ON THE FROZEN FORCE-FREE MAGNETIC FIELD

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ABSTRACT

In this paper, the general features of the axial symmetrical force-free field with large magnetic Reynolds number (R > 10^3) are discussed. By introducing the magnetic surface function, it is possible to derive a set of fundamental equations describing force-free fields of frozen type from both the conditions of force-free field and magnetic island equations. The configuration of a magnetic field is actually of a force force-free field only if all the three relations of the fundamental equations are satisfied. We present several consistent solutions. For the static cases, the freeze-in factor η of the field need not be a constant. On condition that η is a constant, these solutions are satisfied also as if the azimuthal velocity component was a function of the magnetic surface function. The consistent solutions of some steady three-dimensional motion and the similar solutions of the unsteady motion are obtained. Finally, the structure and the evolution of the magnetic rope are analyzed, and some more general results than the Alfvén model are obtained.

1. On the Force-Free Field

The region of a strong magnetic field is known as the force-free field when the local electric current density runs parallel to the local magnetic field vector. The configuration of the magnetic field have received considerable considerations. In most parts of the earth’s magnetosphere, the magnetic field configurations satisfy the condition of force-free field. In the solar atmosphere, especially in the sunspots of solar active regions, the magnetic field should be force-free. The essential problem of the solar flare is to study the configuration and evolution of the force-free field. The magnetic field in other stellar atmosphere may also be force-free. There has also been discussion on the configuration of the force-free field in the magnetosphere of the pulsars. The force-free field model of a galactic magnetic field is an interesting item too. On the other hand, the configuration of the force-free field is a technical and theoretical problem of importance in the production and set-up of a strong magnetic field, especially in the experiment of plasma physics. Thus, the idea of force-free field is gradually built up on a great deal of study on geophysics, astrophysics and experimental physics.

To put it precisely, the force-free field refers to a field, of which the Lorentz force equals zero, i.e.

\[ \text{curl} \mathbf{B} \times \mathbf{B} = 0. \]  (1.1)
This formula is also expressed as:

$$\text{curl } B = \alpha B,$$

(1.2)

where \( \alpha \) is a scalar function of position and time, which can be called the force-free factor. Further, the magnetic field must satisfy the condition:

$$\text{div } B = 0.$$

(1.3)

The condition (1.1) is obtained from the relation of moment conservation. When the dynamical pressure and the thermodynamical pressure of the macroscopic motion and the stress of the turbulent motion are much smaller than the magnetic pressure in order, then no force will be equal to non-zero Lorentz force, and the electric current must run parallel to the local magnetic field everywhere. Thus, the relation the force-free field expresses should be satisfied by the strong magnetic field. In the low \( \beta \) problems of plasma physics, configurations of force-free field should be of a magnetic field in the case of \( \beta \ll 1 \).

A vector is determined completely if both of its divergence and curl are given under certain boundary conditions. The relationship of force-free field (1.3) gives the divergence of the magnetic field, and (1.1) gives the relationship of the curl to projecting on the direction of B. The magnetic field could be given only if we present the relationship of the curl B projecting on the direction perpendicular to B.

$$\text{Curl } B \cdot B = \alpha B,$$

i.e. the force-free field could be determined only when we can give the distribution of the force-free factor \( \alpha \). The key of the problem lies in the discussion of the law governing the change of \( \alpha \).

When the theory of magnetohydrodynamics is applied to the study of the interaction between the magnetic field and the plasma, it implies both the mechanical effect of the magnetic field acting on plasma and the electro-magnetic effect of the plasma acting on the magnetic field. The former leads to force-free condition (1.1), while the latter is expressed by the induced equation, i.e.,

$$\frac{\partial B}{\partial t} = \text{curl} (\nabla \times B) + \gamma \alpha B,$$

(1.4)

where \( \gamma \) is the magnetic diffusion coefficient. Evidently, the force-free field must also satisfy the relationship of (1.4). This shows that the plasma, in the relationship of moment conservation, may have arbitrary velocity distribution, but the electro-magnetic effect of the plasma to the field will change the configurations of the magnetic field, and the velocity distribution of the force-free field is not arbitrary. This demands that the configurations of the magnetic field which are changed by the Draining field should satisfy the conditions of force-free field in time and space, that is, Eqs. (1.1), (1.3), and (1.4) must be consistent. At this point, \( \alpha (r, t) \) must satisfy certain limiting conditions.

Much work is confined to the discussion of the static study of the force-free field. This may be viewed as the result of \( \mathbb{E} \approx 0 \). Jette demonstrated that the necessary condition for the static force-free field in the case of finite conductivity must be a constant \( \alpha \), which will not change with time and space, i.e.
\( a(r, t) = \text{constant} \). \hspace{1cm} (1.5)

Using Eqs. (1.2) and (1.4), we find

\[ \Delta B + \dot{a} B = 0, \hspace{1cm} (1.6) \]
\[ \frac{\partial B}{\partial t} = - \nabla \times B. \hspace{1cm} (1.7) \]

Therefore, the static force-free field is a decaying field, and from (1.7) we see that

\[ B(r, t) = e^{-\omega t} B_0(\mathbf{r}), \hspace{1cm} (1.8) \]

where \( B \) is determined by (1.5). This type of the configurations of the magnetic field has many limitations, it being unable to describe the processes of the increase of the magnetic energy and the change of the space structure with the change of time.

Alfvén and others have studied the type of force-free field with simple motion, which they called magnetic rope\(^{15}\). The kinetic features of force-free field with \( \omega_0 \ll 1 \) have also been studied recently\(^{16}\).

In many practical problems, \( \omega_0 \) is usually very large. In this case, the diffusion effect of magnetic field is less important, and the induced equation (1.4) is simplified as

\[ \frac{\partial B}{\partial t} = \text{curl} (\nabla \times B), \hspace{1cm} (1.9) \]

i.e. the magnetic field will satisfy the "frozen" condition. We call the force-free field at the large \( \omega_0 \), the frozen force-free field, which is generally related with the kinetic and dynamical state of the plasma.

