gravity from the fixed point o be h_1 and h_2, respectively. According to Eq. (3.19), and by using the following lemma:

Lemma 1. *Let $Z = e + if$ $(e > 0)$ and a be any real number, then*

$$(1)\ \mathrm{Re}\,\{-Z \pm \sqrt{Z^2 - a^2}\} < 0, \tag{4.1}$$

$$(2)\ 0 < \max\,\{\mathrm{Re}\,[-Z \pm \sqrt{Z^2 + a^2}\,]\} < a, \tag{4.2}$$

the criteria of stability may be obtained as follows:

Theorem 1. *If all the eigenvectors $\boldsymbol{\omega}$ satisfy the equation*

$$\begin{aligned} L(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = {} & \Omega_0^2[(C_1 - B_1 + C_2 - B_2)\omega_{10}\omega_{10}^* + (C_1 - A_1 + C_2 - A_2)\omega_{20}\omega_{20}^*] \\ & - (M_1 g h_1 + M_2 g h_2)(\omega_{10}\omega_{10}^* + \omega_{20}\omega_{20}^*) > 0, \end{aligned} \tag{4.3}$$

then the rotation of a solid body containing liquid must be stable.

Proof. According to the theory of stability of viscous fluid motion, when the imaginary parts of all eigenvalues are positive, the motion is stable[13,14]. Now we may prove the theorem by the reduction to absurdity. If there existed a certain eigenvector $\boldsymbol{\omega}_K$ satisfying $L(\boldsymbol{\omega}_K, \omega_K^*) < 0$, we might obtain

$$\mathrm{Re}\,(i\lambda) = \frac{1}{A + I}\,\mathrm{Re}\,\{-Z + \sqrt{Z^2 + a^2}\} > 0$$

from Eq. (4.2), in which we set

$$e = \frac{\Phi(\boldsymbol{V}, \boldsymbol{V}^*)}{2} > 0, \quad if = \frac{B(\boldsymbol{\omega}_K, \boldsymbol{\omega}_K^*) + \Psi(\boldsymbol{V}, \boldsymbol{V}^*)}{2},$$

$$a^2 = -\,[A(\boldsymbol{\omega}_K, \boldsymbol{\omega}_K^*) + I(\boldsymbol{V}, \boldsymbol{V}^*)] \cdot L(\boldsymbol{\omega}_K, \boldsymbol{\omega}_K^*) > 0,$$

(where $\boldsymbol{V}$ may be an arbitrary continuous function). If the eigenvelocity function $\boldsymbol{V}_K$ corresponding to $\boldsymbol{\omega}_K$ were substituted for $\boldsymbol{V}$, λ would be the eigenvalue λ_K corresponding to $\boldsymbol{\omega}_K$ and $\boldsymbol{V}_K$. The imaginary part of this λ_K would be negative. This is obviously contrary to the assumption for stability.

Corollary 1. *When it holds that*

$$\Omega_0^2(C_1 - A_1 + C_2 - A_2) > M_1gh_1 + M_2gh_2, \tag{4.4}$$

where $A_1 \geqslant B_1$, $A_2 \geqslant B_2$, the rotation of the solid body containing liquid must be stable.

This result has been obtained by Rumyantsev only under the stability definition referred to a part of variables ([4], Chap. 3, §3).

Corollary 2. *When $h_1 = h_2 = 0$ and if*

$$C_1 - A_1 + C_2 - A_2 > 0, \quad C_1 - B_1 + C_2 - B_2 > 0 \tag{4.5}$$

holds, the free gyroscopic motion of a solid body containing liquid must be stable, namely, the free gyration around its shortest principal axis of inertia is stable.

This result has been obtained by Chernouseko ([6] Chap. 4.) and Smirnova[10] only in the case of a small viscosity and a special shape of the cavity.

By the procedures entirely analogous to the proof of Theorem 1, we may derive theorems of instability as follows:

Theorem 2. *If there exists at least an eigenvector $\boldsymbol{\omega}$ satisfying*

$$\begin{aligned} L(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = {} & \Omega_0^2[(C_1 - B_1 + C_2 - B_2)\omega_{10}\omega_{10}^* + (C_1 - A_1 + C_2 - A_2)\omega_{20}\omega_{20}^*] \\ & - (M_1gh_1 + M_2gh_2)(\omega_{10}\omega_{10}^* + \omega_{20}\omega_{20}^*) < 0, \end{aligned} \tag{4.6}$$

the rotation of a solid body containing liquid must be unstable.

Corollary 3. *When*

$$\Omega_0^2(C_1 - A_1 + C_2 - A_2) < M_1gh_1 + M_2gh_2 \tag{4.7}$$

holds, where $A_1 \leqslant B_1$, and $A_2 \leqslant B_2$, the rotation of a solid body containing liquid must be unstable.

Corollary 4. *When $h_1 = h_2 = 0$ and if*

$$C_1 - A_1 + C_2 - A_2 < 0 \quad (A_1 < B_1, A_2 < B_2), \tag{4.8}$$

then, the free gyration of a solid body containing liquid must be unstable, namely, the free gyration around its longest principal axis of inertia is unstable.

