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FRACTURE CRITERIA FOR COMBINED MODE CRACKS

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ABSTRACT

The complex variable method is employed to analyse the energy release rate for combined mode cracks. A functional integral equation, which contains no singularity, is derived for a branched crack problem by a functional transformation. The integrand $\varphi_1'(z)$ is expanded in eigenfunctions. The energy fracture criterion for the combined mode (K_I and K_{II}) cracks is then derived when the propagation branch is made to approach zero. An energy fracture criterion is also presented for the case that K_{III} is present. In addition, a new fracture criterion for combined mode cracks based on the stress parameters is proposed.

I. INTRODUCTION

The linear elastic fracture mechanics (LEFM) has been successfully employed in solving the problem of the unstable growth of the opening mode cracks. But in engineering practice cracks are usually in a combined mode state of deformation in which all K_I, K_{II}, and K_{III} are present. Crack branching will take place in cases where the loading is unsymmetrical, or the crack is in an unsymmetrical position, or the material is anisotropic, or the crack is propagating at a high velocity. Therefore, the investigation into the fracture criteria for the combined mode crack is of great significance in respect of theoretical study and such a criterion has a wide application in practice as well.

There are two kinds of criteria of the combined mode fracture: the energy release rate criteria[1-3] and the stress parameter criteria[4,5].

The problem of crack branching was analysed by Anderson[1], who was among the first to make an attempt to solve the problem by a complex variable method. Hussain *et al.*[3] gave a detailed analysis of the energy release rate criterion, but it appears to the author that there are some points in his derivation remaining questionable.

The complex variable method is employed in this paper to analyse the energy release rate for combined mode cracks. A functional integral equation, which contains no singularity, is derived to solve the branched crack problem by a functional transformation. The integrand $\varphi_1'(z)$ is expanded in eigenfunctions. The energy fracture criterion for the combined mode (K_I and K_{II}) cracks is then derived when the propagation branch is made to approach zero. An energy fracture criterion is also presented for the case that K_{III} is present. In addition, a new fracture criterion for combined mode cracks based on the stress parameters is proposed.

II. Fundamental Equation and Its Transformation

A crack branch which makes an angle γ with the main crack is considered as shown in Fig. 1. According to [3], we have the following formulas for the mapping function $\omega(\zeta)$:

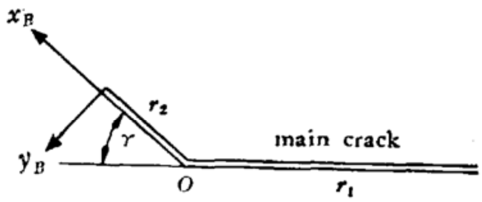

Fig. 1. Crack with branch.

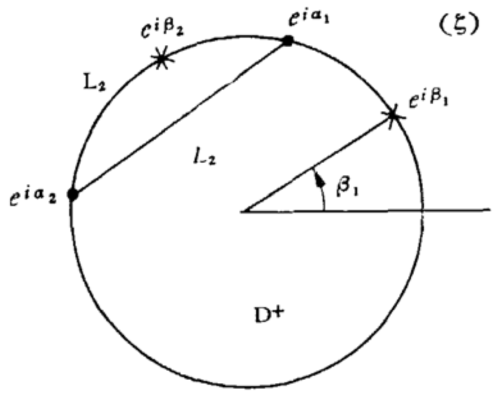

Fig. 2. ζ-plane.

$$\omega(\zeta) = \frac{A}{\zeta}(\zeta - e^{i\alpha_1})^{\lambda_1}(\zeta - e^{i\alpha_2})^{\lambda_2}, \tag{1}$$

$$\lambda_1 = (1 - \gamma/\pi), \quad \lambda_2 = (1 + \gamma/\pi), \tag{2}$$

$$\left.\begin{array}{l} \alpha_1\lambda_1 + \alpha_2\lambda_2 = 2\pi, \\ \lambda_1 \operatorname{ctg}\left(\dfrac{\alpha_1 - \beta_1}{2}\right) + \lambda_2 \operatorname{ctg}\left(\dfrac{\alpha_2 - \beta_1}{2}\right) = 0, \\ \lambda_1 \operatorname{ctg}\left(\dfrac{\alpha_1 - \beta_2}{2}\right) + \lambda_2 \operatorname{ctg}\left(\dfrac{\alpha_2 - \beta_2}{2}\right) = 0. \\ r_1 = 4A\left(\sin\dfrac{\alpha_1 - \beta_1}{2}\right)^{\lambda_1}\left(\sin\dfrac{\alpha_2 - \beta_1}{2}\right)^{\lambda_2}, \\ r_2 = 4A\left(\sin\dfrac{\beta_2 - \alpha_1}{2}\right)^{\lambda_1}\left(\sin\dfrac{\alpha_2 - \beta_2}{2}\right)^{\lambda_2}. \end{array}\right\}. \tag{3}$$

Denoting

$$\varepsilon = \frac{\alpha_2 - \beta_2}{2}, \qquad \delta = \frac{\beta_2 - \alpha_1}{2}, \tag{4}$$

we have,

$$\left.\begin{array}{l} \delta = \operatorname{tg}^{-1}\left(\dfrac{\lambda_1}{\lambda_2}\operatorname{tg}\varepsilon\right), \\ \beta_1 = (\varepsilon - \delta) - (\varepsilon + \delta)\gamma/\pi, \\ \beta_2 = (\delta - \varepsilon) - (\varepsilon + \delta)\gamma/\pi + \pi, \\ r_1 = 4A(\cos\varepsilon)^{\lambda_1}(\cos\delta)^{\lambda_2}, \\ r_2 = 4A(\sin\delta)^{\lambda_1}(\sin\varepsilon)^{\lambda_2}. \end{array}\right\}. \tag{5}$$

In the limit as ε approaches zero, r_2, δ, and β_1 approach zero, α_1, α_2, and β_2 approach π, and r_1 approaches $4A$. The boundary value problem of elasticity can be reduced to the problem of finding $\varphi(\zeta)$ and $\psi(\zeta)$,

$$\sigma_x + \sigma_y = 4\mathrm{Re}\{\varphi'(\zeta)/\omega'(\zeta)\}, \tag{6}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\{[\overline{\omega(\zeta)}/\omega'(\zeta)][\varphi'(\zeta)/\omega'(\zeta)]' + \psi'(\zeta)/\omega'(\zeta)\}. \tag{7}$$

Being holomorphic in the exterior of a unit circle, $\varphi(\zeta)$ and $\psi(\zeta)$ satisfy the following boundary conditions:

$$\varphi^-(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'^-(\sigma)} + \overline{\psi^-(\sigma)} = 0. \qquad \sigma \in L \tag{8}$$

Denoting

$$\varphi_*(\zeta) = (\zeta - e^{i\beta_1})(\zeta - e^{i\beta_2})\varphi(\zeta), \tag{9}$$

we obtain, after some manipulation (Appendix 1),

$$\varphi_*(\zeta) = G_\infty(\zeta) - M_0(\zeta) + G_0'(\zeta) + \frac{(1-e^{-2\gamma i})}{2\pi i}\int_{L_2} \frac{\overline{\varphi'^-(\sigma)}}{(\sigma-\zeta)} \frac{g_*(\sigma)}{\sigma}\, d\sigma, \quad \zeta \in D^- \tag{10}$$

where

$$\left\{\begin{aligned} G_\infty(\zeta) &= (\zeta - e^{i\beta_1})(\zeta - e^{i\beta_2})(\Gamma A\zeta + A_0) \\ &\quad + A_1(\zeta - \gamma_1 - \gamma_2) + A_2, \\ M_0(\zeta) &= \bar{\Gamma}' A e^{i(\beta_1+\beta_2)}/\zeta, \\ G_0'(\zeta) &= \bar{\Gamma} A e^{i(\alpha_1+\alpha_2)}/\zeta. \end{aligned}\right. \tag{11}$$

Eq. (10) is the fundamental equation after the transformation. The coefficients Γ, Γ', A_0, A_1, and A_2 are determined by the behaviour of functions $\phi(\zeta)$ and $\psi(\zeta)$ at the infinity.