II. THE BASIC EQUATIONS OF AXIALLY SYMMETRIC
FROZEN FORCE-FREE FIELD

When the cylindrical coordinate system is used, the divergence of (1.2) gives:

\[ (B \cdot \nabla) a = 0. \hspace{1cm} (2.1) \]

This shows that the force-free factor \( a \) is invariant along the force lines. In the axially symmetric problem, every force line relating around the \( z \) axis gives a magnetic surface. Formula (2.1) shows that the values of \( a \) on every magnetic surface is constant. Let us introduce into the function of the magnetic surface \( \varphi \) and make

\[ B_0 e_r + B_1 e_z = - \frac{1}{r} \varphi \hat{\phi} \times e_z, \hspace{1cm} (2.2) \]

where \( e_r, e_\phi, e_z \) are the unit vectors. Expanding (2.2), we shall have

\[ B_0 = \frac{1}{r} \frac{\partial \hat{\phi}}{\partial z}, \hspace{1cm} B_1 = - \frac{1}{r} \frac{\partial \hat{\phi}}{\partial r}. \hspace{1cm} (2.3) \]
Thus, every magnetic surface may be described as:

$$\phi(r, t, \theta) = \text{constant.} \quad (2.4)$$

Equation (1.3) is satisfied by the definition of $\phi$. Substituting (2.3) into (2.1), we have

$$\frac{\partial \phi}{\partial r} \frac{\partial r}{\partial s} - \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial s} = 0. \quad (2.5)$$

This shows that the value of $s$ on every magnetic surface remains invariant, although the value of $\phi$ for the same magnetic surface at different times may not be the same. Thus, many detailed functions could be transformed into the function of $\phi$.

1. The Demonstration of Basic Equations

Substituting (2.3) into force-free condition (1.2), we have

$$\frac{\partial (r B_0)}{\partial r} = -\frac{\partial u}{\partial t}, \quad \frac{\partial (r B_0)}{\partial \theta} = -\frac{\partial \phi}{\partial r}, \quad (2.6)$$

$$r B_0 = \frac{1}{\alpha} \mathcal{L} (\phi), \quad (2.7)$$

where the operator $\mathcal{L} (\phi)$ is equal to $r B_0$. It may be expressed as

$$\mathcal{L} (\phi) = \frac{\partial \phi}{\partial r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial t}. \quad (2.8)$$

From both relationships of (2.6), we derive

$$\frac{\partial (r B_0)}{\partial r} \frac{\partial r}{\partial s} - \frac{\partial (r B_0)}{\partial \theta} \frac{\partial \theta}{\partial s} = 0. \quad (2.9)$$

Thus $r B_0$ is the function of $\phi$ and includes the parameter $t$. This may be expressed as

$$r B_0 = G(\phi, t). \quad (2.10)$$

Substituting (2.10) into (2.6), we can easily get

$$\phi(r, t) = -\frac{\partial G(\phi, t)}{\partial \phi}. \quad (2.11)$$

The result of (2.11) agrees with (2.5), which gives an expression of the force-free vector $A$. Using (2.10) and (2.11) in (2.3), we obtain

$$\mathcal{L} (\phi) = -G(\phi, t) \frac{\partial G(\phi, t)}{\partial \phi}. \quad (2.12)$$

This is an elliptic partial differential equation of the second order, the right-hand side could be non-linear, and its solution is existent and unique for certain boundary values. The three relationships (2.10)—(2.12) are equivalent to the conditions of force-free field (1.2) and (1.3) in the cases of axial symmetry. Much of the work in the
study of force-free field is limited to the discussion of the property of formula (2.23), with the usual assumption that \( n \) is a constant. Obviously, besides the condition of force-free field, we have to take into consideration the magnetic induced equation at the same time.

By using formula (1.3), the force-free relation can be reduced to

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{V} - (\nabla \cdot \mathbf{V}) \mathbf{B}. 
\]

(2.23)

Substituting relation (2.3) of the magnetic surface function \( \psi \) into the above formula, we have

\[
\begin{align*}
\frac{\partial \mathbf{B}}{\partial t} &+ \mathbf{V} \frac{\partial \mathbf{B}}{\partial t} + \mathbf{B} \frac{\partial \mathbf{V}}{\partial t} = - \left( \frac{\partial \mathbf{B}}{\partial t} \right) \frac{\partial \mathbf{B}}{\partial t} + \left( \frac{\partial \mathbf{B}}{\partial t} \right) \frac{\partial \mathbf{B}}{\partial t}, \\
\frac{\partial \mathbf{B}}{\partial \mathbf{v}} &+ \mathbf{V} \frac{\partial \mathbf{B}}{\partial \mathbf{v}} + \mathbf{B} \frac{\partial \mathbf{V}}{\partial \mathbf{v}} = - \left( \frac{\partial \mathbf{B}}{\partial \mathbf{v}} \right) \frac{\partial \mathbf{B}}{\partial \mathbf{v}} + \left( \frac{\partial \mathbf{B}}{\partial \mathbf{v}} \right) \frac{\partial \mathbf{B}}{\partial \mathbf{v}}, \\
\frac{\partial \mathbf{B}}{\partial \mathbf{r}} &+ \mathbf{V} \frac{\partial \mathbf{B}}{\partial \mathbf{r}} + \mathbf{B} \frac{\partial \mathbf{V}}{\partial \mathbf{r}} = - \left( \frac{\partial \mathbf{B}}{\partial \mathbf{r}} \right) \frac{\partial \mathbf{B}}{\partial \mathbf{r}} + \left( \frac{\partial \mathbf{B}}{\partial \mathbf{r}} \right) \frac{\partial \mathbf{B}}{\partial \mathbf{r}}.
\end{align*}
\]

(2.24)

Exs. (2.14) and (2.16) give the relationship between \( \psi \) and the meridional velocity components \( u \) and \( w \), with \( R \) and \( v \) absent in these equations. It is not difficult to find that (2.14) and (2.16) are consistent, and that they can be simplified. Thus,

\[
\begin{align*}
\frac{\partial \psi}{\partial t} + \mathbf{V} \frac{\partial \psi}{\partial t} + \mathbf{B} \frac{\partial \mathbf{V}}{\partial t} &= 0, \\
\frac{\partial \psi}{\partial \mathbf{v}} + \mathbf{V} \frac{\partial \psi}{\partial \mathbf{v}} + \mathbf{B} \frac{\partial \mathbf{V}}{\partial \mathbf{v}} &= 0.
\end{align*}
\]

(2.17)

(2.18)

From the above two equations, we obtain the first integral relationship:

\[
\frac{\partial \psi}{\partial t} + \mathbf{V} \frac{\partial \psi}{\partial t} + \mathbf{B} \frac{\partial \mathbf{V}}{\partial t} = f(t),
\]

(2.19)

where \( f(t) \) is an arbitrary function of time. We may combine \( f(t) \) into the initial value of \( \psi \), and take the right-hand side of (2.19) to be zero. This corresponds to the transformation of a new magnetic surface function, namely,

\[
\begin{align*}
\phi &= \phi - \int f(t) dt, \\
\mathcal{O}(\phi, t) &= \mathcal{O}(\psi, t).
\end{align*}
\]