For the axisymmetric case, i.e. $A_1 = B_1$, $A_2 = B_2$, according to Theorems 1 and 2, the following theorem immediately follows:

Theorem 3. *When a solid body containing liquid rotates uniformly around the symmetric axis and if it satisfies*

$$\Omega_0^2(C_1 - A_1 + C_2 - A_2) > M_1gh_1 + M_2gh_2 \quad (A_1 = B_1, A_2 = B_2), \tag{4.9}$$

the motion is stable. If it satisfies

$$\Omega_0^2(C_1 - A_1 + C_2 - A_2) < M_1gh_1 + M_2gh_2 \quad (A_1 = B_1, A_2 = B_2), \tag{4.10}$$

the motion is unstable.

The results, obtained by Sobolev[11], Ishliskii and Temcheko[12] and others, are the special cases of this theorem.

Theorem 4. *When the axisymmetric solid body containing liquid rotates freely around the symmetric axis, the sufficient and necessary conditions for the motion to be stable are that the symmetric axis must be the shortest principal axis of inertia.*

V. Remarks on Zhukovskii's Theorem

Zhukovskii proved that the motion of the ideal liquid completely filled in the cavities of a solid body is equivalent to that of a rigid body (see [4], Chap. 2, §2). The conclusion for stability of uniform rotation of a solid body with cavities completely filled with ideal liquid is analogous to the conclusion for stability of rotation of a rigid body without cavities[4,15], which may be stated as follows:

The rotation around the longest or the shortest principal axis of inertia is stable, only the rotation around the intermediate axis is unstable.

We have studied the effect of the viscosity on the stability of rotation of a solid body with cavities completely filled with liquid and proved that only the rotation around the shortest principal axis of inertia is stable. Rumyantsev proved that this is the sufficient condition for the stability theory referred to a part of variables (see [4], Chap 3, §4). Hence, Zhukovskii's conclusion should be revised correspondingly as follows: A solid body with cavities completely filled with liquid, only in rotating about its shortest principal axis of inertia, does acquire the "gyroscopic stabilizing effect".

VI. Applications

The variational method developed in this paper introduces a new method for treating the theory of stability of motion of a solid body containing liquid. It may also be used, in principle, for the problem of stability of a small perturbed motion as described in [4]—[6]. In treating the case with viscosity, this method is more effective. Its limitation lies in its inability to treat the nonlinear problem.

Now, some possible applications of the method can be listed as follows:

(1) Solving the Columbus problem. This is a classical problem, its solution has found practical application in the theory of liquid gyroscope and geophysics. We will treat this problem in another paper.

(2) Dealing with the problem of stability of motion of a solid body with cavities partly filled with liquid[4,5].

(3) Treating the cases where the shell with liquid-filled cavities is elastic or connects with an elastic frame[16,17].

The author is deeply grateful to Professor Tan Haosheng (H. S. Tan 谈镐生) for his guidance and help in this work.

References

[1] 徐硕昌：旋转磁流体力学系统的变分原理及其应用，《科学通报》，**20** (1975)，372—378.

[2] Lamb, H.: *Hydrodynamics*, 6th ed., Cambridge, (1932), §384.

[3] Rumyantsev, V. V.: Stability of motion of solid bodies with liquid-filled cavities by Lyapunov's methods, *Advances in Applied Mechanics*, Vol. VIII, Academic Press, New York and London, (1964).

[4] Моисеев, Н. Н. и Румянцев, В. В.: *Динамика тела с полостями, содержащими жидкость*, М., изд-во «Наука», (1965).

[5] Микишев, Г. Н. и Рабинович, Б. И.: *Динамика твердого тела с полостями, частично заполненными жидкостью*, Изд. Машиностроние, (1968).

[6] Черноусько, Ф. Л.: *Движение твёрдого тела с полостями, содержащими вязкую жидкость.*, М., В. Ц. АН СССР, (1968).

[7] Greespan, H. P.: On the general theory of contained rotating fluid, *J. Fluid Mech.*, **22** (1965), 449—462.

[8] Couront, R. & Hilbert, D.: *Methods of Mathematical Physics*, Vols. 1, 2, Interscience, New York, (1962).

[9] Prasad, S. N. & Herrmann, G.: Adjoint variational method in non-conservative stability problem, *Int. J. Solid & Struct.*, **8** (1972), 29—40.

[10] Смирнова, Е. М.: Устойчивости свободного вращенися волчка, содержащего тороидамкую полость с жидком малой вязкоми, *Изв. АНСССР, МТТ*, **5** (1976).

[11] Соболев, С. Л.: О движении симметричного волчка с полостью, наполненной жидкостью, *ПМТФ*, **3** (1960), 20—55.

[12] Ишлинский, А. Ю. и Темчеко, М. Е.: О малых колебаниях вертикальной оси волчка, имеющего полость, целиком наполненную идеальной несжимаемой жидкостью, *ПМТФ*, **3**(1960), 65—75.

[13] 谷一郎，粘性流体の理論，岩波講座，1957.

[14] Ландау, Л. Д. и Лифшиц, Е. М.: *Механика сплошных сред*, Гостехиздат, (1953).

[15] Четаев, Н. Г.: *Устойчивость движения*, Гостехиздат, (1955).

[16] Meirovitch, L.: A stationarity principle for the eigenvalue problem for rotating structure, *AIAA*, **14** (1976), 1387.

[17] Руменцев, В. В.: О движинии и устойчивости упругого тела с полостью, содержащей жидкость, *ПММ*, **33** (1969).