A further manipulation gives that

$$\varphi'^-(\gamma_2) = \varphi_0'(\gamma_2) + \frac{1}{(\gamma_2-\gamma_1)} \cdot \left\{\frac{1}{2} f_0''^-(\gamma_2) - \frac{f_0'^-(\gamma_2)(\gamma_2-\gamma_1) - f_0^-(\gamma_2) + f_0^-(\gamma_1)}{(\gamma_2-\gamma_1)^2}\right\}, \tag{12}$$

where

$$\gamma_1 = e^{i\beta_1}, \qquad \gamma_2 = e^{i\beta_2},$$

$$\varphi_0(\zeta) = \Gamma A\zeta + A_0 - \frac{A(\bar{\Gamma} + \bar{\Gamma}')}{\zeta}, \tag{13}$$

$$f_0(\zeta) = \frac{(1-e^{-2\gamma i})}{2\pi i}\int_{L_2} \frac{\overline{\varphi'^-(\sigma)}g_*(\sigma)}{\sigma(\sigma-\zeta)}\, d\sigma, \tag{14}$$

$$g_*(\sigma) = (\sigma - e^{i\alpha_1})(\sigma - e^{i\alpha_2}). \tag{15}$$

In the limit as the length of the branch goes to zero, it can be shown (Appendix 2) that

$$\varphi'^-(\gamma_2) = \varphi_0'(\gamma_2) - \frac{1}{4}(1-e^{-2\gamma i}) \cdot C^* \cdot \overline{\varphi'^-(\gamma_2)}\,, \tag{16}$$

where

$$C^* = C_1^* + iC_2^*, \tag{17}$$

$$C_1^* = \left(\frac{\lambda_1}{\lambda_2}\right)^{\gamma/2\pi} \cdot \left\{P(t_2) + \frac{1}{2}\Omega(t_2)P_2(t_2)\right\},$$

$$C_2^* = \frac{\Omega(t_2)}{\pi}\left(\frac{\lambda_1}{\lambda_2}\right)^{\gamma/2\pi} \cdot \left\{\frac{1}{\Omega(t_2)}\int_0^1 \frac{P(t)-P(t_2)}{(t-t_2)}dt - \left[\int_0^{t_2-\xi} + \int_{t_2+\xi}^1\right]\frac{dt}{(t-t_2)^3P(t)}\right.$$
$$+ \frac{1}{2}\int_{t_2-\xi}^{t_2+\xi}\frac{P_2(t)-P_2(t_2)}{t-t_2}dt - \frac{1}{2\xi}[P_1(t_2+\xi)+P_1(t_2-\xi)]$$
$$\left. + \frac{1}{2\xi^2}\left[\frac{1}{P(t_2+\xi)} - \frac{1}{P(t_2-\xi)}\right] + \frac{P(t_2)}{\Omega(t_2)}\ln\frac{(1-t_2)}{t_2}\right\}. \tag{18}$$

Functions $\Omega(t)$, $P(t)$, $P_1(t)$, and $P_2(t)$ are given in Appendix 2. The result given in [3] is equivalent to the case $C^* = 1$. The calculated values of $C_1{}^*$ and $C_2{}^*$ are listed in Table 1. As the length of the branch approaches zero, the stress intensity

Table 1

Values of C_1^* and C_2^*

γ	0°	5°	10°	15°	20°
C_1^*	1.00	1.0003	1.0010	1.0023	1.0042
$-C_2^*$	0	4.137×10^{-3}	8.297×10^{-3}	1.250×10^{-2}	1.678×10^{-2}
γ	25°	30°	35°	40°	45°
C_1^*	1.0066	1.0095	1.0131	1.0173	1.0222
$-C_2^*$	2.116×10^{-2}	2.566×10^{-2}	3.031×10^{-2}	3.515×10^{-2}	4.022×10^{-2}
γ	50°	55°	60°	65°	70°
C_1^*	1.0279	1.0343	1.0417	1.0500	1.0594
$-C_2^*$	4.555×10^{-2}	5.118×10^{-2}	5.718×10^{-2}	6.361×10^{-2}	7.054×10^{-2}
γ	75°	80°	85°	90°	
C_1^*	1.0700	1.0821	1.0957	1.1110	
$-C_2^*$	7.804×10^{-2}	8.624×10^{-2}	9.524×10^{-2}	0.1052	

factors at the branch tip approach the following limiting values:

$$K_I - iK_{II} = \frac{(\alpha_0 - \bar{\alpha}_0\beta_0)}{1-\beta_0\bar{\beta}_0}, \tag{19}$$

where

$$\alpha_0 = (\mathring{K}_I - i\mathring{K}_{II})e^{\gamma i}\left(\frac{\lambda_1}{\lambda_2}\right)^{\gamma/2\pi}, \tag{20}$$

$$\beta_0 = \frac{1}{4}(e^{2\gamma i} - 1) \cdot C^*, \tag{21}$$

and $\mathring{K}_I$ and $\mathring{K}_{II}$ are the stress intensity factors of a crack which does not have a branch.

III. Energy Release Rate and Energy Fracture Criterion

In the vicinity of any crack tip, the stresses and the strains are determined by

$$\begin{aligned}
\sigma_r &= \frac{1}{2\sqrt{2\pi r}}\left\{K_I(3-\cos\theta)\cos\frac{\theta}{2} + K_{II}(3\cos\theta - 1)\sin\frac{\theta}{2}\right\},\\
\sigma_\theta &= \frac{1}{2\sqrt{2\pi r}}\{K_I(1+\cos\theta) - K_{II}\cdot 3\sin\theta\}\cos\frac{\theta}{2},\\
\tau_{r\theta} &= \frac{1}{2\sqrt{2\pi r}}\{K_I\sin\theta + K_{II}(3\cos\theta - 1)\}\cos\frac{\theta}{2};
\end{aligned} \tag{22}$$

$$\begin{aligned}
u_r &= \frac{1}{4\mu}\sqrt{\frac{r}{2\pi}}\left\{K_I\left[(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] - K_{II}\left[(2\kappa-1)\sin\frac{\theta}{2} - 3\sin\frac{3\theta}{2}\right]\right\},\\
u_\theta &= \frac{1}{4\mu}\sqrt{\frac{r}{2\pi}}\left\{K_I\left[-(2\kappa+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] - K_{II}\left[(2\kappa+1)\cos\frac{\theta}{2} - 3\cos\frac{3\theta}{2}\right]\right\},
\end{aligned} \tag{23}$$

from which it can be seen that the displacements on the upper and the lower edges are equal in magnitude and opposite in sign (apart from a uniform displacement of the crack tip). When a branch of length r_2 at an angle θ to the main crack is developed from the main crack, the energy released from the elastic system is equal to

$$\begin{aligned}
G\cdot r_2 &= \frac{1}{2}\int_0^{r_2}\{\mathring{\sigma}_\theta u_\theta^{(1)} + \mathring{\tau}_{r\theta}u_r^{(1)}\}dr - \frac{1}{2}\int_0^{r_2}\{\mathring{\sigma}_\theta u_\theta^{(2)} + \mathring{\tau}_{r\theta}u_r^{(2)}\}dr\\
&= \int_0^{r_2}\{\mathring{\sigma}_\theta u_\theta^{(1)} + \mathring{\tau}_{r\theta}u_r^{(1)}dr\} = \frac{(\kappa+1)}{16\mu}r_2\{K_I\mathring{f}_1 + K_{II}\mathring{f}_2\}.
\end{aligned}$$