(2.20)

We may neglect the superscript \( - \) and express the magnetic surface function as \( \psi \), and (2.19) thus becomes

\[
\frac{\partial \psi}{\partial t} + \mathbf{V} \frac{\partial \psi}{\partial t} + \mathbf{B} \frac{\partial \psi}{\partial t} = 0.
\]

(2.21)

The partial differential equation (2.21) is of the first order, its characteristics being
\[ \frac{dt}{1} = \frac{dr}{v} = \frac{dz}{w} \]  
\[ (2.22) \]

From this, we obtain two first integral relationships:
\[ x_\theta(r, z, t) = c_\theta, \quad x_r(r, z, t) = c_r. \]  
\[ (2.21) \]

From (2.21), we obtain the general solution of the magnetic surface function:
\[ \phi(r, z, t) = P[x(r, z, t), x_\theta(r, z, t)]. \]  
\[ (2.24) \]

The function \( P \) is determined by initial value or boundary value. So \( \psi \) could be solved from (2.22)-(2.24) in principle, if the distribution of \( w \) and \( w \) is known.

Substituting (2.24) into (2.3), we can obtain the distribution of meridian magnetic field:
\[ B_r = -\frac{1}{r} \left( \frac{\partial P}{\partial z} \frac{\partial \theta}{\partial r} + \frac{\partial P}{\partial r} \frac{\partial \theta}{\partial z} \right), \quad B_\theta = \frac{1}{r} \left( \frac{\partial P}{\partial r} \frac{\partial \theta}{\partial r} + \frac{\partial P}{\partial z} \frac{\partial \theta}{\partial z} \right). \]

Let us now discuss Eq. (2.15). We may substitute (2.19) into (2.15), considering that
\[ \frac{\partial (r \Phi)}{\partial t} = \frac{\partial (r \Phi)}{\partial z} + \frac{\partial (\Phi)}{\partial \Phi} \frac{\partial (r \Phi)}{\partial r}, \quad \frac{\partial (r \Phi)}{\partial \Phi} \frac{\partial (r \Phi)}{\partial z}, \quad \frac{\partial (r \Phi)}{\partial \Phi} \frac{\partial (r \Phi)}{\partial r}, \]
and using (2.5) and (2.21), we derive
\[ (2.35) \]

On the axially symmetric frozen force-free field, we have obtained the basic equation, composed of three equations (2.12), (2.21) and (2.25). This system is an overdetermined system. If we want to solve the five unknown functions \( u, v, w, \psi, G \), it is necessary to analyse the velocity and magnetic field of some special type to satisfy the configurations of the frozen force-free field.

2. The Consistency of Basic Equations

It could be demonstrated that the basic equations will always be consistent, if no limitation is added to \( V \), although the velocity so determined is not always consistent physically. If the distribution of velocity is given, Eq. (2.3) will be a determined system, of which the solution is not always consistent with (1.1) and (1.3). We have to solve the problems of the consistency of the basic equations under reasonable boundary conditions. There is inevitably some relation between the force-free factor and the velocity and magnetic field.

It could be estimated from the condition of force-free field that this kind of configuration has certain stationary properties. The mathematical forms of the quasi-linear elliptic equation of the 2nd order (2.12) also confirm such properties. The solution of (2.12) makes the configuration of the magnetic field at any specific time satisfy the condition of force-free field, but does not surely satisfy magnetic induced equations.
The induced equations under frozen conditions may be summarized into (2.21) and (2.25). The type of (2.21) can be discussed on the basis of (2.14) and (2.16). The characteristic equation of (2.14) is

\[
\begin{vmatrix}
-\lambda & 0 & \frac{1}{2} \\
0 & -\lambda & \frac{a}{2} \\
\frac{1}{2} & \frac{a}{2} & \lambda - i
\end{vmatrix}
= \lambda^3 - \lambda^2 + \frac{1}{2} = 0. \tag{2.26}
\]

It is not difficult to find the three characteristic roots, which are respectively

\[
\lambda_1 = 0, \quad \lambda_2 = \frac{1}{2} (\sqrt{1 + w^2} + w), \quad \lambda_3 = \frac{1}{2} (\sqrt{1 + w^2} - w). \tag{2.27}
\]

Similarly, from the characteristic equation of (2.16), we can obtain its characteristic roots:

\[
\lambda_1 = 0, \quad \lambda_2 = \frac{1}{2} (a + \sqrt{1 + a^2} + a^2), \quad \lambda_3 = \frac{1}{2} (a - \sqrt{1 + a^2} + a^2). \tag{2.28}
\]

So, the characteristic roots of (2.14) and (2.16) are a zero, a positive, and a negative root. Eq. (2.21) is parabolic for an unsteady problem, and hyperbolic for a steady problem.

If we want Eqs. (2.12) and (2.21) to be consistent, then we must be sure that the solution of (2.12) under certain initial value has the same distribution as the boundary value problem of (2.21) for time \( t \). It is not likely that we shall be able to find the existence of solving the basic equations in given velocity, considering that these two equations are of different type, that they are possibly non-linear, and that they have to satisfy formula (2.25) at the same time.

In demonstrating (2.22)—(2.24), we assume that \( u \) and \( w \) are independent of \( \psi \).

If the meridian velocity is the function of \( \psi \), it will be easy for formula (2.21) to have discontinuity solutions. If these discontinuity solutions propagates in the region, it will be impossible to satisfy the conditions of force-free field. If the discontinuity surface is steady, there will be different configurations of force-free field in different regions, the \( u \) and \( \psi \) being discontinuity while crossing the discontinuity surface. It seems that in some structure of magnetic neutral sheets there exist such discontinuity configurations.