Therefore, the energy release rate is

$$G = \frac{\kappa+1}{16\mu}\{K_I\mathring{f}_1 + K_{II}\mathring{f}_2\}, \tag{24}$$

$$\begin{aligned}
\mathring{f}_1 &= \{\mathring{K}_I(1+\cos\theta) - \mathring{K}_{II}\cdot 3\sin\theta\}\cos\frac{\theta}{2},\\
\mathring{f}_2 &= \{\mathring{K}_I\sin\theta + \mathring{K}_{II}(3\cos\theta - 1)\}\cos\frac{\theta}{2},
\end{aligned} \tag{25}$$

where the superscript $\circ$ is used to denote the functions and the physical quantities of the crack without a branch. The case of Fig. 1 is equivalent to the case $\theta = -\gamma$.

According to the energy fracture criterion, the crack will propagate in the direction where the energy release rate is maximum, and the crack will start to propagate when this maximum energy release rate G_{max} reaches a critical value. The calculation of

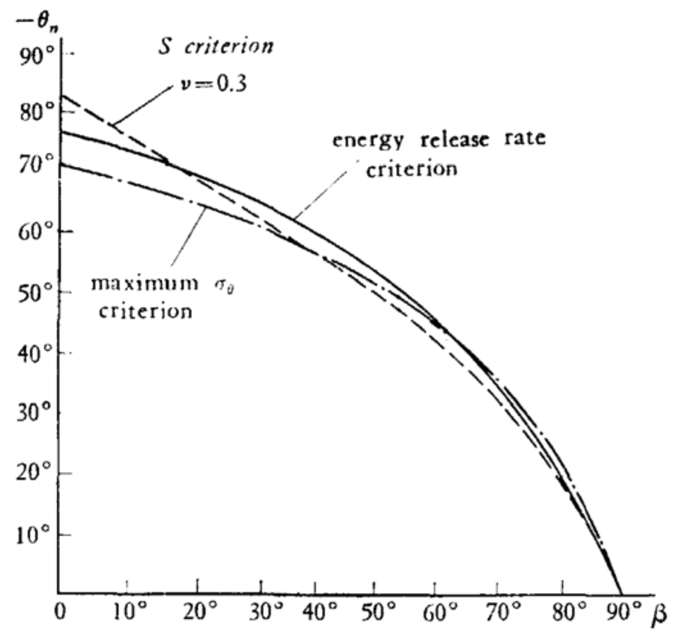

Fig. 3. Fracture angles for inclined crack under uniaxial tension.

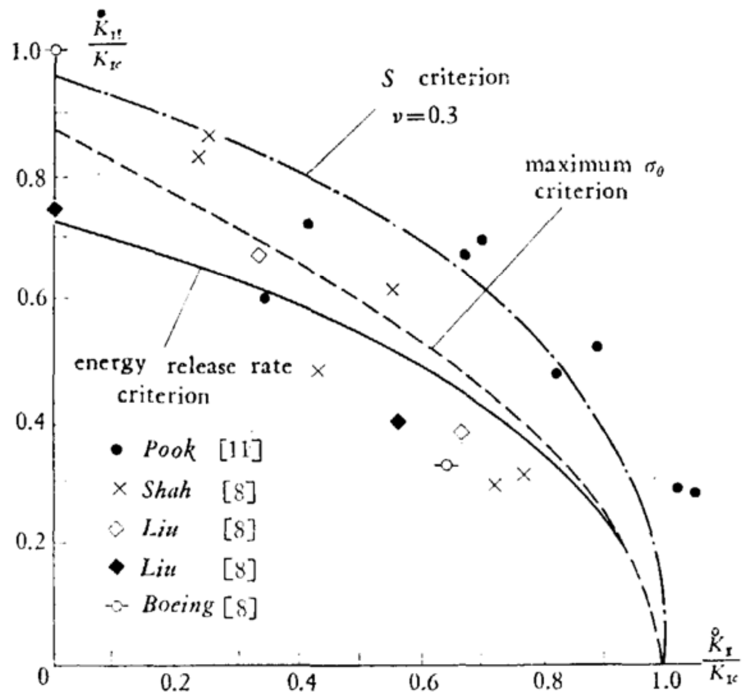

Fig. 4. Critical $\mathring{K}_I$ and $\mathring{K}_{II}$ for inclined crack under uniaxial tension.

Eq. (24) leads to the following results: for a crack of the sliding mode, the fracture angle is $\gamma = 76.2°$, and $K_{IIc} = 0.724\ K_{Ic}$, while according to the maximum σ_θ criterion, $K_{IIc} = 0.87\ K_{Ic}$ and the fracture angle is $\gamma = 70.5°$, and the criterion of the minimum strain energy density gives $K_{IIc} = 0.96\ K_{Ic}$ and $\gamma = 82.3°$ (with $\nu = 0.3$).

For the case of the uniaxial tension with an inclined crack, the fracture angles are shown in Fig. 3, and the correlation curve of $\mathring{K}_I$ and $\mathring{K}_{II}$ in the critical state is shown in Fig. 4. Also shown in these figures are the experimental results available, which scatter in a rather wide range.

IV. Energy Fracture Criterion Incorporating K_{III}

As shown in Fig. 5, due to the combined action of the axial stress σ and the antiplane shear stress τ at the infinity, all $\mathring{K}_I$, $\mathring{K}_{II}$, and $\mathring{K}_{III}$ are present and they are

$$\left.\begin{aligned}\mathring{K}_I &= \sigma\sqrt{\pi a}\sin^2\beta,\\ \mathring{K}_{II} &= \sigma\sqrt{\pi a}\sin\beta\cos\beta,\\ \mathring{K}_{III} &= \tau\sqrt{\pi a}\sin\beta.\end{aligned}\right\} \tag{26}$$

Since the antiplane shear produces only the displacement w, in the direction perpendicular to the plane, the in-plane displacements u and v are both equal to zero for the case that K_{III} alone is present. As the axial traction is only responsible for the strains in the plane, it has no contribution to the strains in the direction perpendicular to the plane. When an infinitesimal branch is developed, the stress intensity factors at the branch tip are

$$K_I - iK_{II} = \frac{\alpha_0 - \bar{\alpha}_0\beta_0}{1-\beta_0\bar{\beta}_0}, \tag{27}$$

$$K_{III} = \mathring{K}_{III}\left(\frac{1-m}{1+m}\right)^{m/2}, \tag{28}$$

where $m=\gamma/\pi$. Eq. (28) was derived in [9].

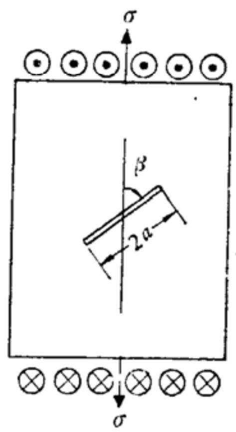

Fig. 5. Combined action of axial stress σ and antiplane shear stress τ.

According to [7], the total potential energy released during the formation of the new crack surfaces C_1' and C_2' can be calculated by

$$-\Delta\Pi = -\frac{1}{2}\int_{C_1'+C_2'} T_i\Delta u_i dS, \tag{29}$$

where T_i are the tractions acted on the surfaces C_1' and C_2' before the crack has extended, and Δu_i are the additional displacements produced after the crack has extended. With the action of the antiplane shear, the stresses and the strains at the crack tip are

$$\tau_{zr} - i\tau_{z\theta} = \frac{K_{III}}{\sqrt{2\pi r}}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right), \tag{30}$$

$$w = \frac{1}{\mu}\sqrt{\frac{2r}{\pi}} \cdot K_{III}\sin\frac{\theta}{2}. \tag{31}$$

Considering the propagation branch as shown in Fig. 1, the stresses along OB before the crack extension are

$$\mathring{\tau}_{zr} - i\mathring{\tau}_{z\theta} = -\frac{\mathring{K}_{III}}{\sqrt{2\pi r}}\left(\sin\frac{\gamma}{2} + i\cos\frac{\gamma}{2}\right), \tag{32}$$

and the additional displacements after the crack extension are

$$w = \pm\frac{1}{\mu}\sqrt{\frac{2(r_2 - r)}{\pi}}\,K_{III}, \tag{33}$$

where K_{III} is the stress intensity factor at the tip of the propagation branch B after the extension. Substituting Eqs. (32) and (33) into (29), the energy release rate is obtained:

$$G = \lim_{r_2\to 0} -\left(\frac{\Delta\Pi}{r_2}\right) = \frac{1}{2\mu}K_{III}\mathring{K}_{III}\cos\frac{\gamma}{2}. \tag{34}$$

Combining Eqs. (34) and (24), we obtain the energy release rate under the combined action of K_I, K_{II}, and K_{III} as follows:

$$G = \frac{(1-\nu^2)}{E}\left\{\frac{1}{2}\left[K_I\mathring{f}_1 + K_{II}\mathring{f}_2\right] + \frac{1}{(1-\nu)}K_{III}\mathring{K}_{III}\cos\frac{\gamma}{2}\right\}. \tag{35}$$

According to the energy fracture criterion and Eq. (35), it follows that

$$K_{IIIc} = \sqrt{(1-\nu)}K_{Ic}. \tag{36}$$

The correlation curves of $\mathring{K}_I$, $\mathring{K}_{II}$, and $\mathring{K}_{III}$ in the critical state are shown in Fig. 6. The

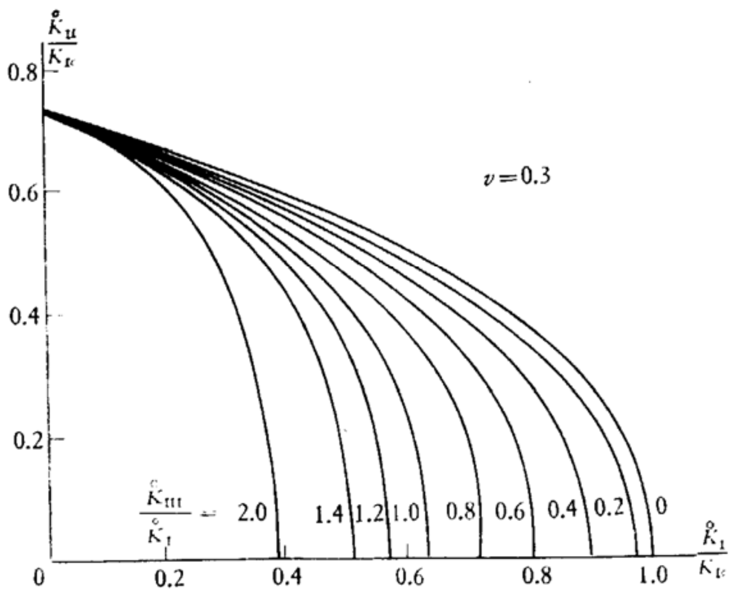

Fig. 6. Correlation curves of $\mathring{K}_I$, $\mathring{K}_{II}$ and $\mathring{K}_{III}$.

correlation curve of $\mathring{K}_I$ and $\mathring{K}_{III}$ with $\mathring{K}_{II}$ equal to zero is shown in Fig. 7. The curve can be represented by the following equation:

$$\left(\frac{\mathring{K}_I}{K_{Ic}}\right)^2 + \left(\frac{\mathring{K}_{III}}{K_{IIIc}}\right)^2 = 1. \tag{37}$$

It can be seen from Fig. 7 that the theory is in fairly good agreement with the experimental data[11].

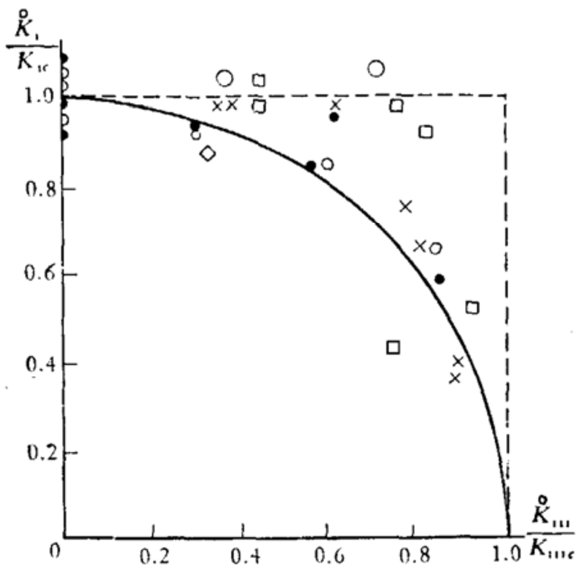

Fig. 7. Theoretical and experimental results of crack under the action of $\mathring{K}_I$ and $\mathring{K}_{III}$.

V. Stress Parameter Criterion for Combined Mode Fracture

Among the stress parameter criteria for the combined mode fracture, the maximum σ_θ criterion and the minimum strain-energy-density criterion are commonly used[4]. Both are based on a comparison of the mechanical quantities on the circles with the crack tip as their centre. This kind of comparison has a clear geometrical significance, but it can be argued that the different points on the circle are not in the same mechanical state (**Fig.** 8).

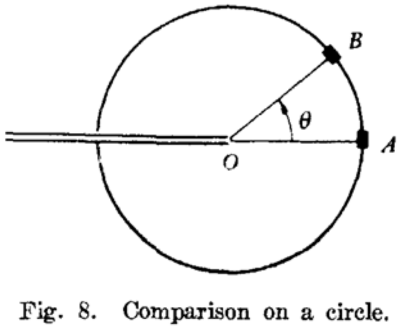

Fig. 8. Comparison on a circle.

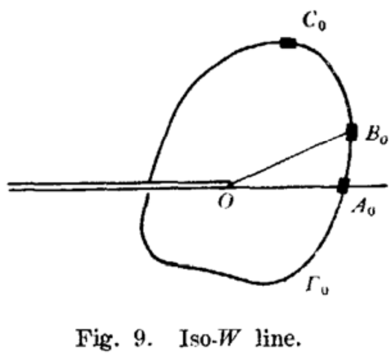

Fig. 9. Iso-W line.