In the same problem of physics, both the equations of (2.12) and (2.21) must be either linear or non-linear of \( \psi \) at the same time. For the linear problem, the right-hand side of (2.12) may be reduced to

\[
\sigma(\psi, \xi) \frac{\partial \psi}{\partial \psi} = \sigma(\xi, \psi) \psi + \frac{1}{2} \tilde{a}(\xi),
\]

or solved as

\[
\sigma(\psi, \xi) = \pm \left[ \sigma(\xi, \psi) \psi' + \sigma(\xi, \psi) \phi' + \sigma(\xi, \psi) \xi' \right].
\]
If \( G(0, t) = 0 \), then \( G(t) + a(t) = 0 \). This implies

\[
G(\phi, t) = \frac{1}{2} \left( a(t) \phi^2 + a(t) \phi \right),
\]

(2.29)

Using (2.11), we obtain the following conclusion on consistence: when the meridian velocity is independent of the magnetic surface function \( \phi \), a necessary condition of the basic equations is that the force-free factor \( a \) satisfies

\[
a(\phi, t) = \frac{1}{2} \left( a(t) \phi^2 + a(t) \phi \right).
\]

(2.30)

A type of special linear problem may be now discussed. If \( a(t) = 0 \), then the force-free factor is only a function of time, namely, \( a(t) = \sqrt{\lambda(t)} \). In this case, the basic equation are reduced to the following three linear equations:

\[
\begin{aligned}
\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial r} + w \frac{\partial \phi}{\partial x} &= 0, \\
\frac{\partial \phi}{\partial x} &= 0,
\end{aligned}
\]

(2.31)

\[
\begin{aligned}
\left( \frac{\partial \phi}{\partial x} \right) + \left( \frac{\partial \phi}{\partial r} \right) \frac{\partial \phi}{\partial r} + \left( \frac{k}{\partial t} + s \left( \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial x} \right) \right) &= 0.
\end{aligned}
\]

Let us make the variable transformation, and change \((r, x, t) \) into \((\xi, \eta, \tau) \):

\[
\xi = a(t) r, \quad \eta = a(t) x, \quad \tau = a(t) t.
\]

(2.32)

So long as \( a(t) = \text{constant} \), we can cancel \( a \) from (2.31). Therefore, these solutions are similar in transformation, if there exists the consistent solution of (2.31). If the physical space is enlarged \( a \) times, the spatial configuration of the field will have the similar structures.

The basic equations derived above may be taken as a criterion of frozen force-free field, and used to study many other special problems, such as the kinetic features of the force-free field, the low governing evolution, the relation between the plasma energy and magnetic energy, and others. These are all very important topics.

III. Solutions of Frozen Force-Free Field

1. Static Problems

For the static problems, we have

\[
\eta = v = \mu = 0.
\]

(3.1)

Using (3.1) in (2.21) and (2.25), we obtain

\[
\frac{\partial \phi}{\partial t} = 0, \quad \frac{\partial G(\phi, t)}{\partial t} = 0.
\]

(3.2)

Thus, the static problems are steady process, namely,
\[ \phi = \phi(r, z), \quad \mathbf{Q} = Q(r). \] 

(3.3)

Introducing (3.3) into (2.12), we reduce the static problems to the following elliptic partial differential equations of the 2nd order under certain boundary value:

\[ \nabla^2 \phi - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) - \frac{\partial^2 \phi}{\partial z^2} = 0. \] 

(3.4)

In physics, the fact that the magnetic field does not change with time, means that the typical time of the problems under discussion is much shorter than the diffusion time of the field.

Jette has proved that the value of \( \alpha \) for the static force-free field of \( B_0 \equiv 1 \) must be a constant which does not vary with time and space. On the basis of these points he has criticized the models of static force-free field with non-constant value. However, according to our analysis of the frozen force-free field, any \( \phi \) that satisfies formula (3.4) is the solution of self-consistence in static problems, of which \( \alpha \) may be a constant or an arbitrary function of \( \phi \). These configurations of force-free field have all physical significance. Therefore, we must make concrete analysis of some works published before.

In the following, we shall discuss the general static problems when \( \alpha \) is a constant. This is applicable to cases of both large \( B_0 \) and small \( B_0 \). When \( B_0 \) is small, the field obtained corresponds to the \( B_0 \) portion in formula (1.8). When \( \alpha \) is a constant, both (2.11) and (3.4) imply

\[ \nabla^2 \phi - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) - \frac{\partial^2 \phi}{\partial z^2} = 0. \] 

(3.5)

Introducing (3.5) into (3.4), we obtain the linear differential equation:

\[ \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \phi = 0. \] 

(3.6)

This equation could be solved by the method of separation variables, so let

\[ \phi(r, z) = \sum_{\lambda=1}^{\infty} R_\lambda(r) Z_\lambda(z), \] 

(3.7)

and we get

\[ \frac{\partial^2 R_\lambda}{\partial z^2} = \lambda \lambda R_\lambda, \] 

(3.8)

\[ \frac{\partial^2 R_\lambda}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial R_\lambda}{\partial r} \right) + (\alpha^2 + \lambda^2) R_\lambda = 0, \] 

(3.9)

where the characteristic root \( \lambda \) is determined by certain boundary condition. By transformation,

\[ R_\lambda(\nu) = r Q_\lambda(\nu). \] 

(3.10)

Formula (3.3) could be reduced to the Bessel equation, namely,

\[ \frac{\partial^2 Q_\lambda}{\partial \nu^2} + \frac{1}{\nu} \frac{\partial Q_\lambda}{\partial \nu} + \left( \nu^2 + \lambda^2 \right) Q_\lambda = 0. \] 

(3.11)
In Eqs. (3.8)-(3.11), the plus and minus signs represent two different types of boundary values respectively. If we take the above signs, we shall obtain the general solution:

\[ \phi(r, \beta) = r \sum_{n} \left[ \left\{ c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} I_n(\beta, r) + \left\{ c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} Y_n(\beta, r) \right], \]  

(3.12)

where \( c_n \) is the coefficient to be determined by the boundary values in the axial direction, \( I_n \) and \( Y_n \) are the Bessel functions, and

\[ \beta_n = \sqrt{\kappa^2 + \beta^2}. \]  

(3.13)

Using (3.12), we could obtain the distribution of the magnetic field:

\[ B_r = \sum_{n} \beta_n \left[ \left\{ -c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} I_n(\beta, r) + \left\{ c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} Y_n(\beta, r) \right], \]  

(3.14)

\[ B_\theta = -\sum_{n} \beta_n \left[ \left\{ c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} I_n(\beta, r) + \left\{ c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} Y_n(\beta, r) \right], \]  

(3.15)

\[ B_\phi = -\sum_{n} \beta_n \left[ \left\{ c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} I_n(\beta, r) + \left\{ c_n e^{i n \beta} + c_n^* e^{-i n \beta} \right\} Y_n(\beta, r) \right]. \]  

(3.16)

Similarly, if we take the signs below in (3.8) and (3.11), we obtain the general solutions of other type of boundary value:

\[ \phi(r, \beta) = r \sum_{n} \left[ \left\{ D_n \sin(\lambda_n \beta) + D_n^* \cos(\lambda_n \beta) \right\} I_n(\gamma, r) \right. \]  

\[ \left. + \left\{ D_n \sin(\lambda_n \beta) + D_n^* \cos(\lambda_n \beta) \right\} Y_n(\gamma, r) \right], \]  

(3.17)

where \( D_n \) is determined by the boundary values, \( I_n \) and \( Y_n \) are modified Bessel functions. The taking of characteristic roots makes

\[ \gamma_n = \sqrt{\Delta^2 - \beta^2} > 0. \]  

(3.18)

We may similarly derive the distribution of the magnetic field corresponding to solution (3.17). On the basis of these results, we have discussed the configurations and kinematic features of the magnetic field of the solar spot.