Consider the strain energy density in the front of the crack

$$W = \frac{1}{\pi r}(a_{11}K_I^2 + 2a_{12}K_IK_{II} + a_{22}K_{II}^2), \tag{38}$$

where

$$a_{11} = \frac{1}{16\mu}\{(1 + \cos\theta)(\kappa - \cos\theta)\},$$
$$a_{12} = \frac{1}{16\mu}\{2\cos\theta - (\kappa - 1)\}\sin\theta, \tag{39}$$
$$a_{22} = \frac{1}{16\mu}\{(\kappa + 1)(1 - \cos\theta) + (1 + \cos\theta)(3\cos\theta - 1)\},$$

and

$$\kappa = \begin{cases} 3 - 4\nu, & \text{(for plane strain)} \\ \dfrac{3-\nu}{1+\nu}. & \text{(for plane stress)} \end{cases} \tag{40}$$

We choose the strain energy density W as a mechanical measure to characterize the brittle fracture and consider the lines with equal strain-energy-densities (the iso-W lines) (Fig. 9). For example, if $W = a_0$ on an iso-W line Γ_0, the points A_0, B_0, and C_0 on the line will have the same strain-energy-density. Since the elements, with the points A_0, B_0, C_0 etc. as their centres, contain the same quantity of the strain energy, these points can be compared with each other and along the direction of the point where the circumferential stress σ_θ is maximum and the fracture is most apt to occur. Thereby a new criterion is obtained to determine the direction along which the crack will start to propagate, that is, the crack will start to grow in the direction where the circumferential stress σ_θ is maximum on an iso-W line. Let the fracture angle be θ_0, then

$$(\sigma_\theta)_{\theta=\theta_0} = \max_{W=a_0}(\sigma_\theta). \tag{41}$$

The load at which the crack will start to grow can be determined by

$$\lim_{r\to 0} \sqrt{2\pi r}\,(\sigma_\theta)_{\theta=\theta_0} = K_{Ic}. \tag{42}$$

On the iso-W lines we have

$$W = \frac{S}{r} = a_0, \tag{43}$$

where S is the strain-energy-density factor given by

$$S = \frac{1}{\pi}(a_{11}K_I^2 + 2a_{12}K_IK_{II} + a_{22}K_{II}^2). \tag{44}$$

In the front of the crack we have

$$\sigma_\theta = \frac{1}{2\sqrt{2\pi r}}\{K_I(1 + \cos\theta) - 3\sin\theta K_{II}\}\cos\frac{\theta}{2}. \tag{45}$$

From Eq. (43) we have

$$r = \frac{S}{a_0}. \tag{46}$$

Substituting Eq. (46) into Eq. (45), we obtain

$$\sigma_\theta = \frac{\sqrt{a_0}}{2\sqrt{2\pi S}}\{K_I(1+\cos\theta) - 3\sin\theta K_{II}\}\cos\frac{\theta}{2}. \tag{47}$$

Eq. (47) gives the relationship between the circumferential stress σ_0 and θ on the iso-W lines. Since a_0 is a positive constant, the fracture angle θ_0 can be determined by the point where the following function f is maximum:

$$f(\theta) = \frac{1}{\sqrt{\pi S}}\{K_I(1+\cos\theta) - 3\sin\theta K_{II}\}\cos\frac{\theta}{2}. \tag{48}$$

The calculated results of the in-plane shear of a plate with a central crack are given in Table 2. A fracture test is proposed in [3] on a 6-inch-wide by 16-inch-long panel of a 0.002-inch-thick steel foil containing a circular crack, where a pure shear state at the crack tip can be realised. The measured fracture angles have an average value of $-75.4°$, which is in good agreement with the theory just described. The fracture angles for the case of the uniaxial tension with an inclined crack are shown in Table 3 and they are in good agreement with the experimental data.

Table 2

$-\theta_0$ for In-Plane Shear (in degrees)

ν	0	0.1	0.2	0.3	0.4
S-criterion	70.5	74.5	78.5	82.3	86.2
Present criterion	70.5	72.3	74.5	76.5	79.5

Table 3

Fracture Angles of Inclined Crack Under Uniaxial Tension (in degrees)

β	30	40	50	60	70	80
Max. σ_θ criterion	60.2	55.7	50.2	43.2	33.2	19.3
S-criterion	63.5	56.7	49.5	41.5	31.8	18.3
Present criterion	62.4	56.2	49.9	42.4	32.6	18.7
Test results[4]	62.4	55.6	51.1	43.1	30.7	17.3

Appendix 1

$\varphi(\zeta)$ and $\psi(\zeta)$ are holomorphic functions in the exterior of a unit circle in the image plane and satisfy the following boundary condition:

$$\varphi^-(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'^-(\sigma)} + \overline{\psi^-(\sigma)} = 0. \quad \sigma \in L \tag{1}$$

According to [1], we have

$$\frac{\omega'(\zeta)}{\omega(\zeta)} = \frac{(\zeta - e^{i\alpha_1})(\zeta - e^{i\alpha_2})}{\zeta(\zeta - e^{i\beta_1})(\zeta - e^{i\beta_2})} = \frac{1}{\zeta g(\zeta)}. \tag{2}$$

By the mapping function $\omega(\zeta)$ a deflected crack in the physical plane is mapped onto a unit circle in the ζ-plane, as shown in Fig. 2, where the arcs L_1 and L_2 are the images of the main crack and the propagation branch in the physical plane, respectively. Hence we have

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \begin{cases} \overline{\sigma g(\sigma)}, & \sigma \in L_1 \\ \overline{\sigma g(\sigma)} e^{-2\gamma i}. & \sigma \in L_2 \end{cases} \tag{3}$$

We locate the branch cut along a secant $\bar{L}_2$ for the mapping function $\omega(\zeta)$, so $\omega(\zeta)$ and $\omega'(\zeta)$ cross continuously the unit circle (apart from two points $e^{i\alpha_1}$ and $e^{i\alpha_2}$).

Introducing a jump function $h(\sigma)$, as

$$h(\sigma) = \begin{cases} 1, & \sigma \in L_1 \\ e^{-2\gamma i}, & \sigma \in L_2 \end{cases} \tag{4}$$

and noting that $\overline{g(\sigma)} = -g(\sigma)$, we can write Eq. (1) as

$$\varphi^-(\sigma) - \frac{g(\sigma)}{\sigma} h(\sigma)\overline{\varphi'^-(\sigma)} + \overline{\psi^-(\sigma)} = 0. \tag{5}$$

Let

$$f_*(\sigma) = (\sigma - e^{i\beta_1})(\sigma - e^{i\beta_2}), \tag{6}$$

$$g_*(\sigma) = (\sigma - e^{i\alpha_1})(\sigma - e^{i\alpha_2}), \tag{7}$$

$$\varphi_*(\sigma) = f_*(\sigma)\varphi(\sigma), \tag{8}$$

and multiply Eq. (5) by the function $f_*(\sigma)$, we have

$$\varphi_*^-(\sigma) - \frac{g_*(\sigma)}{\sigma} h(\sigma)\overline{\varphi'^-(\sigma)} + f_*(\sigma)\overline{\psi^-(\sigma)} = 0. \qquad \sigma \in L \tag{9}$$

Assuming that the function $\psi(\zeta)$ has poles of order one at the points $\zeta = e^{i\beta_1}$ and $\zeta = e^{i\beta_2}$, it can be shown that the function $f_*(\zeta)\,\psi(1/\zeta)$ is holomorphic in the interior of the unit circle, except for the origin. From Eq. (9), using the extended Cauchy's integral formula, we obtain

$$-\varphi_*(\zeta) + G_\infty(\zeta) - \frac{1}{2\pi i}\oint_L \frac{h(\sigma)g_*(\sigma)}{\sigma(\sigma-\zeta)}\overline{\varphi'^-(\sigma)}d\sigma - M_0(\zeta) = 0, \qquad \zeta \in D^- \tag{10}$$

where $G_\infty(\zeta)$ is the main part of the function $\varphi_*(\zeta)$ in the neighbourhood of $\zeta = \infty$ and $M_0(\zeta)$ is the main part of the function $f_*(\zeta)\,\psi(1/\zeta)$ in the neighbourhood of $\zeta = 0$.

Assume that in the infinity we have

$$\varphi(\zeta) = \Gamma A\zeta + A_0 + \frac{A_1}{\zeta} + \frac{A_2}{\zeta^2} + \cdots, \tag{11}$$

$$\psi(\zeta) = \Gamma' A\zeta + B_0 + \frac{B_1}{\zeta} + \frac{B_2}{\zeta^2} + \cdots. \tag{12}$$

From Eq. (10) we can obtain

$$\varphi_*(\zeta) = G_\infty(\zeta) - M_0(\zeta) + G_0'(\zeta) + \frac{(1 - e^{-2\gamma i})}{2\pi i}\int_{L_*} \frac{\overline{\varphi'^-(\sigma)}}{(\sigma - \zeta)} \frac{g_*(\sigma)}{\sigma} d\sigma, \tag{13}$$

where $G_0'(\zeta)$ is the main part of the function $\frac{g_*(\zeta)}{\zeta}\bar{\varphi}'\left(\frac{1}{\zeta}\right)$, holomorphic in the interior of the unit circle, in the neighbourhood of $\zeta = 0$. From Eqs. (11) and (12), we have

$$G_\infty(\zeta) = (\zeta - \gamma_1)(\zeta - \gamma_2)(\Gamma A\zeta + A_0) + A_1(\zeta - \gamma_1 - \gamma_2) + A_2, \tag{14}$$

$$M_0(\zeta) = \bar{\Gamma}' A\gamma_1\gamma_2/\zeta, \tag{15}$$

$$G_0'(\zeta) = \bar{\Gamma} A\sigma_1\sigma_2/\zeta, \tag{16}$$

where

$$\gamma_1 = e^{i\beta_1}, \quad \gamma_2 = e^{i\beta_2}, \quad \sigma_1 = e^{i\alpha_1}, \quad \sigma_2 = e^{i\alpha_2}.$$

Let

$$f_0(\zeta) = \frac{(1 - e^{-2\gamma i})}{2\pi i}\int_{L_2} \frac{\overline{\varphi'^-(\sigma)}}{(\sigma - \zeta)} \frac{g_*(\sigma)}{\sigma} d\sigma. \tag{17}$$

Eq. (13) becomes

$$\begin{aligned}\varphi_*(\zeta) = &(\zeta - \gamma_1)(\zeta - \gamma_2)(\Gamma A\zeta + A_0) + (\zeta - \gamma_1 - \gamma_2)A_1 + A_2 \\ &+ \frac{A}{\zeta}(\sigma_1\sigma_2\bar{\Gamma} - \gamma_1\gamma_2\bar{\Gamma}') + f_0(\zeta).\end{aligned} \tag{18}$$

In the limit as ζ approaches γ_1 and γ_2 from outside of the unit circle, we have

$$\left.\begin{aligned}- A_1\gamma_2 + A_2 + \frac{A}{\gamma_1}(\sigma_1\sigma_2\bar{\Gamma} - \gamma_1\gamma_2\bar{\Gamma}') + f_0^-(\gamma_1) = 0, \\ - A_1\gamma_1 + A_2 + \frac{A}{\gamma_2}(\sigma_1\sigma_2\bar{\Gamma} - \gamma_1\gamma_2\bar{\Gamma}') + f_0^-(\gamma_2) = 0,\end{aligned}\right\} \tag{19}$$

from which we obtain

$$A_1 = -A(\bar{\Gamma} + \bar{\Gamma}') - \frac{f_0^-(\gamma_2) - f_0^-(\gamma_1)}{(\gamma_2 - \gamma_1)},$$

and

$$A_2 = \frac{\gamma_1 f_0^-(\gamma_1) - \gamma_2 f_0^-(\gamma_2)}{(\gamma_2 - \gamma_1)}. \tag{20}$$

Substituting Eq. (20) into Eq. (18), after rearrangement, we have

$$\varphi(\zeta) = \varphi_0(\zeta) + \frac{1}{(\gamma_2 - \gamma_1)}\left\{\frac{f_0(\zeta) - f_0^-(\gamma_2)}{(\zeta - \gamma_2)} - \frac{f_0(\zeta) - f_0^-(\gamma_1)}{(\zeta - \gamma_1)}\right\}, \tag{21}$$

where

$$\varphi_0(\zeta) = \Gamma A\zeta + A_0 - \frac{A}{\zeta}(\bar{\Gamma} + \bar{\Gamma}'), \tag{22}$$

and

$$\varphi'(\zeta) = \varphi_0'(\zeta) + \frac{1}{(\gamma_2 - \gamma_1)}\left\{\frac{f_0'(\zeta)(\zeta - \gamma_2) - f_0(\zeta) + f_0^-(\gamma_2)}{(\zeta - \gamma_2)^2} - \frac{f_0'(\zeta)(\zeta - \gamma_1) - f_0(\zeta) + f_0^-(\gamma_1)}{(\zeta - \gamma_1)^2}\right\}. \tag{23}$$

Using Taylor's formula with remainder, we have

$$f_0(\zeta_1) = f_0(\zeta) + f_0'(\zeta)(\zeta_1 - \zeta) + \frac{1}{2} f_0''(\zeta + \theta(\zeta_1 - \zeta))(\zeta_1 - \zeta)^2. \quad \zeta, \zeta_1 \in D^- \tag{24}$$

As ζ_1 goes to γ_2, we obtain

$$f_0^-(\gamma_2) = f_0(\zeta) + f_0'(\zeta)(\gamma_2 - \zeta) + \frac{1}{2} f_0''(\zeta + \theta(\gamma_2 - \zeta))(\gamma_2 - \zeta)^2. \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (23) and let ζ go to γ_2, we obtain

$$\varphi'^-(\gamma_2) = \varphi_0'(\gamma_2) + \frac{1}{(\gamma_2 - \gamma_1)} \cdot \left\{\frac{1}{2} f_0''^-(\gamma_2) - \frac{f_0'^-(\gamma_2)(\gamma_2 - \gamma_1) - f_0^-(\gamma_2) + f_0^-(\gamma_1)}{(\gamma_2 - \gamma_1)^2}\right\}. \tag{26}$$

Appendix 2

Let

$$f_0(\zeta) = \frac{(1 - e^{-2\gamma i})}{2\pi i}\int_{L_2} \frac{\overline{\varphi'^-(\sigma)} g_*(\sigma)}{\sigma(\sigma - \zeta)} d\sigma. \qquad \zeta \in D^- \tag{27}$$

If the Goursat functions in the physical plane (z-plane) are $\varphi_1(z)$ and $\psi_1(z)$, we have

$$\varphi(\zeta) = \varphi_1(\omega(\zeta)), \quad \varphi'(\zeta) = \varphi_1'(\omega(\zeta))\omega'(\zeta).$$

Let the region between the arc L_2 and the secant $\bar{L}_2$ be denoted by T_2, as shown in Fig. 10, then the function $\bar{\omega}(1/\zeta)$ is holomorphic in T_2 and takes the same values on L_2 as

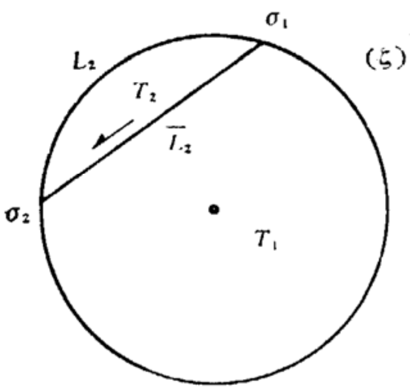

Fig. 10. Circle on the ζ-plane.