The relationship of (3.2) and (3.3) obtained from the static condition (3.1) is obviously too strong. If the angular velocity component is uniform on every magnetic surface at special time, namely,

\[ \omega = \tilde{\omega}(\phi, \beta), \]  

(3.19)

then the right-hand side of (2.25) will be zero. Furthermore, we add the condition,

\[ u = u = 0, \]  

(3.20)

and use conditions (3.19) and (3.20) to replace the static condition (3.1) then we shall obtain the entirely same results as those in the static problems.

Furthermore, (3.4) is quite similar to the second equation of (2.31). The difference lies in the fact that \( \alpha \) in (3.6) is a constant. The remaining two equations of
(2.3) are satisfied in the cases of static problems, the similar transformations as (2.22) existing in this case. These similar relationships may be viewed as the basic characteristics of the static frozen-force-free field.

2. The Steady Kinetic Problems

In discussing the three-dimensional steady problems, we have now

$$\frac{\partial \psi}{\partial t} = 0,$$  \hspace{1cm} (3.21)

$$\frac{\partial \mathbf{u} \times \mathbf{B}}{\partial t} = 0.$$  \hspace{1cm} (3.22)

The basic equations could be reduced to

$$\begin{cases}
\frac{\partial \psi}{\partial r} + \frac{\hat{w}}{r} \frac{\partial \phi}{\partial \theta} = 0, \\
\frac{\partial \mathbf{u}}{\partial r} + \frac{\hat{w}}{r} \frac{\partial \mathbf{u}}{\partial \theta} = \mathbf{r} \left[ \frac{\partial}{\partial r} \left( \frac{\psi}{r} \right), \frac{\partial}{\partial \theta} \left( \frac{\psi}{r} \right), \frac{\partial}{\partial z} \left( \frac{\psi}{r} \right) \right], \\
\mathbf{\Omega} \times \mathbf{u} = -\nabla \phi \frac{\partial \mathcal{L}(\phi)}{\partial \phi}.
\end{cases}$$  \hspace{1cm} (3.23)

If the angular velocity $\mathbf{\Omega} = \mathbf{e}/r$ is uniform on every magnetic surface, namely,

$$\frac{\mathbf{u}}{r} = \mathbf{\Omega}(\phi),$$  \hspace{1cm} (3.24)

then, Eq. (3.24) is reduced to

$$\frac{\partial \psi}{\partial r} \left( \frac{\hat{w}}{r} \right) + \frac{\partial \phi}{\partial \theta} \left( \frac{\hat{w}}{r} \right) = 0.$$

(3.25)

Let us use function $\chi(r, \theta)$ and make

$$w = -\frac{\partial \chi}{\partial \theta}, \quad w = -\frac{\partial \chi}{\partial \theta},$$

(3.26)

then (3.27) is satisfied. Introducing (3.26) into (3.24), we obtain the general solution of $\psi$ as [see (2.21)-(2.24)]

$$\psi = f(\chi(r, \theta)).$$  \hspace{1cm} (3.27)

The consistence of the basic equations requires that the solution of (3.25) under certain boundary value must be a function of $\chi(r, \theta)$. It is not difficult to satisfy this requirement. Introducing the general solution (3.26) into (3.25), we derive

$$-\nabla \phi \frac{\partial \mathcal{L}(\phi)}{\partial \phi} = \frac{dF}{d\chi} \frac{\partial \mathcal{L}(\chi)}{d\chi} + \frac{dF}{d\ell} \left[ \frac{\partial \mathcal{L}(\chi)}{d\ell} + \frac{\partial \mathcal{L}(\chi)}{d\ell} \right].$$

(3.28)

This means that the right-hand side of (3.28) must be the function of either $\psi$ or $x$. If this requirement is satisfied, the integral of (3.28) gives
\[-\frac{1}{2} \delta_{\varepsilon}(\phi) + C = \left\{ \frac{dF}{d\varepsilon} \delta_{\varepsilon}(\phi) + \frac{\partial F}{\partial \varepsilon} \left( \frac{\partial \delta_{\varepsilon}(\phi)}{\partial r} \right) + \frac{\partial F}{\partial \varepsilon} \left( \frac{\partial \delta_{\varepsilon}(\phi)}{\partial z} \right) \right\} \frac{dF}{d\varepsilon} dx, \tag{3.31} \]

where \( C \) is a constant. Eq. (3.31) gives the relationship between \( \phi \) and \( \varepsilon \).

If the problem is of one variable, it could be expressed by
\[
\phi = \phi(x), \quad \varepsilon = \varepsilon(x), \tag{3.32}
\]
or
\[
\phi = \phi(z), \quad \varepsilon = \varepsilon(z). \tag{3.33}
\]

These types of problems can always satisfy the conditions of existence, and they represent problems of very wide features.

In the case of problems of two variables, if
\[
\chi(r, z) = \chi(r^2 + z^2), \tag{3.34}
\]
then
\[
\delta_{\varepsilon}(\chi) = 4(r^2 + z^2) \chi \frac{dF}{d\varepsilon} + 2(z^2 + 2r^2 + z^2) \chi \frac{dF}{d\varepsilon}, \tag{3.35}
\]
where the differentiation of \( \chi \) is for the variable of \( r^2 + z^2 \). The right-hand side of the above equation is a function of \( \chi \) and the distribution of \( \delta_{\varepsilon}(\chi) \) could be solved by integration. Especially, if we take
\[
\chi(r^2 + z^2) = (r^2 + z^2)^{-1}, \tag{3.36}
\]
where \( a \) is an appropriate constant, or let us take
\[
\chi(r^2 + z^2) = \exp\left(-\frac{r^2 + z^2}{a}\right) - 1, \tag{3.37}
\]
then these typical flowing fields are solutions of existence. On the basis of these results, many configurations of magnetic fields with slow variation could be studied.

2. The Similar Solutions of Unsteady Flow

For the study of the similar solutions of unsteady flow, let us assume
\[
\begin{align*}
\frac{u}{u} &= f(x, z, t), \\
\frac{v}{v} &= f(x, z, t), \\
\frac{w}{w} &= f(x, z, t), \\
\frac{\phi}{\phi} &= f(x, t) \psi, \\
G &= k \psi. \tag{3.38}
\end{align*}
\]

Introducing (3.38) into the basic equations, we could obtain the solution of existence as
\[
\begin{align*}
c_1 &= c_2 = c_3 = -1, \\
k &= \left(\frac{c_1}{c_2} - 1\right) c_3, \quad \text{(when } m = 1\text{)}, \\
0 &= c_3, \quad \text{(when } m = 1\text{).} \tag{3.39}
\end{align*}
\]

and
\[
\begin{align*}
f \frac{\partial \phi}{\partial t} + f \frac{\partial \phi}{\partial z} - \frac{k}{m + 1} f i = 0, \\
\left[ \delta_{\varepsilon}(L) - \frac{\partial L}{\partial z} \right] \frac{\partial \phi}{\partial t} = \frac{1}{h} \left[ \frac{\partial L}{\partial z} \frac{\partial \phi}{\partial r} + \frac{\partial L}{\partial z} \frac{\partial \phi}{\partial r} \right]. \tag{3.41}
\end{align*}
\]
The second equation of (3.41) gives \( f_i \); then a linear relationship between \( f_i \) and \( f_i \) is obtained from the first equation of (3.41); if some physical condition is added, the solution of consistence could be obtained from the third equation of (3.41). In the following, we shall still discuss the similar solutions in the form of power series, assuming that \( f_i \) in (3.38) has the form:

\[
f_i(x, r) = k r^i e^{\lambda r}, \quad (i = 1, 2, 3, 4). \tag{3.42}
\]

Using (3.42) in the first equation of (3.41), we have:

\[
\lambda^2 \rho^e (r^e)^e + \lambda \rho^e (r^e)^e + \lambda^2 = \frac{k}{m - 1} r^e e = 0; \tag{3.43}
\]

and it is solved that

\[
e_1 = 1, \quad e_2 = 0, \quad b_1 = 0, \quad b_2 = 1; \tag{3.44}
\]

and

\[
a_0 + b_1 = \frac{k}{m - 1} = 0. \tag{3.45}
\]

Then, using (3.43) in the second equation of (3.41), we could easily obtain

\[
a_0 (a_0 - 2) e_a e_a + a_0 (a_0 - 1) e_a e_a = - \omega \rho (e_a e_a + \omega e_a e_a + \omega e_a e_a). \tag{3.46}
\]

It may be seen that if both \( a_0 \) and \( b_1 \) are not zero, there will be no consistent solutions. In the case of \( a_0 = 0 \), we obtain

\[
a_0 = 0, \quad b_1 = - \frac{1}{m - 1}, \quad (\omega = 1), \tag{3.47}
\]

and

\[
a_0 = 0, \quad b_1 = \frac{1}{m - 1}, \quad (\omega = 1). \tag{3.48}
\]

Similarly, if \( b_1 = 0 \), we have

\[
a_0 = 0, \quad b_1 = 0, \quad (\omega = 1); \tag{3.49}
\]

and

\[
a_0 = 0, \quad b_1 = 0, \quad (\omega = 1). \tag{3.50}
\]

Using (3.42) in the third equation of (3.41), we have:

\[
a_0 = 1 + (m - 1) e_1, \quad a = 1 + (m - 1) b_1 \tag{3.51}
\]

\[
a_0 (a_0 - 2) e_a e_a + a_0 (a_0 - 1) e_a e_a = \omega \rho (e_a e_a + \omega e_a e_a + \omega e_a e_a). \tag{3.52}
\]

Thus, we have obtained two groups of consistent solutions in the form of power series.

(i) The case of \( a_0 = 0 \). The solutions are

\[
u = - \lambda \frac{1}{r}, \quad v = \lambda \frac{1}{r}, \quad w = - \lambda \frac{1}{r}, \quad \phi = \lambda \frac{1}{r}, \quad G = \lambda^2. \tag{3.53}
\]

These solutions require \( m = 0 \), hence \( G(\rho, t) = G(1) \), namely, force-free factor \( \alpha = 0 \).

The configuration of the magnetic field now is the potential field without electric current.
(ii) The case of \( b_0 = 0 \). The solutions are

\[
\begin{align*}
    u &= -\frac{k}{2} t, \\
    v &= \pm \frac{k + b_2}{(u - 1)^{1/2}} \sqrt{\frac{1 - 2m}{u}} \phi^{-(u-1)/2}, \\
    v &= \pm \frac{1}{(u - 1)^{1/2}(u - m)} \sqrt{\frac{1 - 2m}{u}} \phi^{1 - m},
\end{align*}
\]

(3.54)

In order to make the coefficients in Eq. (3.54) all real numbers, it is required that the value of \( m \) should satisfy

\[
0 < m < \frac{1}{2}.
\]

(3.55)

The magnetic field now is force-free field with electric current. The coefficient \( k \) could be determined if the initial value is given at \( t = t_0 \). The solutions of both (3.53) and (3.54) are not singular, and may be applied to the study of some processes of evolution.

The evolution of thermodynamic parameters could also be discussed. If we introduce the solution of potential field (3.53) into the continuity equation of matter, we may derive

\[
\rho \frac{\partial \rho}{\partial t} + k_T \frac{\partial \rho}{\partial r} - k_T \frac{\partial \rho}{\partial z} = (b - 2k_0) \rho, \quad (3.56)
\]

where \( \rho \) is the density of plasma. It will not be difficult to obtain the general solution of (3.56),

\[
\rho = e^{\lambda t} P(r^{m1}, t^1), \quad (3.57)
\]

If the initial condition is expressed as

\[
\rho|_{t_0} = \rho_0 (r^{1}, t^1), \quad (3.58)
\]

then the special solution (3.57) which satisfies the initial value (3.58) will be

\[
\rho = \left( \frac{1}{k_T} \right) \rho_0 \left( r^1 \right)^{m_1} \left( \frac{t^1}{t_0} \right)^{\lambda_1}. \quad (3.59)
\]

Similarly, we can find out the processes of evolution for the other thermodynamic parameters, such as temperature and pressure. Through such analyses, we can have a vivid picture of the kinematic features of the force-free field.