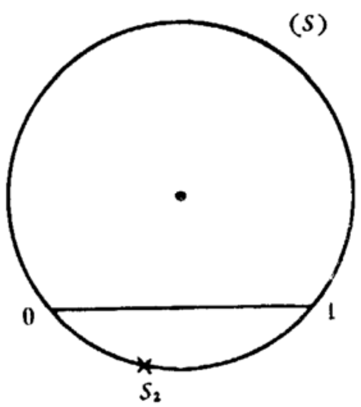

Fig. 11. s-plane.

the function $\omega(\zeta)e^{2\gamma i}$, sectionally holomorphic with the cut $\bar{L}_2$. Therefore in T_2 we have this identity:

$$\bar{\omega}\left(\frac{1}{\zeta}\right) = \omega(\zeta)e^{2\gamma i}. \qquad \zeta \in T_2 \tag{28}$$

Function $\varphi'(1/\zeta)$ is also holomorphic in T_2, therefore,

$$f_0(\zeta) = \frac{(1 - e^{-2\gamma i})}{2\pi i}\int_{\bar{L}_2} \frac{\bar{\varphi}'(1/\sigma)g_*(\sigma)d\sigma}{\sigma(\sigma - \zeta)}, \qquad \zeta \in D^- \tag{29}$$

where the integration path is already shifted to the secant $\bar{L}_2$. We have the following relations:

$$\bar{\varphi}'\left(\frac{1}{\zeta}\right) = \overline{\varphi'\left(\frac{1}{\bar{\zeta}}\right)} = \overline{\varphi_1'\left(\omega\left(\frac{1}{\bar{\zeta}}\right)\right)\omega'\left(\frac{1}{\bar{\zeta}}\right)}$$

$$= -\overline{\varphi_1'(\overline{\omega(\zeta)}\,e^{-2\gamma i})}\frac{\zeta(\zeta - \gamma_1)(\zeta - \gamma_2)}{(\zeta - \sigma_1)(\zeta - \sigma_2)}\,\omega(\zeta)e^{2\gamma i}. \qquad \zeta \in T_2 \tag{30}$$

It is well known that the function $\varphi_1(z)$ can be expanded into the following series:

$$\varphi_1(z) = \sum_{n=1}^{\infty} A_n(z - z_2)^{\nu_n}, \tag{31}$$

$$\varphi_1'(z) = \sum_{n=1}^{\infty} \nu_n A_n(z - z_2)^{\nu_n - 1}, \qquad \nu_n = \frac{n}{2} \tag{32}$$

in the neighbourhood of the propagation branch tip $z = z_2$. Hence

$$\overline{\varphi_1'\left(\omega\left(\frac{1}{\bar{\zeta}}\right)\right)} = \sum_{n=1}^{\infty} \nu_n \bar{A}_n\left(\bar{\omega}\left(\frac{1}{\zeta}\right) - \bar{z}_2\right)^{\nu_n - 1} = \sum_{n=1}^{\infty} \nu_n \bar{A}_n(z - z_2)^{\nu_n - 1}e^{2\gamma i(\nu_n - 1)},$$

when $\zeta \in T_2$. Substituting this expression into Eq. (29), we have

$$f_0(\zeta) = -\frac{(1 - e^{-2\gamma i})}{2\pi i}\int_{\bar{L}_2} \frac{\omega^-(\sigma)(\sigma - \gamma_1)(\sigma - \gamma_2)}{(\sigma - \zeta)}\sum_{n=1}^{\infty} \nu_n \bar{A}_n(\omega^-(\sigma) - \omega(\gamma_2))^{\nu_n - 1}e^{2\gamma i\nu_n}d\sigma$$

$$= -(1 - e^{-2\gamma i})\sum_{n=1}^{\infty} \nu_n \bar{A}_n f_n(\zeta)e^{2\gamma i\nu_n}, \tag{33}$$

where

$$f_n(\zeta) = \frac{1}{2\pi i}\int_{\bar{L}_2} \frac{\omega^-(\sigma)(\sigma - \gamma_1)(\sigma - \gamma_2)}{(\sigma - \zeta)}\,[\omega^-(\sigma) - \omega(\gamma_2)]^{\nu_n - 1}d\sigma, \tag{34}$$

and $\omega^-(\sigma)$ refers to the values of $\omega(\zeta)$ on the secant $\bar{L}_2$ as ζ goes to L_2 from inside of the region T_2. Introduce the following linear transformation

$$\zeta = \sigma_1 + s(\sigma_2 - \sigma_1), \tag{35}$$

by which the exterior of the unit circle in the ζ-plane is mapped onto the exterior of the circle L^* in the s-plane, and the secant $\overline{\sigma_1\sigma_2}$ onto the segment $(0, 1)$ on the real axis. Then

$$\omega(\zeta) = Ae^{-\pi\lambda_2 i}\frac{(\sigma_2 - \sigma_1)^2}{\sigma_1}\,\Omega(s), \tag{36}$$

$$\Omega(s) = \frac{s^{\lambda_1}(1-s)^{\lambda_2}}{(1+es)}, \tag{37}$$

and

$$e = \frac{\sigma_2 - \sigma_1}{\sigma_1}. \tag{38}$$

Substituting Eqs. (36), (37), and (38) into Eq. (34), we obtain

$$f_n(\zeta) = \left(\frac{Ae^{-\pi\lambda_2 i}}{\sigma_1}\right)^{\nu_n} \frac{(\sigma_2-\sigma_1)^{2\nu_n+1}}{2\pi i} \cdot \int_0^1 \frac{\Omega(t)(t-s_2)}{(t-s)} [\sigma_1 - \gamma_1 + t(\sigma_2-\sigma_1)][\Omega(t) - \Omega(s_2)]^{\nu_n - 1} dt. \tag{39}$$

$f_n(\zeta)$, $f_n'(\zeta)$, and $f_n''(\zeta)$ exist everywhere in the exterior of the unit circle and on the unit circle, including the point $\zeta=\gamma_2$, except for the points σ_1 and σ_2. Using above expressions, we can easily show that for the case $n \geqslant 2$, $f_n(\zeta)$, $f_n'(\zeta)$, and $f_n''(\zeta)$ all approach zero in the limit as the length of the propagation branch goes to zero[13]. Hence, in order to find $f_0(\zeta)$ in the limiting case it is only necessary to calculate $f_1(\zeta)$, which is

$$f_1(\zeta) = (\sigma_2-\sigma_1)^2 \sqrt{\frac{Ae^{-\pi i\lambda_2}}{\sigma_1}} \int_0^1 \frac{\Omega^-(t)(t-s_2)\{\sigma_1-\gamma_1+t(\sigma_2-\sigma_1)\}}{(t-s)\sqrt{\Omega^-(t)-\Omega(s_2)}} dt,$$

and

$$f_1''(\zeta) = 2\sqrt{\frac{Ae^{-\pi i\lambda_2}}{\sigma_1}} \int_0^1 \frac{\Omega^-(t)(t-s_2)\{\sigma_1-\gamma_1+t(\sigma_2-\sigma_1)\}}{(t-s)^3\sqrt{\Omega^-(t)-\Omega(s_2)}} dt, \tag{40}$$

where s_2 is the image of γ_2:

$$s_2 = \frac{(\gamma_2-\sigma_1)}{(\sigma_2-\sigma_1)}.$$

On the other hand, since the numerator of the function $\Omega^-(t)$ takes real values when t varies on the interval $[0, 1]$ of the real axis, $\Omega^-(t)$ can be extended analytically from the lower half plane to the upper half plane through the interval $[0, 1]$. Therefore, the function $\Omega(s)$ can be expanded into a Taylor's series in the neighbourhood of $s=s_2$ with a circle of convergence including some part of the interval $[0, 1]$. Since $\omega'(\gamma_2) = 0$ and $\Omega'(s_2)=0$, we have