IV. The Structure and Evolution of a Magnetic Rope

The magnetic line of axial symmetric frozen force-free field is generally a spatial spiral. To a magnetic surface,

\[
\phi = C, \quad \text{(constant)}; \quad (4.1)
\]

according to (2.10), the tangent component of the magnetic field on the magnetic surface is
\[ B_s = \frac{1}{r} G(r, \theta). \]  

(4.2)

If \( G \) is the linear function of \( \psi \), hence \( a_1 = 0 \) in Eq. (2.9), then

\[ B_s = -\frac{\Omega}{r} \varphi(r). \]  

(4.3)

This shows that, the screw of magnetic lines is a general feature of the force-free field (\( a = 0 \)) and the degree of the screw is closely dependent on the value of the force-free factor \( a \).

Alfvén and others have studied the magnetic rope\(^{[5,6]}\) and applied it to explain many filament and fibril structures of astrophysics. But their model is steady and has finite conductivity, therefore it has limitation in application. The evolute problems about the so-called Alfvén model will be studied specially in this section.

Alfvén has studied the structure of the magnetic rope for the plasma with middle density in a uniform electric field. In this case, the force-free field satisfies

\[ \begin{cases} v = w = 0, & n_r = 0, \\ j_r(x)/j_z(x) = n_r(x)/n_z(x) = K(x), \end{cases} \]  

(4.4)

(4.5)

and takes the initial value \( K(0) = 0 \). Let us use the concept of frozen force-free field, and assume that both (4.4) and (4.5) are operable.

According to the condition that \( K = 0 \), the function of the magnetic surface is independent of \( r \), namely,

\[ \frac{\partial \phi}{\partial r} = 0, \quad \psi = \psi(r, t). \]  

(4.6)

This is an unstable one-dimensional problem, and the basic equations may be reduced to

\[ \frac{\partial \phi}{\partial t} + \frac{\partial \psi}{\partial r} = 0, \]  

(4.7)

(4.8)

(4.9)

Obviously, these systems are determined. By using the relationship in Alfvén’s model (4.5) and those in (2.3) and (2.10), we obtain

\[ \Omega(\phi, \psi) = -\frac{\partial K}{\partial r} \frac{\partial \psi}{\partial r}. \]  

(4.10)

For the differential (4.10) of time \( t \)

\[ \frac{\partial \psi}{\partial t} = \frac{\partial \Omega(\phi, \psi)}{\partial t} + \frac{\partial \Omega(\phi, \psi)}{\partial \phi} \frac{\partial \phi}{\partial t} - K(r) \frac{\partial \phi}{\partial r}. \]  

(4.11)
Introducing (4.11) into (4.9), and cancelling the term $\frac{\partial \mathbf{E}}{\partial t}$ in (4.7) and the term $\frac{\partial \mathbf{B}}{\partial t}$ in (4.8), we have

$$\frac{\partial \Phi}{\partial t} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\rho \mathbf{E}}{\rho} \right) - \frac{1}{\rho} \frac{\partial \mathbf{B}}{\partial \rho} = 0. \tag{4.12}$$

To integrate with $\rho$, we derive

$$\frac{\partial \Phi}{\partial \rho} = A_\rho(t) \exp \left[ - \int_{\rho_0}^{\rho} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right], \tag{4.13}$$

where

$$A_\rho(t) = \rho E_\rho(\rho, r, t). \tag{4.14}$$

To integrate again with (4.13), we obtain the function of the magnetic surface:

$$\Phi = A_\rho(t) + A_\phi(t) \int_{\rho_0}^{\rho} \exp \left[ - \int_{\rho_0}^{r} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right] \, dr, \tag{4.15}$$

where

$$A_\phi(t) = \Phi(\rho_0, t). \tag{4.16}$$

Furthermore, introducing (4.13) into (4.10), we have

$$\Phi(\rho, t) = A_\rho(t) \int_{\rho_0}^{\rho} \exp \left[ - \int_{\rho_0}^{r} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right] \, dr. \tag{4.17}$$

Using (4.15) and (4.17), we obtain the following relationships:

$$u_\rho(r, t) = - A_\rho(t) \int_{\rho_0}^{r} \exp \left[ - \int_{\rho_0}^{r} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right] \, dr, \tag{4.18}$$

$$A_\phi(t) \int_{\rho_0}^{r} \exp \left[ - \int_{\rho_0}^{r} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right] \, dr \left\{ \begin{array}{l}
B_\rho = \rho \frac{K(r)}{K(r') + 1} A_\rho(t) \exp \left[ - \int_{\rho_0}^{r} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right], \\
B_\phi = - \frac{1}{r} A_\phi(t) \exp \left[ - \int_{\rho_0}^{\rho} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right] \end{array} \right. \tag{4.19}$$

In order to find out the distribution of $K(r)$, we have to make use of (4.8). If we substitute (4.13) for operator $\mathcal{L}(\phi)$, it will not be difficult to obtain

$$\mathcal{L}(\Phi) = - \frac{\rho}{r} \frac{K(r)}{K(r) + 1} A_\phi(t) \exp \left[ - \int_{\rho_0}^{r} \frac{1}{r} \frac{K(r') - 1}{K(r') + 1} \, dr' \right]. \tag{4.20}$$

Considering (4.8) and (4.17), we may from (4.20) derive

$$u = - \frac{\partial \Phi(r, t)}{\partial \phi} = \frac{\rho}{r} \frac{K(r)}{K(r) + 1} \tag{4.21}$$

Now, $u$ is only a spatial function and independent of time. In solving $K(r)$, we must at the same time satisfy Eqs. (4.17), (4.21) and (4.15), because
\[
\frac{\partial \phi \phi}{\partial r} = \frac{\partial \phi^2}{\partial r} - \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r}
\]

Differentiating (4.17) with respect to \( r \) and introducing (4.13) into the above equation, we have

\[
\frac{\partial \phi \phi}{\partial r} = - \frac{dK}{dr} \frac{K^2(r) - 1}{r^2} \cdot (4.22)
\]

The difference between (4.22) and (4.21) is only one minus sign, and from there we may derive

\[
\frac{dK}{dr} \frac{K^2(r) - 1}{r^2} = 0. (4.23)
\]

It is noted at once that

\[
K(r) = K_i \frac{r}{r_i}, \quad (4.24)
\]

where \( K_i = K(r_i) \) is a constant, and requires that \( r_k > 6 \). If we use (4.24) in the above relationships, we obtain the final results, namely,

\[
\begin{align*}
\phi &= \frac{A_i(t)}{2} \left( 1 + \frac{K}{K_i^2} \ln \left( \frac{1 + K_i^2/r_i^2}{1 + K^2/r^2} \right) \right), \\
G &= -\frac{A_i(t)}{2} \left( \frac{r}{r_i} \right)^2 \left( 1 + \frac{K_i^2}{K^2} \right), \\
\alpha &= -\frac{A_i(t)}{2} \left( \frac{r}{r_i} \right)^2 \left( \frac{1 + K_i^2}{K^2} \right) \ln \left( \frac{1 + K_i^2/r_i^2}{1 + K^2/r^2} \right).
\end{align*}
\]

(4.25)

(4.26)

(4.27)

On the basis of these results, we shall be able to analyse the evolution and dynamical features of the magnetic rope. Omitting the less important term \( A_i \), we may take \( A_i = 0 \). Then we obtain from (4.27):

\[
w(r, t) = -\frac{A_i(t)}{2K_i^2} \left( \frac{r}{r_i} \right)^2 \left( 1 + \frac{K_i^2}{K^2} \right) \ln \left( \frac{1 + K_i^2/r_i^2}{1 + K^2/r^2} \right). \]

(4.28)

The above formula shows that if \( |A_i(t)| \) is increased, the radial velocity will be negative and the plasma will flow inward; otherwise, \( \alpha \) will be positive and the plasma will flow outward.

From the above Eqs. (4.25) and (4.26), we obtain that the density of the magnetic energy in the magnetic rope is

\[
\sigma_n = \frac{A_i(t)}{8\pi r_i^2} \left( \frac{1 + K_i^2}{K^2} \right) \ln \left( \frac{1 + K_i^2/r_i^2}{1 + K^2/r^2} \right). \]

(4.29)

Thus, the total magnetic energy in \( r_i < r < K_i \) is

\[
E_n = \int_{r_i}^{K_i} \sigma_n 2\pi r \sin \theta \, dr = \frac{A_i(t)}{4\pi K^2} \left( 1 + K_i^2 \right) \ln \left( \frac{1 + K_i^2/r_i^2}{1 + K^2/r^2} \right). \]

(4.30)
Both relations (4.28) and (4.30) show that if $|A(t)|$ is increased with the increase of time, the magnetic energy density and the total magnetic energy of the magnetic region in a finite space will be increased during the increase of $t$, and vice versa.

The reason is that the plasma and magnetic field are frozen together, the magnetic energy is increased when the plasma flows inward, and decreased when the plasma flows outward.

The change of the thermodynamical parameters could also be discussed. The continuous equation of plasma for the unsteady one-dimensional problem is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho \mathbf{w}}{\partial r} + \frac{1}{r} \frac{\partial (\rho \mathbf{w}^2)}{\partial r} \rho = 0. \quad (4.31)$$

Using (4.28) in the above formula, we derive

$$\frac{A(\xi)}{A(0)} \frac{\partial \rho}{\partial t} - \frac{\rho_0}{2K_0} \left( 1 + K \xi^2 \right) \left( \frac{\sigma_0}{\sigma} \right) \ln \left( 1 + \frac{K \xi^2}{1 + K} \right) \frac{\partial \sigma}{\partial r}$$

$$- \left[ 1 + \ln \left( \frac{1 + K \xi^2}{1 + K} \right) \right] \rho = 0. \quad (4.32)$$

By discussing the characteristic equation of (4.32), two solutions of first integral relationships are given as

$$\left[ \left( 1 + K \xi^2 \right) \frac{\exp \left[ A(\xi) \right]}{1 + K} \right] = C_1 \quad (4.33)$$

$$\rho \left( \frac{1 + K \xi^2}{1 + K} \right) \ln \left( 1 + K \xi^2 \right) = C_1. \quad (4.34)$$

Therefore, the general solution of the plasma density is:

$$\rho = C \left[ \left( 1 + K \xi^2 \right) \frac{\exp \left[ A(\xi) \right]}{1 + K} \right] + P \left( \frac{1 + K \xi^2}{1 + K} \right)^{1/4} \sigma_{\xi \xi},$$

where $C$ is an arbitrary constant, and $P$ is an arbitrary function; both are determined by the initial value. If the initial density is uniform,

$$\rho_{\xi \xi} = \rho_0 \text{ (constant)} \quad (4.35)$$

This implies that we have the following special solution, which will satisfy the initial values (4.35):

$$\frac{\rho}{\rho_0} = 1 + \left[ \left( 1 + K \xi^2 \right) \frac{\ln \left( 1 + K \xi^2 \right)}{1 + K} \right] \frac{\exp \left[ -\frac{A(\xi)}{A(0)} \right]}{1 + K} \sigma_{\xi \xi}. \quad (4.36)$$

Eq. (4.36) also demonstrates that if $|A(t)|$ is increased with the increase of time, the density will be increased; otherwise, the density will be decreased. This corresponds with the fact that the plasma is concentrated when the plasma flows inward, and rarified during outward flow.
If we assume that the plasma is a perfect gas in the adiabatic process, then the thermodynamic relationships give
\[ \frac{p}{\rho} = \left( \frac{\rho}{\rho_0} \right)^\gamma, \quad \frac{\rho}{\rho_0} = \left( \frac{\rho}{\rho_0} \right)^{1-\gamma}. \] \hspace{1cm} (4.37)

We can find the evolution relation between the pressure and the temperature. The p and T are increased as the plasma is compressed and heated when the plasma flows inward, and vice versa.

In the discussion of what is presented above, we have made use of the conditions (4.4) and (4.5) for Alfven's model of a magnetic rope. But, if (4.4) is substituted by
\[ \sigma = \sigma(r, t), \quad u = u(r, t), \quad B = 0, \] \hspace{1cm} (4.38)

namely, \( \sigma \) and \( u \) are the arbitrary functions of \( r, t \), the results we describe above still satisfy the basic equations. The reason is that in the problem of irrotational one-dimensional axial symmetry, the magnetic surface is a cylindrical one, and the uniform rotation and axial motion on every surface do not in any way affect the configurations of the magnetic field. Thus, we may develop the magnetic rope to ease of three-dimensional flow.

Furthermore, Alfven has only studied the case where the plasma is flowing inward. Now, the plasma is concentrated at the inner region, and does not satisfy the condition of force-free field when \( r = 0 \). Their calculation is taken at the boundary \( r = 0 \) and \( K(0) = 0 \). Our analysis demonstrates that the magnetic field could be force field in the region \( r < r_0 \), but the region of \( r > r_0 \) is force-free field, and \( r_0 \) varies with the variance of time. The evolution process of the magnetic rope could only be discussed with the analysis involving time.

As the magnetic Reynolds number is usually very large in the problems of astrophysics, geophysics and even in some problems of plasma physics, the results of frozen force-free field discussed in this paper have a general significance.

References