$$P_0(t) = \frac{\Omega^-(t)-\Omega(s_2)}{(t-s_2)^2} = \sum_{n=2}^{\infty} \frac{\Omega^{(n)}(s_2)}{n!}(t-s_2)^{n-2} \tag{41}$$

in the interval. Denoting

$$P(t) = -i\sqrt{P_0(t)}, \tag{42}$$

and

$$Q(s_2) = \frac{1}{2\pi}\int_0^1 \frac{-\Omega^-(t)dt}{(t-s_2)^3 P(t)}, \tag{43}$$

and integrating them by part, we obtain

$$Q(s_2)=\left\{\int_0^1\frac{P(t)-P(s_2)}{(t-s_2)}dt+P(s_2)\ln\left(\frac{s_2-1}{s_2}\right)\right\}\frac{1}{2\pi}$$

$$-\Omega(s_2)\left\{\left[\int_0^a+\int_b^1\right]\frac{dt}{(t-s_2)^3P(t)}+\frac{1}{2}\int_a^b\frac{\left[\frac{1}{P(t)}\right]''-\left[\frac{1}{P(t)}\right]''_{t=s_2}}{(t-s_2)}dt\right.$$

$$\left.+\frac{1}{2}[P^{-1}(t)]''_{t=s_2}\ln\frac{(b-s_2)}{(a-s_2)}-\frac{1}{2}\frac{\left[\frac{1}{P(t)}\right]'}{(t-s_2)}\Bigg|_a^b-\frac{1}{2}\frac{\left[\frac{1}{P(t)}\right]}{(t-s_2)^2}\Bigg|_a^b\right\}\frac{1}{2\pi},\tag{44}$$

where all the integrals are the Riemann integrals in the ordinary sense. In the limit as the length of the propagation branch approaches zero, we have

$$s_2\to t_2=\frac{\lambda_1}{2},\qquad \Omega(t)\to t^{\lambda_1}(1-t)^{\lambda_2},\tag{45}$$

and

$$D=\operatorname*{Lim}_{r_2\to 0}Q(s_2)=\frac{\Omega(t_2)}{2\pi}\left\{\frac{1}{\Omega(t_2)}\int_0^1\frac{P(t)-P(t_2)}{t-t_2}dt-\left[\int_0^{t_2-\xi_0}+\int_{t_2+\xi}^1\right]\frac{dt}{(t-t_2)^3P(t)}\right.$$

$$+\frac{1}{2}\int_{t_2-\xi}^{t_2+\xi}\frac{P_2(t)-P_2(t_2)}{t-t_2}dt-\frac{1}{2\xi}[P_1(t_2+\xi)+P_1(t_2-\xi)]$$

$$+\frac{1}{2\xi^2}\left[\frac{1}{P(t_2+\xi)}-\frac{1}{P(t_2-\xi)}\right]+\frac{P(t_2)}{\Omega(t_1)}\ln\frac{(1-t_2)}{t_2}$$

$$\left.-\pi i\left(\frac{P(t_2)}{\Omega(t_2)}+\frac{1}{2}P_2(t_2)\right)\right\},\tag{46}$$

where

$$P_1(t)=\frac{1}{2}\frac{P_0'(t)}{P_0(t)}\frac{1}{P(t)},\tag{47}$$

$$P_2(t)=\frac{1}{2}\left\{P_0(t)P_0''(t)-\frac{3}{2}P_0'(t)P_0'(t)\right\}\frac{1}{P_0^2(t)P(t)},\tag{48}$$

and ξ is an arbitrary positive number that satisfies the following condition:

$$2\xi\leqslant\lambda_0=\min\{\lambda_1,\lambda_2\}.\tag{49}$$

It is easily shown that[13]

$$|\Omega^{(n)}(t_2)|\leqslant\Omega(t_2)|\cdot(n+1)!\left(\frac{2}{\lambda_0}\right)^n.\quad n\geqslant 3\tag{50}$$

Hence the following Taylor's series exists:

$$\Omega(t)=\Omega(t_2)+\Omega'(t_2)(t-t_2)+\cdots+\frac{\Omega^{(n)}(t_2)}{n!}(t-t_2)^n+\cdots\tag{51}$$

when $|t-t_2|\leqslant\xi$. From Eqs. (40) and (46), we have

$$\operatorname*{Lim}_{\varepsilon\to 0}f_1''^{-}(\gamma_2)=-4\sqrt{\frac{A\,e^{-\pi\lambda_2 i}}{\sigma_1}}D,\tag{52}$$

and

$$\operatorname*{Lim}_{\varepsilon\to 0}f_1'^{-}(\gamma_2)=0.\tag{53}$$

From Eq. (26), we have

$$\varphi'^{-}(\gamma_2) = \varphi_0'(\gamma_2) - \frac{1}{4} f_0''^{-}(\gamma_2), \tag{54}$$

as $\varepsilon \to 0$. Using Eq. (33), we can obtain

$$\begin{aligned}\frac{1}{4} f_0''^{-}(\gamma_2) &= -\frac{1}{4}(1 - e^{-2\gamma i})\frac{1}{2}\bar{A}_1 e^{\gamma i} f_1''^{-}(\gamma_2)\\ &= \frac{1}{2}(1 - e^{-2\gamma i})\bar{A}_1\sqrt{A}\, e^{\gamma i} e^{-\pi(\lambda_2+1)i/2} D.\end{aligned} \tag{55}$$

It was shown in [1] that

$$\frac{1}{\sqrt{e^{\pi\lambda_1 i}}} A_1 = \frac{1}{\sqrt{2\pi}}(K_I - iK_{II}) = \frac{\sqrt{2}\,\varphi'^{-}(\gamma_2)}{\sqrt{e^{\pi\lambda_1 i}\,\omega''(\gamma_2)}}. \tag{56}$$

Substituting Eq. (56) into Eqs. (54) and (55), we have

$$\varphi'^{-}(\gamma_2) = \varphi_0'(\gamma_2) - \frac{1}{4}(1 - e^{-2\gamma i})C^* \overline{\varphi'^{-}(\gamma_2)}, \tag{57}$$

where

$$C^* = 2\left(\frac{\lambda_1}{\lambda_2}\right)^{\gamma/2\pi} Di = C_1^* + iC_2^*, \tag{58}$$

$$C_1^* = \left(\frac{\lambda_1}{\lambda_2}\right)^{\gamma/2\pi}\left\{P(t_2) + \frac{1}{2}\Omega(t_2)P_2(t_2)\right\}, \tag{59}$$

$$\begin{aligned}C_2^* = \frac{\Omega(t_2)}{\pi}\left(\frac{\lambda_1}{\lambda_2}\right)^{\gamma/2\pi}&\left\{\frac{1}{\Omega(t_2)}\int_0^1 \frac{P(t) - P(t_2)}{(t - t_2)}dt - \left[\int_0^{t_2-\xi} + \int_{t_2+\xi}^1\right]\frac{dt}{(t - t_2)^3 P(t)}\right.\\ &+ \frac{1}{2}\int_{t_2-\xi}^{t_2+\xi}\frac{P_2(t) - P_2(t_2)}{(t - t_2)}dt - \frac{1}{2\xi}[P_1(t_2 + \xi) + P_1(t_2 - \xi)]\\ &\left.+ \frac{1}{2\xi^2}\left[\frac{1}{P(t_2 + \xi)} - \frac{1}{P_2(t_2 - \xi)}\right] + \frac{P(t_2)}{\Omega(t_2)}\ln\frac{(1 - t_2)}{t_2}\right\}.\end{aligned} \tag{60}$$

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